Shifting assignments between infinite sets

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Abstract:

Shifting assignments between infinite sets is creating a disturbance within the assignment itself which cannot be removed. Assignments carrying such a disturbance cannot be seen or taken as a static assignment.

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Shifting and reassigning of sets playing an important role in set theory and are leading to surprising results used with infinite sets. They are widely used in proofs like the Banach-Tarski-Theorem (The Banach-Tarski-Theorem, in a nutshell, seems to prove that you can disassemble a ball into 6 peaces and reassemble those parts into two identical balls of the same size as the original ball!)
Used with infinite sets you can have up to infinite gains and losses which are still unexplained.

This essay is meant to give an answer on the question where those gains and losses come from.

Most commonly known in this matter is probably the virtual hotel Infinity, which can be used very easily to show the assumptions and logics of those operations with infinite sets.

The hotel Infinity consists two infinite sets:
The infinite set of hotel rooms and the infinite set of guests.
Those sets are assigned in a 1 to 1 relationship. Therefore the hotel has no vacancies. Every guest has exactly one room and in each room is exactly one guest.

Now even booked up you can show that by shifting or reassigning, you are able to create infinite new rooms.

We can fix as prerequisites:
1. The hotel has infinite rooms (set of rooms)
2. The hotel is booked up (1 to 1 assignment between the guests and rooms)

If now a new guest arrives at hotel Infinity the clever hotel manager advices the guests as follows: the inhabitant of room #1 should move on into room #2. The inhabitant of room #2 should move to room #3 and so on.

Instantly you have a free room available for the new guest, room #1. It looks like you have created a new room as the hotel was booked up before!
There are multiple variants existing of this principle which can create up to infinite new space (rooms). As all of them are based on the same principle we stick to this easy example.

Now where is this extra room coming from?

The hotel manager simply shifts all his guests' one room ahead into infinity.
This looks easy on a first view but is rather problematic during execution.

In order to get a better understanding about this shifting we define some rules which a static assignment should (must) comply:

Rule #1: Every guest is only allowed to stay in a hotel room no where else (there is no hallway)

Rule #2: In every room only one guest is allowed

Those rules are a direct consequence of the pre assessment of a 1 to 1 assignment between the guests and rooms and I think unchallenged.

As a first breakup of the problem: by defining those rules the idea comes up where this gains and losses come from. As many rooms as are created, there are always as many guests in a temporary double occupation of rooms or on a virtual hallway, as we will see.

There are two possibilities how you can execute such a shifting:

You can try under respect of rule #1:
The guest from room #1 enters the room #2, followed by the guest from room #2 entering the room #3 and so on. You can see as guest #1 is entering the room #2 there are two guests in the room #2. Guest #2 is soon leaving the room but is entering then room #3 so that there 2 guests in room #3 and so on. At all times two guests will be together in one room if the set is infinite. It's like a disturbance in the assignment which move towards infinite bur never ends. This is not a static assignment anymore and it never will become again a static assignment.

Or you try under respect of rule #2:
The guest from room #1 leaves room #1 (onto a virtual hallway) followed from the guest from room #2 leaving the room. The guest from room #1 enters now the room #2. The guest from room #3 leaves the room. The guest from room #2 enters the room #3 and so on. In this case you do not have a double occupancy of hotel rooms but there will always be a guest on the virtual hallway. At no time all the guests are in a room. Again you lost the attribute of a static 1 to 1 assignment.

There is no way to transform a static assignment (an assignment which comply with both rules) complete into a new static assignment through a shifting or reassignment, if performed on infinite sets.

The results of such a shifting or reassignment are taken as new static assignments. But that's not the case. There are always guests in a double occupancy or on the virtual hallway. The double occupancy or the usage of the hallway is temporary only for the single guest but for the infinite set of guests and rooms permanent therefore you cannot take it as static assignment.

A shifted or reassigned infinite set contains a disturbance of the assignment which can not be removed while a static assignment has a fixed and not moving assignment.

A static assignment can only be transferred into a new static assignment through shifting or reassignment if this process can be completed.

A way out would be if all elements of the whole infinite set are shifted at the same time. All the guests are moving from one room to the other room together.
By doing so none of the rules get broken (no double occupancy no usage of the hallway).
In this case a rule is needed which is valid for all elements of the set at the same time.
This brings us back to the hotel manager and his advice to the guests.
It was implicitly assumed that he made his advice in form of a rule like:

Move into the next room and pass this rule to the guest you find in the room.

This is a serial rule which cannot be applied to all the guests at the same time. The rule is passed from one guest to another. All guests must be informed in front to execute this advice at the exact same time. All the guests must get this information but you are facing the problem that the very last guest never gets the information as there is no such thing as the last guest. It is just a relocation of the problem. In this case the preparation of the shifting or reassignment cannot be finished and therefore never be executed.

Basically this means you would have to write out the defined assignment function for all elements (of the infinite sets).
To simply define such a conversion without any prove of existence makes no sense.

All examples of gains and losses with infinite sets are based on the assumption that we are able to transform one static assignment between infinite sets into another static assignment between the infinite sets but the very existence of such assignments does not include the possibility to transform them into each other.

Within the Banach-Tarski-Theorem the cutting lanes are seen as a set of infinite points which are moved or rotated against each other, shifting their assignments. Those rotations or movements are the source for the gains as they are equal to the shifting into infinity described obove. The assumption is made that there is a ***static 1 to 1 relationship*** between the sets of points from the cutting lanes before **and** after the shifting. But that’s not the case as we have seen. Without these virtual movements or shifts the Banach-Tarski-Theorem does not work.

Our common sense tricks us as we can imagine such different relationships between infinite sets. But to shift or reassign them is a task which cannot be finished and without the completeness of the shifting or reassignment there is no such thing as a gain or loss.