Indeterminacy in arithmetic, well-known to logicians, is missing from quantum theory

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Abstract. This article is one of a series explaining the nature of mathematical undecidability discovered within quantum theory. Significantly, a formula's undecidability certifies its indeterminacy and vice versa. This paper describes the algebraic environment in which the undecidability and indeterminacy originate; provides proof of their existence; and demonstrates the role these play in a three-valued logic which is free to permeate mathematical physics via this algebra.

The radical idea applied in this research is taken from very well-known results in mathematical logic. All scalars engage in the arithmetic of scalars by way of a single algebra. But in terms of validity, these scalars partition into sets which are logically distinct: those with valid existence with respect to this algebra, and those with indeterminate existence. Failure of mathematical physics to notice this distinction is the reason why quantum theory is logically at odds with quantum experiments.

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1. Introduction

Inherent within quantum measurement experiments is a decision process which current theory fails to express and does not explain. Each act of measurement decides on one value from a spectrum of definite options. Nature executes this decision according to two influences. One is the physical charactistic of the experiment. The other is an *indeterminate* component not 'caused' by any *physical* influence. Profoundly, this statement rules out any possibility of physical influences of which we may be in ignorance [1]. Consequently, rules for this indeterminacy may possibly exist in Nature but are not to be found in Physics.

This removal of physical influences from the arena of explanations might direct us toward theories rooted in some non-classical logic, or other. Such notions have motivated an extensive history of study scrutinising quantum mathematics for clues. Nevertheless, the absence of references in the physics literature indicates that this scrutiny does not extend as far as the non-classical logic inherent within arithmetic beneath quantum theory, upon which the theory rests. Yet most curiously, personal experience of mathematical logicians reveals them to be acquainted with elements of this logic, to the extent that they regard them as obvious and self-evident.

The discrepancy between experiment and theory is traceable to a logical detail of arithmetic, not encoded in mathematical physics. The arithmetic in question is that of *scalars*. And the logic in question concerns distinct qualities or modes of existence in which scalars occur in the mathematics. This logic of scalars is synonymous with the 3-valued quantum logic postulated by Hans Reichenbach. Reichenbach showed in detail how implementation of his logic resolves the discrepancies [17].

Scalars are the mathematical objects whose rules of algebra are the *Field Axioms*. See Table 1. Mathematical physics assumes the *a priori* existence of scalars. In this article, apriority is transferred to the Field Axioms themselves. This initiative promotes mathematical physics from a *semantic theory* to a *logical theory*. Validity in the logical theory has greater complexity. Well known theorems of model theory, a branch of mathematical logic, set the Field Axioms within a rigorous environment that naturally differentiates between scalars that Axioms *derive*, distinct from scalars that *satisfy* them. Scalars which are not derivable exist *independent of Axioms* [9, 15, 18]. They contain information not present in Axioms.

Model theory proves existence of such independent scalars are *mathematically undecidable* and *logically indeterminate*. Logical indeterminacy and mathematical undecidability are complementary aspects of a logical condition present in certain axiomatised mathematical theories [4, 5, 15]. In such theories, indeterminacy describes the state of validity of propositions that are neither valid nor invalid. Undecidability refers to the provability of these indeterminate propositions, being neither provable nor disprovable.

In 1944, Hans Reichenbach proposed a quantum logic consisting of values: true, false and *indeterminate*. This was in response to 'causal anomalies' evident in the results of quantum experiments. His logic is an adaptation of the 3-valued logic of Jan Łukasiewicz [8, 13], which Reichenbach gives certain truth tables, conjunctions, disjunctions, tautology etc,. During its formation, Reichenbach arrived at the particular qualities of his indeterminate middle through detailed, reasoned analysis of results of quantum experiments.

He found that his 3-valued logic 'suppresses' the causal anomalies [8, 16, 17]. It furnishes a consistent epistemology for *prepared* and *measured* states: typically the question of what we may know about the state of a photon immediately before measurement. It derives *complimentary* propositions: if statement A is either true or false, statement B is indeterminate, and vice versa. Such statements correspond to measurements of complimentary pairs such as momentum and position. And his logic also overcomes the problem of *action at a distance*, a paradox identified by Einstein, Podolsky & Rosen [12].

Though his results are compelling, Reichenbach's logic is hypothetically based and is not in simple agreement with mainstream quantum logics based on the quantum postulates, originating with Birkhoff and von Neumann [2]. Acceptance of these would tend to imply the unacceptability of Reichenbach's logic. That said, Hardegree argues that these logics are not in opposition but describe different things [11]. While the mainstream logics are based on Hilbert space quantum theory, Reichenbach's logic is a framework for an alternative formulation. This present paper expounds foundation for Reichenbach's alternative.

Gödel's First Incompleteness Theorem proves that mathematical undecidability necessarily exists in arithmetic [4, 6, 7, 19]. This is not the kind of undecidability forced upon us through ignorance of information; the distinction is that information necessary for decision does not exist. Chaitin takes this informational approach to Gödel's Theorem. He argues: 'if a theorem [proposition] contains more information than a given set of axioms, then it is impossible for the theorem [proposition] to be derived from the axioms' [6]. Svozil uses Turing's proof of Gödel's Theorem to argue that undecidability exists in Physics [19].

The subject matter of this paper is the *Field Axioms*, the theory they derive, and the indeterminacy inherent within. I call this the *Theory of Fields* to distinguish from *Field Theory* in Physics, which is not under study. Interest in the Theory of Fields is motivated in knowledge of the fundamental position occupied by the arithmetic of the Field Axioms; and in consideration of the non-classical logic this theory must proliferate, most profoundly, throughout applied mathematics, mathematical physics and quantum theory. The object of the investigation is to confirm the existence of indeterminate propositions within the Theory of Fields and establish distinct conditions that guarantee either determinacy, or otherwise, indeterminacy. An inherent 3-valued logic consisting of values: valid, invalid and indeterminate, will become apparent.

Propositions under consideration are mathematical statements proposing the existence of particular scalars, written as formulae in *first-order logic* [3, 4, 5]. Examples of interest are:

$$\exists\alpha\,(\alpha\times\alpha=4)\,; \tag{1}$$

$$\exists\alpha\,(\alpha\times\alpha=2)\,; \tag{2}$$

$$\exists\alpha\,(\alpha\times\alpha=-1)\,; \tag{3}$$

$$\exists\alpha\,\left(\alpha^{-1}=0\right). \tag{4}$$

For the sake of accessibility, I have used a slightly relaxed form here. Strictly, existence of the negative sign and the inverse should also to be proposed.

Naturally, certain propositions are valid and provable, while others are invalid and disprovable. But of special interest is a further set which are neither of these. Instead, these propositions are in independent of Axioms; they are indeterminate and they are mathematically undecidable. For instance: of the four propositions above, the Field Axioms prove only (1). Each of propositions (2) and (3) is neither provable nor disprovable. In both these cases, existence of α can neither be confirmed nor denied. Even so, the square roots of 2 and -1 are nevertheless objects that satisfy the Field Axioms and therefore, they do engage in the arithmetic. Proposition (4) is disproved by the Field Axioms. Together, (1), (2), (3) and (4) illustrate examples of provable, undecidable and disprovable propositions.

These claims of provability and validity are explained in terms of Model theory. This is a branch of Mathematical Logic that considers validity of propositions in relation to associated mathematical structures. In the context of the Field Axioms these associated structures are *fields*: not to be confused with fields in quantum field theory. Normal interpretation of the $+$ and $\times$ operators restricts these to the infinite fields, of which there are at least three: the complex plane $\mathbb{C}$, the real line $\mathbb{R}$ and the smallest, the rational field $\mathbb{Q}$. Each is a closed structure; but jointly they form a field-subfield hierarchy where the smallest field is special because it is a subfield of each. This fact has critical influence on which propositions are valid and provable, and which are indeterminate and undecidable.

2. Algebraic and logical environment

The radical observation of this paper came while noticing the distinction between *necessary* existence, entailing *derivation* from Axioms, and *possible* existence that entails *satisfying* those Axioms. This distinction spurns two related logics. One is notionally *causal* where *n*ecessary and possible, together with necessarily-not constitute a *modal logic* [8]. The other is notionally *existential*, consisting of logically valid, logically invalid and logically indeterminate; identifiable with Reichenbach. The environment in which this second logic emerges from the Field Axioms is now discussed.

From the perspective of applied mathematics, the Field Axioms are seen as a selection of combination rules for addition and multiplication, applied ad hoc, in our most familiar arithmetic. These rules of combination are regarded as properties belonging to *scalars* and so significance, meaning and 'reality' is placed on scalars, with Axioms taking an incidental, appended role. In this applied mathematical scenario, scalars are the semantic interpretations of the objects: $\alpha, \beta, \gamma, 0, 1, \ldots$ in Table 1. This interpretation arises when the mathematician designates $\alpha, \beta, \gamma, 0, 1, \ldots$ to the real line or complex plane, whichever suits the application. This interpretational approach deals in *semantic information.* And the act of such designation irreversibly discards *logical information* imparted by the

	The Field Axioms	
	Additive Group	
FA0	$\forall\alpha\forall\beta\exists\gamma\,(\gamma=\alpha+\beta)$	Closure
FA1	$\exists 0\forall\alpha\,(0+\alpha=\alpha)$	Identity 0
FA2	$\forall\alpha\exists\beta\,(\alpha+\beta=0)$	Inverses
FA3	$\forall\alpha\forall\beta\forall\gamma\,((\alpha+\beta)+\gamma=\alpha+(\beta+\gamma))$	Associativity
FA4	$\forall\alpha\forall\beta\,(\alpha+\beta=\beta+\alpha)$	Commutativity
	Multiplicative Group	
FM0	$\forall\alpha\forall\beta\exists\gamma\,(\gamma=\alpha\times\beta)$	Closure
FM1	$\exists 1\forall\alpha\,(1\alpha=\alpha 1=\alpha\wedge 0\neq 1)$	Identity 1
FM2	$\forall\alpha\exists\beta\,(\alpha\times\beta=1\wedge\alpha\neq 0)$	Inverses
FM3	$\forall\alpha\forall\beta\forall\gamma\,((\alpha\times\beta)\times\gamma=\alpha\times(\beta\times\gamma))$	Associativity
FM4	$\forall\alpha\forall\beta\,(\alpha\times\beta=\beta\times\alpha)$	Commutativity
FAM	$\forall\alpha\forall\beta\forall\gamma\,(\alpha\times(\beta+\gamma)=(\alpha\times\beta)+(\alpha\times\gamma))$	Distributivity

Table 1. The Field Axioms written as sentences in *first-order logic*. The variables: $\alpha, \beta, \gamma, 0, 1$ represent mathematical objects complying with these axioms. The semantic interpretations of these objects are known as *scalars*.

Axioms.

In contrast to the emphasis of applied mathematics on the existence of scalars, *first-order theory* places precedence on Axioms. The Theory of Fields is the first-order theory under the Field Axioms. This poses a quite different scenario in which Axioms define and generate the objects $\alpha, \beta, \gamma, 0, 1, \ldots$ along with their arithmetical behaviour. First-order theory is a stricter and stronger system of derivation than applied mathematics. It takes full account of all logical information imparted by Axioms, including information that is indeterminate. That said, no indeterminate information, independent of Axioms, can be proved to exist by the first-order theory itself. Proof that (1) is a theorem may indeed be established by direct derivation from the Axioms. But direct proof that (2) and (3) *are*, or *are not* theorems is impossible because no information in the Axioms proves or negates them.

In order to confirm the existence of indeterminate information, theorems of Model Theory are applied to the fields. Fields are mathematical structures *satisfying* the Field Axioms [14]. Despite this trivial relationship linking fields and Field Axioms, their logical relationship is not straightforward. Note that: *satisfying* the Field Axioms is a condition of possible existence and not a condition of necessary existence. We shall see that the smallest field necessarily exists while others possibly exist.

Model Theory demands that any given proposition is proved by the Field Axioms, if and only if, it is true across *all* fields. All indeterminate propositions, independent of Axioms always have mixed true/ false values, disagreeing between some field and another. For existential propositions such as (1), (2), or (3), the condition of agreeing truths is satisfied only for scalars in the *rational field.* Therefore, only (1) is a theorem because it is the only case where α is rational. Figure 1 gives a preview of how this works. A consequence is that Axioms prove the existence of all rational scalars; existence of other scalars is undecidable. These

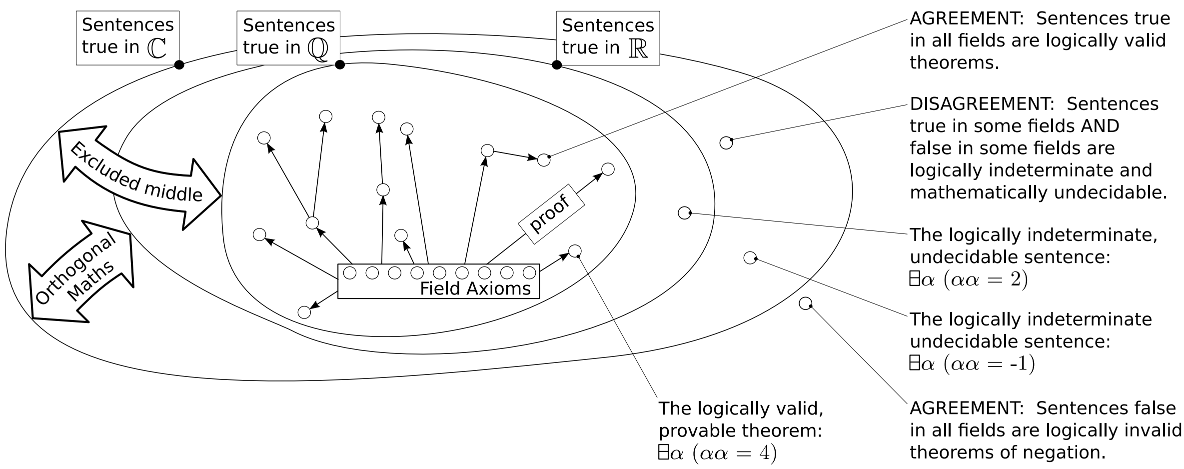

Figure 1. Validity under the Field Axioms. Due to theorems of Model Theory, sentences (small circles) such as $\exists\alpha\,(\alpha\alpha = 4)$, whose semantic validities agree are *logically valid* and are *theorems.* Sentences such as $\exists\alpha\,(\alpha\alpha = -1)$, whose semantic validities disagree are logically indeterminate and are *mathematically undecidable.* These exhaust all possibilities.

are surprising facts considering nothing in the arithmetic distinguishes the rational scalars.

The non-rational scalars are *logically independent* of the Axioms. That is to say: scalars of the non-rational fields express extraneous information, absent in the Axioms. The rational scalars, whose existence *can* be proved, contain no such extraneous information; they contain only information already in the Axioms. In short, the Field Axioms are unable to prove or disprove the existence of logically independent scalars. Logical independence is synonymous with mathematically undecidability.

3. Concepts

True is a semantical reference, synonymous with **semantically valid**. A proposition modelled by a given mathematical structure is true when interpreted in that structure.

Valid is a logical reference. It is more fully referred to as **logically valid.** A proposition is logically valid if: purely symbolically, independent of interpretation, by following rules of inference, Axioms imply the proposition.

Connectives: $\wedge$ = and; $\neg$ = negates; $\vdash$ = derives; $\models$ = models.

First-order theories comprise formulae written as propositions in *first-order logic*. The term *first-order* refers to depth of recursion of logical operations; it is not reference to approximation. Any first-order theory is specified in a set of axiom sentences, drawn up for the purpose. A crucial feature that distinguishes first-order theories from applied mathematics is their strict accounting of logical information. Variables satisfy all axiom sentences but are attributed with nothing more. They are purely abstract and meaningless. If this is misunderstood, the integrity of any derivation is at risk. In particular, the mathematician may not introduce new information, logically independent of Axioms, without recording the fact in an account of *assumed* dependencies. She may not, for example, assign a variable to the real line, simply by saying so, as is done in applied mathematics. Effectively, a first-order theory is a computational machine that runs according to a programme of axiom sentences. Output from this machine exhibits richer conceptualisations of theorem and validity than does applied mathematics. In absence of any *logically independent* input, output of the machine consists solely of theorems. In cases when there is logically independent input, output relying on that independency is always undecidable and indeterminate.

Bound variable: when we write the equation:

$$\alpha + \beta = \beta + \alpha\,, \tag{5}$$

this is an informal use of *bound variables*. Notice this relation *specifies* something about the algebraic behaviour of the objects α and β rather than suggesting the performance of some arithmetic. Bound variables occur where there is specification. When writing the formal version of (5), quantifiers $\forall$ are shown. These explicitly state the logic but also do the job of highlighting the fact that *specification* is intended rather than arithmetic. Thus:

$$\forall\alpha\forall\beta\,(\alpha+\beta=\beta+\alpha)\ . \tag{6}$$

The format of parenthesisation is typical of formulae in first-order logic. Quantifiers $\forall\alpha$ and $\forall\beta$ apply to every occurrence of α and β within the brackets.

Sentence: formulae such as (6), where every variable is bound, are known as *sentences.* (6) happens to be the sentence adopted as axiom FA4 in Table 1. An example of a formula which is not a sentence is the formula:

$$\forall\beta\exists\alpha\,(\alpha=\beta+\vartheta)\ . \tag{7}$$

In this ϑ is not bound.

Free variable: In (7), ϑ is a *free variable* as opposed to a bound variable. It is free to be substituted by a particular value; thus inviting the performance of some arithmetic rather than specification.

The Field Axioms are listed in Table 1. They comprise a set of axiom sentences formed by the union of axioms for the Additive Group and the Multiplicative Group. In addition to these, there is one axiom for distributivity. In these Axioms, different possibilities of interpretation exist for the symbols $+$ and $\times$. For example, modulo arithmetics are options, but these are not under consideration here. In this paper, $+$ and $\times$ are interpreted in the usual way, as symbols of an unbounded (infinite) arithmetic.

Model: This is a *mathematical structure* that satisfies a sentence. It is usual to say that such a structure *models* the sentence. As an illustration, consider the axiom sentence FA4 from Table 1, specifying additive commutativity. This is modelled by any of the sets: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\{1\}$, $\{1,-1\}$, $\{1,2,3\}$, $\{$all 4×3 matrices$\}$, etc.. As well as individual sentences, sets of sentences also have models. To illustrate, take two sentences. As before, take axiom FA4 from Table 1, but now model axiom FM4 also. Together these two sentences specify both additive *and* multiplicative commutativity. The addition of this second sentence eliminates the former inclusion of 4×3 matrices from the set of models.

Semantic interpretation: Bound variables, such as the objects $\alpha,\beta,\gamma,0,1,\ldots$ complying with Axioms in Table 1, convey no more meaning than the properties bestowed upon them by those Axioms. That said, they may be *interpreted* as elements of a particular model of the Axioms. For instance, these objects might be interpreted as members of the real line $\mathbb{R}$. This would be a *semantic interpretation* of $\alpha,\beta,\gamma,0,1,\ldots$, and would involve an injection of information originating not from the Axioms but from elsewhere.

Field: This is the general name for mathematical structures that model the Field Axioms. There are at least three infinite fields. These are the complex plane $\mathbb{C}$, the real line $\mathbb{R}$ and the rational field $\mathbb{Q}$. The term *field* is likely to cause confusion. In quantum field theory, fields are entities associated with the mechanics of elementary particles. This meaning is not intended here. In this paper, definition is taken from Linear Algebra.

Scalar: An element of a field. Semantic interpretation of the objects $\alpha, \beta, \gamma, 0, 1, \ldots$ in Table 1 are *scalars*: either complex scalars, real scalars or rational scalars, depending on the field elected. The term *scalar* is likely to cause confusion. In relativity, a scalar is a zero rank tensor: under change of inertial reference frame, an object that transforms as a constant number. In this paper, definition is taken from Linear Algebra.

4. Model Theory

Our specific interest in Model Theory is the Soundness Theorem and its converse, the Completeness Theorem. These are two standard theorems in model theory which apply to all first-order theories [4, 5]. We shall see that jointly, they isolate an excluded middle of mathematically undecidable sentences, from the set of all other sentences which are theorems.

4.1. Standard theorems

The Soundness Theorem:

$$\Sigma \vdash \mathcal{S} \Rightarrow \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right). \tag{8}$$

If structure $\mathcal{M}^{\Sigma}$ models axiom-set Σ, and Σ derives sentence $\mathcal{S}$, then every structure $\mathcal{M}^{\Sigma}$models $\mathcal{S}$.
Alternatively**:** *If a sentence is a theorem, provable under an axiom-set, then that sentence is true for every model of that axiom-set.*

The Completeness Theorem:

$$\Sigma \vdash \mathcal{S} \Leftarrow \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right). \tag{9}$$

If structure $\mathcal{M}^{\Sigma}$ models axiom-set Σ, and every structure $\mathcal{M}^{\Sigma}$ models sentence $\mathcal{S}$, then Σ *derives sentence $\mathcal{S}$.*
Alternatively**:** *If a sentence is true for every model of an axiom-set, then that sentence is a theorem, provable under that axiom-set.*

4.2. Proofs

We now proceed to prove further theorems of model theory. Jointly, (8) and (9) result in the 2-way implication:

Validity Theorem:

$$\Sigma \vdash \mathcal{S} \Leftrightarrow \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right). \tag{10}$$

If structure $\mathcal{M}^{\Sigma}$ models axiom-set Σ, then axiom-set Σ derives sentence $\mathcal{S}$, if and only if all structures $\mathcal{M}^{\Sigma}$ model sentence $\mathcal{S}$.
Alternatively**:** *A sentence is provable under an axiom-set, if and only if, that sentence is true for all models of that axiom-set.*

Furthermore, for every sentence $\mathcal{S}$ there is a sentence $\neg\mathcal{S}$; hence, jointly, (8) and (9) also guarantee a second 2-way implication:

Invalidity Theorem:

$$\Sigma \vdash \neg\, \mathcal{S} \Leftrightarrow \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \neg\, \mathcal{S} \right). \tag{11}$$

If structure $\mathcal{M}^{\Sigma}$ models axiom-set Σ, then axiom-set Σ derives the negation of sentence $\mathcal{S}$, if and only if all structures $\mathcal{M}^{\Sigma}$ model the negation of $\mathcal{S}$.
Alternatively**:** *A sentence is disprovable under an axiom-set, if and only if, that sentence is false for all models of that axiom-set.*

Each of (10) and (11) excludes the sentences of the other. And moreover, together they isolate sentences excluded by both. In the left hand sides of (10) and (11), there is no indication of other sentences existing which satisfy neither, that is: sentences that are neither provable nor disprovable. And so, it is of particular interest that the right hand sides of (10) and (11) do indeed imply the existence of sentences that correspond precisely to this condition. These are the sentences excluded by the right hand sides of (10) and (11) and so satisfy the following condition on modelling:

$$\neg \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right) \wedge \neg \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \neg\, \mathcal{S} \right). \tag{12}$$

The aim now is to find the status of provability for sentences excluded by (12). We firstly deduce (13) and (14), the negations of (10) and (11):

$$\neg\, (\Sigma \vdash \mathcal{S}) \Leftrightarrow \neg \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right); \tag{13}$$

$$\neg\, (\Sigma \vdash \neg\, \mathcal{S}) \Leftrightarrow \neg \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \neg\, \mathcal{S} \right); \tag{14}$$

and combine these, so as to construct:

$$\neg\, (\Sigma \vdash \mathcal{S}) \wedge \neg\, (\Sigma \vdash \neg\, \mathcal{S}) \Leftrightarrow \neg \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \mathcal{S} \right) \wedge \neg \forall \mathcal{M}^{\Sigma} \left(\mathcal{M}^{\Sigma} \models \neg\, \mathcal{S} \right). \tag{15}$$

This limits sentences that are neither provable nor negatable, to those that are neither true nor false across all structures that model the Axioms. For theories whose axioms are modelled by more than one single structure, where $\mathcal{M}_1^{\Sigma}$ and $\mathcal{M}_2^{\Sigma}$ are distinct, we can assert (16):

Indeterminacy Theorem:

$$\neg\, (\Sigma \vdash \mathcal{S}) \wedge \neg\, (\Sigma \vdash \neg\, \mathcal{S}) \Leftrightarrow \exists \mathcal{M}_1^{\Sigma} \left(\mathcal{M}_1^{\Sigma} \models \mathcal{S} \right) \wedge \exists \mathcal{M}_2^{\Sigma} \left(\mathcal{M}_2^{\Sigma} \models \neg\, \mathcal{S} \right). \tag{16}$$

Axiom-set Σ derives neither $\mathcal{S}$ nor its negation, if and only if there exist structures $\mathcal{M}_1^{\Sigma}$ and *$\mathcal{M}_2^{\Sigma}$ which model axiom-set Σ, such that $\mathcal{M}_1^{\Sigma}$ models sentence $\mathcal{S}$, and $\mathcal{M}_2^{\Sigma}$ models the negation of $\mathcal{S}$.*
Alternatively**:** *A sentence is true for some but not all models of an axiom-set, if and only if, that sentence is undecidable under that axiom-set.*

5. Application

The Theory of Fields is a first-order theory and so the above theorems apply. Our interest in deriving these theorems is the construction of practical tests for the detection of indeterminacy, and validity.

Indeterminacy Test: *when scalars of a given sentence are interpreted, in turn, as members of the complex plane* $\mathbb{C}$, *the real line* $\mathbb{R}$, *the rational field* $\mathbb{Q}$; *the Field Axioms neither prove nor disprove that sentence, if and only if, that sentence is true in at least one case and false in at least one case.*

This reduces to a simple check for disagreement within truth-tables.

The Indeterminacy Test serves the purpose we require, but note that it is not exhaustively comprehensive because it samples only three fields. A Validity Test, constructed form the Validity Theorem is an impractical prospect since it requires the sampling of every infinite field, no matter how obscure. For a realistic test for validity we embrace the model theory characterising direct derivation from Axioms. When a formula asserting existence of some particular number is proved directly from the Field Axioms, that number will be rational. This follows because scope for construction of such numbers is restricted to arithmetical combinations of the numbers 0 and 1, and are therefore limited in form to p/q, where p and q are integers. Moreover, proof of existence for *every* rational derives from the Field Axioms in this way. Hence, every formula asserting existence of a rational number is provable; and therefore, by the Soundness Theorem, is true in every field. And so, any such formulae is true independent of interpretation, and consequently, is valid by definition. These arguments summarise simply as:

Validity Test: *A formula is valid if and only if it is true in the rational field* $\mathbb{Q}$.

5.1. Examples

Existence of scalars Formulae (1), (2), (3) and (4) on page 3, each proposes the existence of a particular scalar. In the context of the Theory of Fields, each of these propositions poses the question: do the Field Axioms derive this formula? These

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha\,(\alpha \times \alpha = 4)$	T	T	T

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha\,(\alpha \times \alpha = 2)$	T	T	F

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha\,(\alpha \times \alpha = -1)$	T	F	F

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha\,(\alpha^{-1} = 0)$	F	F	F

Table 2. Truth-tables for propositions: $\exists\alpha\,(\alpha \times \alpha = 4)$, $\exists\alpha\,(\alpha \times \alpha = 2)$, $\exists\alpha\,(\alpha \times \alpha = -1)$ and $\exists\alpha\,(\alpha^{-1} = 0)$. In these T and F denote true and false.

questions are answered in the four truth-tables of Table 2. In the first, proposition (1) is seen to be true for all three fields, so by the Validity Test, (1) is a theorem. The second two truth-tables show disagreeing truth values; hence, by the Indeterminacy Test, (2) and (3) are undecidable. In the last of these truth tables, all three truth values agree false; hence by the Invalidity Test, proposition (4) is negated.

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha\left(\alpha = \xi^{\mathbb{Q}}\right)$	T	T	T

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha\left(\alpha = \zeta^{\mathbb{R}}\right)$	T	T	F

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists\alpha\left(\alpha = \eta^{\mathbb{C}}\right)$	T	F	F

Table 3. Truth-tables for propositions $\exists\alpha\left(\alpha = \xi^{\mathbb{Q}}\right)$, $\exists\alpha\left(\alpha = \zeta^{\mathbb{R}}\right)$ and $\exists\alpha\left(\alpha = \eta^{\mathbb{C}}\right)$.

Existence of rational scalars The rational field is a subfield of all fields. Consequently, propositions of existence that are true in this smallest field are necessarily true in all fields. See Figure 1. This means that *rational* scalars exist by theorem. Table 3 illustrates the provability of the general rational scalar $\xi^{\mathbb{Q}}$, the undecidability of the general real scalar $\zeta^{\mathbb{R}}$ and the undecidability of the general complex scalar $\eta^{\mathbb{C}}$.

Existence of functions A function in applied mathematics can spurn different first-order propositions; some of which might be theorems and some which might be undecidable. Propositions: $\forall x \exists y\,(y = x^2)$ and $\forall y \exists x\,(y = x^2)$ have quantifiers reversed. This makes an important logical difference. Table 4 shows the first of these two propositions is a theorem, yet the second is undecidable.

	$x, y \in \mathbb{C}$	$x, y \in \mathbb{R}$	$x, y \in \mathbb{Q}$
$\forall x \exists y\,(y = x^2)$	T	T	T

	$x, y \in \mathbb{C}$	$x, y \in \mathbb{R}$	$x, y \in \mathbb{Q}$
$\forall x \exists y\,(y = x^2)$	T	F	F

Table 4. Truth-tables concerning the function $y = x^2$.

Existence of finite polynomials versus transcendental functions Table 5 compares formulae proposing the existence of a finite polynomial with an example of transcendental function: the exponential. The first truth-table in Table 5 is for the proposition: $\forall x \exists y\,(y = p\,(x))$. In this, p is a finite polynomial, so if x is rational then so also is any finite sum of terms $p\,(x)$. Corresponding reasoning applies to real or complex x. In contrast, in the proposition $\forall x \exists y\,(y = \exp\,(x))$ where

$$\exp\,(x) \equiv \lim_{n\to\infty} \left[1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots\right],$$

rational x is not necessarily mapped to a rational point by the exponential function. Hence, $p\,(x)$ exists by theorem but $\exp\,(x)$ exists undecidably.

	$x, y \in \mathbb{C}$	$x, y \in \mathbb{R}$	$x, y \in \mathbb{Q}$
$\forall x \exists y\,(y = p\,(x))$	T	T	T

	$x, y \in \mathbb{C}$	$x, y \in \mathbb{R}$	$x, y \in \mathbb{Q}$
$\forall x \exists y\,(y = \exp\,(x))$	T	T	F

Table 5. Truth-table for finite polynomial: $\forall x \exists y\,(y = p\,(x))$ and the transcendental function: $\forall x \exists y\,(y = \exp\,(x))$.

5.2. Theorems from undecidability

Arithmetical combination Scalars that exist undecidably can be combined to yield scalars that exist by theorem. Consider the two propositions: $\exists \alpha\,(\alpha = 3 + 4\mathrm{i})$ and $\exists \alpha^*\,(\alpha^* = 3 - 4\mathrm{i})$. These are undecidable, but the product of these scalars is the rational scalar: 25, which is logically valid and so exists by theorem. See Table 6.

	$\alpha \in \mathbb{C}$	$\alpha \in \mathbb{R}$	$\alpha \in \mathbb{Q}$
$\exists \alpha\,(\alpha = 3 + 4\mathrm{i})$	T	F	F

	$\alpha^* \in \mathbb{C}$	$\alpha^* \in \mathbb{R}$	$\alpha^* \in \mathbb{Q}$
$\exists \alpha^*\,(\alpha^* = 3 - 4\mathrm{i})$	T	F	F

	$\beta \in \mathbb{C}$	$\beta \in \mathbb{R}$	$\beta \in \mathbb{Q}$
$\exists \beta\,(\beta = \alpha\alpha^*)$	T	T	T

Table 6. Truth-tables for the proposition $\exists \alpha\,(\alpha = 3 + 4\mathrm{i})$, $\exists \alpha^*\,(\alpha^* = 3 - 4\mathrm{i})$ and $\exists \beta\,(\beta = \alpha\alpha^*)$.

Limiting Theorems The limit of an undecidable scalar can exist by theorem. The proposition $\exists y\,(y^2 = -x^2)$ is undecidable. Nevertheless, it has a limiting case:

$\exists y\,(\lim_{x\to 0}[y^2 = -x^2])$ which is a theorem. See Table 7.

	$y \in \mathbb{C}$	$y \in \mathbb{R}$	$y \in \mathbb{Q}$
$\exists y\,(y^2 = -x^2)$	T	F	F

	$y \in \mathbb{C}$	$y \in \mathbb{R}$	$y \in \mathbb{Q}$
$\exists y\,(\lim_{x\to 0}[y^2 = -x^2])$	T	T	T

Table 7. Truth-table for proposition $\exists y\left(y^2 = -x^2\right)$ and its limiting case: $\exists y\left(\lim_{x\to 0}\left[y^2 = -x^2\right]\right)$.

Conclusions

This paper documents profound facts well-known amongst mathematical logicians but never taken on-board by quantum theorists. There is a logical feature hidden, inherent within the everyday arithmetic with which we are most familiar. We commonly understand formulae in algebra to be either true or false, depending on whether they are derived correctly or erroneously and we expect no alternative to these possibilities. But this paper shows there does exist another alternative: that of *indeterminate* or *mathematically undecidable.* This logical information is not picked up by the standard algebraic formalism and perpetuates unnoticed throughout applied mathematics and into quantum mechanics where its absence is conspicuous. Consequently, as it presently stands, physical theory is denied the possibility of linking these theoretical indeterminacies with indeterminacies we observe in Nature. This paper shows that the said logical information is exposed in an existential version of the algebra.

This research is a study in mathematical logic applied to the algebra of the Field Axioms. These Axioms define the algebra or arithmetic of *scalars*: objects basic throughout mathematics. Scalars are realised to exist in two modes. By definition, all scalars *satisfy* the Field Axioms and so all are *possible.* On top of this, a subset of scalars *necessarily* exists because these *derive* directly from the Axioms. Derivation and satisfaction are seen as causally distinct; a distinction not noted in applied mathematics.

The central point on which the above claims rest is proof, given in this paper, of a theorem in model theory that confirms the existence of indeterminacy under the Field Axioms: indeterminacy that cannot be derived directly. Application of this and other closely related theorems in model theory furnish two simple tests identifying those formulae which Axioms render logically indeterminate and those they render logically valid theorems in the Theory of Fields. The said theorems in model theory strictly identify undecidable propositions as those with truth values that do not concur across all semantic interpretations, but disagree. That is: they are not consistently true, or false, when interpreted in turn as members of the complex plane $\mathbb{C}$, the real line $\mathbb{R}$, and the rational field $\mathbb{Q}$. This result is used in various examples of interest, checking truth-tables for agreement or disagreement. Rational scalars are shown to exist by theorem while strictly imaginary or irrational scalars are undecidable. This ultimately follows from the fact that only the rational field is a subfield of all fields.

An important finding of this research is that performance of algebraic operations on undecidable propositions can produce propositions that are theorems. This has ramifications for our understanding of the mechanism for measurement in quantum mechanics and will be explored in greater detail in a subsequent paper.

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