Is magnetic moment variant or invariant in a plasma?

Leonard Freeman
leonard.freeman@ntlworld.com

Abstract
For many years one of the basic tenets of plasma physics has been the invariance, or constancy, of the magnetic moment of a charged particle in a magnetic field that varies slowly in time or space. However it is proposed here that this is invalid and that the magnetic moment is not constant: it is a function of the magnetic flux density. It is shown that there are contradictions within the conventional theory, and that this is due to a missing term in the derivation. A new equation for the variation of the magnetic moment in a collision free plasma is derived.
The implication of this new equation for the loss cone in magnetic mirrors is considered.

Introduction

A charged particle moving in a magnetic field executes Larmor rotations, and so has a magnetic moment. The invariance of the magnetic moment of a particle in a collision free plasma, in a varying magnetic field, was first proposed by Alfven [1] in 1950, and appears in many texts and papers on Plasma Physics: see for example Boyd and Sanderson (2003) [2] or Yi et al (2005) [3]. A key consequence of this conventional theory is that the magnetic flux Φ, through the particle's gyro-orbit must also be constant, as shown for example by Chen (1984) [4] or as indicated here in equation (6).
The way that a charged particle moves in a varying magnetic field is generally considered in two ways. The first case is when the field is time varying only and the second when it is spatially varying only. In both cases the conventional theory concludes that the magnetic moment of the particle is constant, provided that the field changes slowly or adiabatically.
A similar approach is adopted here – and in both cases it is suggested that a missing term in the conventional theory has produced an incorrect conclusion.

Definitions and standard formulas

The radius, r, of the Larmor orbit, where B is the magnetic field, m the mass of the particle and e is its charge is given by the standard formula:

$$r = \frac{mV_\perp}{Be} \qquad (1)$$

$V_\perp$ is the velocity of the particle at right angles to the field.
The magnetic moment μ is defined as:

$$\mu = \frac{mV_\perp^2}{2B} = \frac{W_\perp}{B} \qquad (2)$$

(This also corresponds to the alternative form $\mu = iA$).
$W_\perp$ is the perpendicular kinetic energy of the particle.
(If μ is constant as conventional theory proposes, it can be seen from (2) that that the kinetic energy $W_\perp$, of a charged particle in a circular Larmor orbit increases linearly with magnetic field B.)

Faraday's law of induction states that

$$\varepsilon = -\frac{d\Phi}{dt} \qquad (3)$$

Where ε is the work done per unit charge (or emf induced) around the perimeter of a closed surface through which a flux Φ passes.

Conventional theory: time varying field

Alfven's (and others) analysis is as follows:
Assume that B is changing very slowly, so that that Larmor orbit is almost a closed circle. This allows the flux through it to be easily calculated.

The conventional theory uses Faraday's law to determine the small energy increase $\delta W_\perp$, during one orbit, caused by a small increase in magnetic flux during the time taken for the orbit:

$$\delta W_\perp = \pi r^2 e \frac{dB}{dt} \tag{4}$$

From (4) the rate of increase of energy can be found by dividing by the time taken for one orbit (2πm/Be) and substituting for 'r' from equation (1):

$$\frac{dW_\perp}{dt} = \frac{W_\perp}{B}\frac{dB}{dt} \tag{5}$$

dt is then eliminated to produce

$$\frac{dW_\perp}{W_\perp} = \frac{dB}{B}$$

Integration gives the conventional result: $W_\perp = \mu B$

Where μ is a constant.

It then follows that the magnetic flux through a gyro-orbit, Φ must also be constant.
This can easily be seen as follows:

$$\Phi = BA = B\pi r^2 = B\pi \frac{m^2 V_\perp{}^2}{B^2 e^2} = \pi \frac{m^2 V_\perp{}^2}{Be^2} \tag{6}$$

which is clearly constant if $\frac{V_\perp{}^2}{B}$ =constant, i.e. if the magnetic moment is constant.

So increasing B slowly by a factor of 100 for example, leads to the kinetic energy of the particle also increasing by a factor of 100, but- paradoxically - with no change in magnetic flux!
The energy increase can be extremely large – but without any change of magnetic flux.

Contradictions

So a contradiction exists between the derivation in equation (4) ,which says there **is** a small flux change during each orbit and the final formula (6) which says there is **no** overall flux change.

There is also a contradiction between the formula Φ = constant and Faraday's law of induction, which states that the induced emf is proportional to the rate of change of magnetic flux. If there is no change in flux, then there is no emf or work done on a charged particle and $W_\perp$ cannot increase.

Missing term - time varying field

The following analysis attempts to resolve this paradox, by showing there is a missing term in the conventional theory.
In the simplest case, where B is uniform over a planar area A at any instant of time, and also at right angles to it, the induced emf is given by:

$$\varepsilon = -\frac{d}{dt}(BA) \tag{7}$$

But equation (4) assumes that the area A of a gyro-orbit is constant and so conventional theory states:

$$\varepsilon = -A\frac{dB}{dt}$$

However the area (and Larmor radius) depend upon B: a small increase in B produces a corresponding decrease in radius and area. Thus in determining the change in flux, we need to consider not only the change in B but also the change in area of the Larmor orbit.
Thus a complete differentiation of equation (7) is required:

$$\varepsilon = -\left(B\frac{dA}{dt} + A\frac{dB}{dt}\right) \tag{8}$$

The second term on the right of equation (8) gives the emf due to a changing magnetic field. The first term gives the emf due to a changing area, which occurs if the circuit is changing or moving. In conventional theory, eg [1] this "changing area" term is missing, on the grounds that when the field B is changing sufficiently slowly, the area (or Larmor radius) is considered to be constant.

As another example, Gartenhaus [5] states this explicitly:
"Consider the motion of particle with kinetic energy $W_\perp$ associated with the velocity perpendicular to the slowly varying magnetic field B. The electric field E associated with this time varying magnetic field does work on the particle and in one Larmor period changes $W_\perp$ by an amount say $\delta W_\perp$. This work may be expressed by

$$\delta W_\perp = q\oint E.dL = q\int (\nabla x E).ds$$

$$= -q\int \frac{\partial B}{dt}.ds \approx -q\pi a^2 \left|\frac{dB}{dt}\right|$$

…a is the (approximately constant) Larmor radius of the particle."

Conventional theory ignores the change in area of the orbit, but the following analysis shows that a new result is obtained when this term is included, which also removes the contradictions referred to.
(Note that Gartenhaus has used a slightly different proof compared to Alven – it is suggested further on how the equation used here is incomplete)

New solution

The new result is obtained by directly by solving equation (8), including the extra term.

The area of the Larmor circle A = πr^2, and substituting from (1):

$$A = \pi \frac{m^2 V_\perp{}^2}{B^2 e^2} \tag{9}$$

In order to find dA/dt, we consider both $V_\perp$ and B to be time dependent – as in conventional theory:

$$\frac{dA}{dt} = \frac{\pi m^2}{e^2}\left(V_\perp{}^2(-2B^{-3})\frac{dB}{dt} + \frac{2V_\perp}{B^2}\frac{dV_\perp}{dt}\right)$$

So that

$$B\frac{dA}{dt} = \frac{\pi m^2 V_\perp}{Be^2}\left(-\frac{2V_\perp}{B}\frac{dB}{dt} + 2\frac{dV_\perp}{dt}\right) \tag{10}$$

And

$$\left(B\frac{dA}{dt} + A\frac{dB}{dt}\right) = \frac{\pi m^2 V_\perp}{Be^2}\left(-\frac{2V_\perp}{B}\frac{dB}{dt} + 2\frac{dV_\perp}{dt}\right) + \frac{\pi m^2 V_\perp{}^2}{Be^2}\left(\frac{1}{B}\frac{dB}{dt}\right) \tag{11}$$

So $$\varepsilon = -\left(B\frac{dA}{dt} + A\frac{dB}{dt}\right) = \frac{\pi m^2 V_\perp}{Be^2}\left(\frac{V_\perp}{B}\frac{dB}{dt} - 2\frac{dV_\perp}{dt}\right) \tag{12}$$

The energy gain over one rotation is eε, and given that the period $T = \frac{2\pi}{\omega} = \frac{2\pi m}{Be}$,

the rate of change of energy is $\frac{d}{dt}\left(\frac{mV_\perp{}^2}{2}\right) = \frac{e\varepsilon}{T} = \frac{\varepsilon Be^2}{2\pi m}$

Substituting into (12) gives: $$\frac{d}{dt}\left(\frac{mV_\perp{}^2}{2}\right) = \frac{mV_\perp}{2}\left(\frac{V_\perp}{B}\frac{dB}{dt} - 2\frac{dV_\perp}{dt}\right) \tag{13}$$

or $$\frac{dW_\perp}{dt} = \frac{W_\perp}{B}\frac{dB}{dt} - \frac{dW_\perp}{dt} \tag{14}$$

so $$\frac{dW_\perp}{dt} = \frac{W_\perp}{2B}\frac{dB}{dt} \tag{15}$$

and $$\frac{dW_\perp}{W_\perp} = \frac{dB}{2B} \tag{16}$$

Integrating gives $W_{\perp} = kB^{1/2}$ (17)

or $\mu = \dfrac{k}{B^{1/2}}$ where k is a constant, given by initial conditions. (18)

This new result demonstrates that the "changing area" term cannot be ignored.
It shows that the magnetic moment is variable. It also can be seen that the particle energy has a weaker dependence on B than previously thought, as the particle energy $W_{\perp}$ is proportional to $B^{1/2}$, rather than to B.
The total flux Φ is now not constant, but is proportional to $B^{-1/2}$.

It is also of interest to note that equation (15) gives the rate of change of particle energy, which is one half of the conventional result, seen in equation (5). This smaller rate of change may be interpreted physically: an increasing magnetic field B, produces a decreasing area A, and results in a lower rate of energy increase than conventional theory.

Conventional theory –other methods
Although the original version [1] of conventional theory used the method outlined here, which is based on Faraday's law of induction, other methods or approaches have used the differential form of the Maxwell-Faraday equation:

$$\nabla x E = -\frac{\partial B}{\partial t} \tag{19}$$

e.g. H Qin and R C Davidson [6] and R O Dendy, [7] or Gartenhaus [5].
However this form of the law has to be used carefully when the loop or circuit is moving. In this case the circuit – the Larmor orbit - is changing: when the magnetic field increases, the orbit shrinks. Equation (19) gives the electric field , **E**, as seen by a stationary observer, not the electric field measured in the moving circuit. The electric field $\mathbf{E_m}$ as measured in the moving loop or circuit is given by :

$\mathbf{E_m} = \mathbf{E} + \mathbf{V} \times \mathbf{B}$.

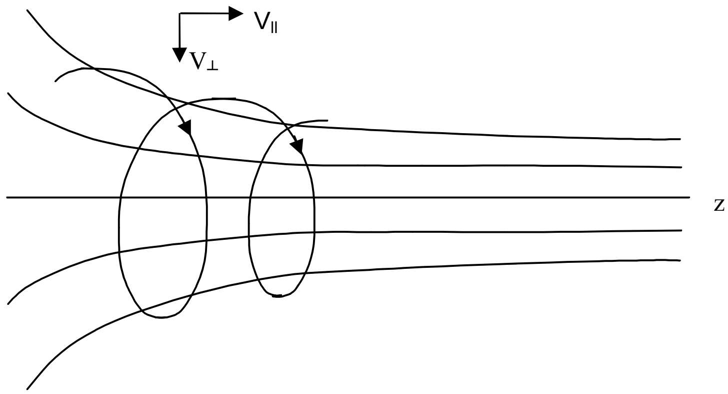
V is the velocity at which the loop or orbit is changing. The overall result is that in using equation (19) we are once again ignoring the changing area of the Larmor orbit.
Thus it would be necessary to use a more general form of the differential equation to include the motional emf, or convective term.
This point is made in texts on electromagnetic theory: see for example Guru et al [8]. Here the movement of the circuit through the field is included. Alternatively Reitz et al (1993) [9] refer to the need for the circuit to be "rigid and stationary" for equation (19) to apply, but in a changing magnetic field, a Larmor orbit is not rigid or stationary.

Removal of the contradiction
The variation of magnetic flux, from this new analysis, means that it is possible to have a gain in the total kinetic energy of a particle, because $\dfrac{d\Phi}{dt} \neq 0$ even when the magnetic field varies slowly, or adiabatically.

Spatially varying field
The analysis that follows uses the result already established for the energy gained by the particle due to induction and motion in a changing magnetic flux, caused in this case by the particle's motion through the field.
The following scenario is similar to the one used by Alven and others.

Consider the situation shown in figure 1. The magnetic field is cylindrically symmetric about the z axis.
A particle is moving to the right into a slowly increasing magnetic field. Assume that the particle has velocity components $V_{\parallel}$ parallel to the field, and $V_{\perp}$ at right angles to the field:

Figure 1



The particle will lose some its V_{ll} velocity, as it experiences a force to the left, caused by the increasing field.
The time rate of change that a particle experiences depends upon its velocity through the field and the spatial rate of change of the field. This can be seen in the full expression for dB/dt:

$$\frac{dB}{dt} = \frac{\partial B}{\partial t} + V_{ll}\frac{\partial B}{\partial z} \tag{20}$$

For a field that varies only spatially, we can find the time variation from

$$\frac{dB}{dt} = V_{ll}\frac{\partial B}{\partial z}$$

With this expression for dB/dt, equations (8) to (18) will still give the relationship between $W_\perp$ and dB/dt , including the final result for magnetic moment μ.

Force on a magnetic dipole and total energy
The standard formula for the force on a dipole in a non-uniform field is:

$$F_z = -\frac{d}{dz}(\mu B) = -\frac{dW_\perp}{dz} \tag{21}$$

(Substituting for μ from equation (2))

So
$$F_z V_{ll} = -\frac{dW_\perp}{dz}V_{ll} = -\frac{dW_\perp}{dt} \tag{22}$$

But $F_z V_{ll}$ is the rate of change of parallel kinetic energy, $\frac{dW_{ll}}{dt}$ so that

$$\frac{dW_{ll}}{dt} = -\frac{dW_\perp}{dt}$$

And
$$\frac{dW_\perp}{dt} + \frac{dW_{ll}}{dt} = 0$$

This simply shows that the total energy of a particle is constant in a magnetostatic field.

Missing term - spatially varying field
The error in the conventional theory appears to be that, as part of the derivation, equation (21) is written as:

$$F_z = -\mu\frac{dB}{dz} \tag{23}$$

(eg Alfven [1]) ,

rather than: $$F_z = -\frac{d}{dz}(\mu B) = -(\mu \frac{dB}{dz} + B\frac{d\mu}{dz})$$

The first term on the right corresponds to the force needed to move a constant magnetic dipole through a magnetic gradient.
However the final term on the right hand side corresponds to the extra force needed if, for example, the magnetic moment is increasing as the particle moves. When this term is omitted, as in conventional theory, it automatically implies that the magnetic moment is constant. No proof for this assumption is given in conventional theory.
The conventional conclusion which is then derived from equation (23), that μ is constant, does not appear to recognize that there is this hidden assumption in the derivation.

Furthermore, although conventional theory shows, as here, how the parallel velocity can be changed by the force Fz, the "constant flux" outcome provides no explanation of **how** $V_\perp$ changes. In the perpendicular plane, an induced electric field is still needed to change the energy $W_\perp$. Equation (20) shows how dB/dt is associated with a spatially varying field, even when the time derivative of B is zero.

So even in the case of a spatially varying field, there will be an induced emf in the gyro-orbit, to provide the mechanism for the energy exchange between $W_\perp$ and W_{ll} .
For a spatially varying field, equations (15) to (18) will apply. Equation (20) gives the expression for dB/dt.

Magnetic mirrors

It is well known that charged particles moving from a weak field to a stronger field, as illustrated in figure 1, may in some circumstances be reflected. As the particle spirals in to the stronger field, it experiences a retarding force F_z as given in equation (21) and gains perpendicular kinetic energy $W_\perp$ at the expense of parallel energy W_{ll}. If the force is sufficiently large, then at some point all of the parallel energy is converted into perpendicular energy. This means that $W_{ll} = 0$ and so the parallel velocity V_{ll} is also zero. The force F_z will then cause the particle to reverse its velocity.
Suppose that a starting position is taken at a point where the field is weak and equal to B_0. Let the perpendicular and parallel energies at this point be $W_{\perp 0}$ and W_{ll0} respectively.
If the particle is reflected, let the kinetic energy at the mirror plane be W_{max}. Then by the conservation of energy

$$W_{\perp 0} + W_{ll0} = W_{max}$$

When the particle is spiralling at the start position, the pitch angle of the spiral, θ is given by

$$\sin^2\theta = \frac{W_{\perp 0}}{W_{\max}}$$

using the general relationship: $W = \frac{1}{2}mv^2$

If the angle θ is less than this, then reflection will not occur. This will apply to particles whose direction of velocity is close to the z axis, or within the angle θ of the axis.

Finally, since $W = kB^{1/2}$ where k = constant

We have $$\sin^4\theta = \frac{B_0}{B_{\max}}$$

Comparing this to the conventional result that $\sin^2\theta = \frac{B_0}{B_{\max}}$

The overall consequence is that the condition for reflection is more stringent than conventional theory predicts, so the loss cone will be larger. For example, under conventional theory, if $\frac{B_0}{B_{\max}} = 0.1$ then particles will be reflected only if the angle θ is greater than 18^0. But for the analysis from this paper, the angle would need to be greater than 34^0 for reflection to occur.

Omissions of conventional theory

In this paper three sources of error in derivations have been identified, including use of the differential form of the Maxwell-Faraday equation. Other theories have been produced which build on Alfven's original result, but are likely to be in error as they implicitly use one of the faulty derivations as a starting point.

Summary

The magnetic moment of a particle is shown to be variable in a magnetic field that is temporally or spatially variable. The energy of the particle is proportional to $B^{1/2}$ rather than B. This means that plasma energy will be increased by magnetic pumping, but not so strongly as previously thought.
The flux through a gyro-orbit is **not** constant, even if the field variation is slow. In a spatially varying field also there is an induced emf, which allows the perpendicular kinetic energy to change.

The result is shown to change the condition of operation of magnetic mirrors, making them less effective in trapping charged particles. The result may also be of importance in other plasmas

References

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Leonard Freeman, M Inst P
23 Hope Street, Cambridge, CB1 3NA, UK