Gödelian features found at quantum indeterminacy: inconsistency, undecidability and self reference

S Faulkner

159a, Weedon Road, Northampton, United Kingdom, NN5 5DA.

Abstract. Standard methods of quantum theory are employed, excepting: quantum theory is initialised by the *a priori* adoption of the Field Axioms; and the square root of minus one is not introduced initially as if axiomatic. Its adoption is postponed until inconsistency in the theory forces its introduction. Entry of this scalar, logically independent of the Axioms, relieves the inconsistency but introduces mathematical undecidability and indeterminacy. Nevertheless, *indeterminate* formulae derive *determinate* probability along with Pythagorean addition. Orthogonality is indicated as the condition around which logical anomalies in quantum physics hinge.

E-mail: `StevieFaulkner@googlemail.com`

1. Introduction

This article is one of a series explaining the nature of mathematical undecidability discovered present within quantum mechanics. The first article [12] describes the algebraic environment in which the said undecidability originates; proves the existence of indeterminacy that complements the undecidability; and demonstrates the part these play in a 3-valued logic which permeates mathematical physics via this algebra. This second article applies these concepts to quantum mechanics in an axiomatised formulation of a famous example in wave mechanics.

The underlying revolutionary idea is well known in Mathematical Logic: that different scalars are logically distinct. The Field Axioms *prove* the existence of certain scalars while a wider set merely *satisfies* them. Consider these examples:

$$\exists\alpha\,(\alpha\times\alpha=4)\,; \tag{1}$$

$$\exists\alpha\,(\alpha\times\alpha=-1)\,. \tag{2}$$

Of these, the Field Axioms prove only (1). (2) is neither provable nor disprovable. Even so, the square root of -1 is nevertheless an object that satisfies the Field Axioms and therefore, it does engage in their arithmetic. Standard quantum theory applies this arithmetic without recognising this distinction. Crucially, standard theory introduces the square root of minus one axiomatically. This elevates (2) from a scalar that satisfies to one which is provable, destroying that scalar's distinct logic.

2. Standard quantum mechanics

2.1. Linear Algebra

The formalism of standard quantum mechanics revolves around a linear algebraic system of eigenvector equations, generally written in notation due to Dirac [10]:

$$\mathbf{a}\,|\alpha\rangle\ = a\,|\alpha\rangle\,. \tag{3}$$

Here $|\alpha\rangle$ is a *vector* in a complex Hilbert space, used to represent a physical state; $\mathbf{a}$ is an Hermitian or self adjoint *linear operator* representing a measurable observable, and a is a *real number* which predicts a possible measurement value for that observable.

The space of vectors $|\,\rangle$ has a dual space of vectors written $\langle\,|$. This pair of vector spaces form inner products, $\langle\,|\,\rangle$ which are complex numbers representing *probability amplitudes.* The probability amplitude $\langle\beta|\alpha\rangle$ is a measure of a system in state $|\alpha\rangle$, to pass into state $|\beta\rangle$.

2.2. Physical principles founding the canonical commutation relation

The algebra of the canonical commutation relation (16) embodies much of the physics of wave mechanics. Derivation of this relation centres on the eigenvalue equation for position:

$$\mathbf{x}\,|\psi\rangle = x\,|\psi\rangle\,, \tag{4}$$

while considering the question: 'How does the position operator $\mathbf{x}$ vary as the reference system is displaced?' The answer lay in the continuous *translation* of $\mathbf{x}$ while basis vectors $|\psi\rangle$ are held fixed using a sequence of *unitary* similarity transformations [19, 22, 36]. These result in the continuum of equivalences between translated operators $\mathbf{x}$, coincident with a sequence of similarity transformed values of $\mathbf{x}$, seen in (13). The principle applied in obtaining this equivalence is *form invariance*: the idea that a transformation representing a symmetry in Nature, leaves physical formulae fixed in *form*, whilst *values* may vary [30].

The symmetry expressed under translation is homogeneity of space. Translation through space leaves vectors of (4) in a different basis. And so, in parallel with this translation, a unitary transformation is chosen that neutralises this basis change. Then a direct comparison is made between the the translated and the unitary transformed position operators. The scheme of transformations is:

$$\begin{array}{ccc} \mathbf{x}\,|\psi\rangle = x\,|\psi\rangle & \xrightarrow[O_{\mathbf{x}}\to O_{\mathbf{x}'}]{\text{translation}} & \mathbf{x}'\,|\psi'\rangle = x'\,|\psi'\rangle \\ {\scriptstyle \psi_x\to\psi_{x'}}\Big\downarrow{\scriptstyle \text{unitarity}} & & \Big\downarrow \\ \mathbf{x}''\,|\psi'\rangle = x\,|\psi'\rangle & \xrightarrow{\qquad\qquad} & \mathbf{x}'' = \mathbf{x}' \end{array} \tag{5}$$

In the detail that follows, the equivalence, $\mathbf{x}'' = \mathbf{x}'$ is derived explicitly at (13). That equation embodies conditionality on the position operator $\mathbf{x}$, due to a *finite* displacement

ϵ, through homogeneous space. The canonical commutation relation (16) is this same conditionality under *infinitesimal* displacement.

2.3. Deriving the canonical commutation relation

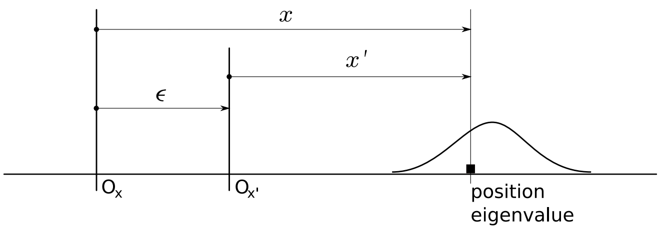

Figure 1. Passive translation: two reference systems, arbitrarily displaced by ϵ, are used to measure position.

2.3.1. The translation Dealing firstly with the top leg of the transformation scheme (5): the translation. The *passive* translation of Figure 1 takes us from position eigenvalue equation (4) for the reference system O_{x}, to a position eigenvalue equation of the same form for the reference system $O_{\mathsf{x}'}$; thus:

$$\mathbf{x}' \left|\psi'\right\rangle = x' \left|\psi'\right\rangle . \tag{6}$$

But the equivalent, *active* translation, in figure 2 gives:

$$\mathbf{x} \left|\psi'\right\rangle = x' \left|\psi'\right\rangle . \tag{7}$$

Principles of relativity tell us that these passive and active versions are identical, indicating that $\mathbf{x}' \equiv \mathbf{x}$. Working from the version in (7) and substituting $x' = x - \epsilon$ gives us:

$$\mathbf{x} \left|\psi'\right\rangle = (x - \epsilon) \left|\psi'\right\rangle .$$

Attributing the wavefunctions with a corresponding position, $\psi \equiv \psi_x$; and noting that the wavefunction translated to $x \mapsto x - \epsilon$ is the waveform $\psi' \equiv \psi_{x+\epsilon}$, giving:

$$\mathbf{x} \left|\psi_{x+\epsilon}\right\rangle = (x - \epsilon) \left|\psi_{x+\epsilon}\right\rangle \tag{8}$$

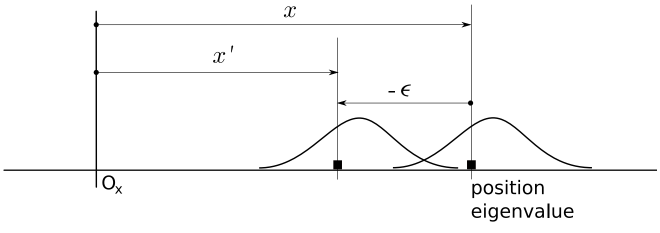

Figure 2. Active translation: a single reference system is used to measure position before and after displacement by $-\epsilon$.

But for any vector $|\rangle$ the trivial eigenvalue equation: $\epsilon\mathbf{1}\,|\rangle = \epsilon\,|\rangle$ is true. Adding $\epsilon\mathbf{1}\,|\psi_{x+\epsilon}\rangle = \epsilon\,|\psi_{x+\epsilon}\rangle$ to (8) produces:

$$(\mathbf{x} + \epsilon\mathbf{1})\,|\psi_{x+\epsilon}\rangle = x\,|\psi_{x+\epsilon}\rangle\,. \tag{9}$$

Putting (9) momentarily on hold and turning attention to unitarity.

2.3.2. Unitarity The left leg of the transformation scheme (5) is the unitary transformation. This is a similarity transformation designed for the purpose of transforming eigenvectors while preserving the spectrum of eigenvalues; with the extra demand that operators are unitary so that probability amplitudes are also preserved. This transforms the original eigenvalue equation (4) to

$$\mathbf{x}''\,|\psi'\rangle = x\,|\psi'\rangle\,.$$

$\mathbf{x}''$ is therefore a unitary, similarity transformation of $\mathbf{x}$. The existence of such a unitary operator is assured by Wigner's Theorem: for vectors in any Hilbert space $\mathcal{H}$, there is a unitary operator that transforms any one vector to any other [22]. Thus:

$$\exists \mathsf{U}\,(\mathsf{U}\,|\psi\rangle = |\psi'\rangle)\,. \tag{10}$$

2.3.3. Single parameter unitarity Any subgroup of the General Linear group has a single parameter subgroup: as does the unitary group [19]. By Making the substitution $\psi = \psi_x \mapsto \psi' = \psi_{x+\epsilon}$ a continuum of bases $|\psi_{x+\epsilon}\rangle$ is furnished, parametrised by all $\epsilon \in \mathbb{R}$. Furthermore, the corresponding unitary group of operators also form a continuum parametrised by ϵ, thus: $\mathsf{U} = \mathsf{U}_{(\epsilon)}$.

$$\begin{aligned} &\exists \mathsf{U}_{(\epsilon)}\left(\mathsf{U}_{(\epsilon)}\,|\psi_x\rangle = |\psi_{x+\epsilon}\rangle\right) \\ &\exists \mathsf{U}_{(\epsilon)}\left(\mathsf{U}^{-1}_{(\epsilon)}\,|\psi_{x+\epsilon}\rangle = |\psi_x\rangle\right). \end{aligned} \tag{11}$$

2.3.4. The unitary transformation Using this parametrised unitary operator $\mathsf{U}_{(\epsilon)}$, we are ready to perform the unitary similarity transformation on the original position eigenvalue equation (4). We write:

$$\begin{aligned} \mathbf{x}\,|\psi_x\rangle &= x\,|\psi_x\rangle \\ \mathbf{x}\mathsf{U}^{-1}_{(\epsilon)}\,|\psi_{x+\epsilon}\rangle &= x\mathsf{U}^{-1}_{(\epsilon)}\,|\psi_{x+\epsilon}\rangle \\ \mathsf{U}_{(\epsilon)}\mathbf{x}\mathsf{U}^{-1}_{(\epsilon)}\,|\psi_{x+\epsilon}\rangle &= x\mathsf{U}_{(\epsilon)}\mathsf{U}^{-1}_{(\epsilon)}\,|\psi_{x+\epsilon}\rangle \\ \mathsf{U}_{(\epsilon)}\mathbf{x}\mathsf{U}^{-1}_{(\epsilon)}\,|\psi_{x+\epsilon}\rangle &= x\,|\psi_{x+\epsilon}\rangle \end{aligned} \tag{12}$$

2.3.5. Comparison of the translated and unitary transformed equations Comparing translated (9) and the unitary transformed (12), uniqueness gives the equivalence:

$$\mathsf{U}_{(\epsilon)}\mathbf{x}\mathsf{U}^{-1}_{(\epsilon)} = \mathbf{x} + \epsilon\mathbf{1}. \tag{13}$$

2.3.6. Exponential representation We now choose a representation for $\mathsf{U}_{(\epsilon)}$. At the same time as being unitary, it must show the additivity of translation of space. Necessary properties are:

$$\forall \epsilon \in \mathbb{R}, \exists \mathsf{U}_{(\epsilon)} \left(\epsilon \mapsto \mathsf{U}_{(\epsilon)}\right)$$
$$\forall \epsilon_1 \forall \epsilon_2 \left(\mathsf{U}_{(\epsilon_1)} \mathsf{U}_{(\epsilon_2)} = \mathsf{U}_{(\epsilon_1+\epsilon_2)}\right) \qquad \text{CLOSURE}$$
$$\forall \mathsf{U}_{(\epsilon)} \exists \mathsf{U}_{(0)} \left(\mathsf{U}_{(\epsilon)} \mathsf{U}_{(0)} = \mathsf{U}_{(0)} \mathsf{U}_{(\epsilon)} = \mathsf{U}_{(\epsilon)}\right) \qquad \text{IDENTITY}$$
$$\forall \mathsf{U}_{(\epsilon)} \exists \mathsf{U}^{-1}_{(\epsilon)} \left(\mathsf{U}^{-1}_{(\epsilon)} = \mathsf{U}^{\dagger}_{(\epsilon)} = \mathsf{U}_{(-\epsilon)}\right). \qquad \text{INVERSE}$$

A theorem due to Marshall H Stone establishes a one-one correspondence between one-parameter families of unitary operators $\mathsf{U}_{(\epsilon)}$ and a self adjoint operator $\mathbf{g}$ [22]:

$$\mathsf{U}_{(\epsilon)} = \mathrm{e}^{\mathrm{i}\epsilon\mathbf{g}}. \tag{14}$$

This exponential satisfies the properties of closure, identity and inverse above and can be used to represent the family of unitary operators defined in (11).

2.3.7. Infinitesimal translation Now taking (13), the condition for a finite translation, and making the substitution (14):

$$\begin{aligned}
\mathsf{U}_{(\epsilon)}\mathbf{x}\mathsf{U}^{-1}_{(\epsilon)} &= \mathbf{x} + \epsilon\mathbf{1}\\
\mathsf{U}_{(\epsilon)}\mathbf{x}\mathsf{U}_{(-\epsilon)} &= \mathbf{x} + \epsilon\mathbf{1}\\
\mathrm{e}^{+\mathrm{i}\epsilon\mathbf{g}}\mathbf{x}\mathrm{e}^{-\mathrm{i}\epsilon\mathbf{g}} &= \mathbf{x} + \epsilon\mathbf{1}\\
\left[\mathbf{1} + \mathrm{i}\epsilon\mathbf{g} + \mathcal{O}\left(\epsilon^2\right)\right]\mathbf{x}\left[\mathbf{1} - \mathrm{i}\epsilon\mathbf{g} + \mathcal{O}\left(\epsilon^2\right)\right] &= \mathbf{x} + \epsilon\mathbf{1}\\
\left[\mathbf{x} + \mathrm{i}\epsilon\mathbf{g}\mathbf{x} + \mathcal{O}\left(\epsilon^2\right)\right]\left[\mathbf{1} - \mathrm{i}\epsilon\mathbf{g} + \mathcal{O}\left(\epsilon^2\right)\right] &= \mathbf{x} + \epsilon\mathbf{1}\\
\mathbf{x} + \mathrm{i}\epsilon\mathbf{g}\mathbf{x} - \mathrm{i}\epsilon\mathbf{x}\mathbf{g} + \mathcal{O}\left(\epsilon^2\right) &= \mathbf{x} + \epsilon\mathbf{1}\\
\mathbf{x} + \mathrm{i}\epsilon\left[\mathbf{g},\mathbf{x}\right] + \mathcal{O}\left(\epsilon^2\right) &= \mathbf{x} + \epsilon\mathbf{1}\\
\left[\mathbf{g},\mathbf{x}\right] &= \frac{\mathbf{1}}{\mathrm{i}} - \mathcal{O}\left(\epsilon\right).
\end{aligned}$$

In the limit, as $\epsilon \to 0$,

$$\left[\mathbf{g},\mathbf{x}\right] \to -\mathrm{i}\mathbf{1}. \tag{15}$$

$\mathbf{g}$ is the infinitesimal generator of translation which we associate with *momentum*. Writing the momentum operator $\mathbf{p} = \mathbf{g}\hbar$, where $\hbar$ has units of action; this balances dimensionality. Thus:

$$\left[\mathbf{p},\mathbf{x}\right] = -\mathrm{i}\hbar\mathbf{1}. \tag{16}$$

2.4. Implementation of the canonical commutator

2.4.1. Representation The canonical commutation relation in (16) is satisfied by the self adjoint representation:

$$\mathbf{p}\left|\psi\right\rangle = -\mathrm{i}\hbar\tfrac{d}{dx}\psi\left(x\right) \qquad \mathbf{x}\left|\psi\right\rangle = x\psi\left(x\right) \qquad \mathbf{1}\left|\psi\right\rangle = 1\psi\left(x\right). \tag{17}$$

These three elements are used in the construction of various linear wave equations representing quantum systems and situations.

2.4.2. Momentum of the free particle Assuming the momentum eigenvalue equation:

$$\mathbf{p}\left|\psi_p\right\rangle = p\left|\psi_p\right\rangle \tag{18}$$

and writing this in terms of the representation (17), we have the differential equation:

$$-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}\psi_p(x) = p\psi_p(x), \tag{19}$$

whose solutions are the momentum eigenfunctions

$$\psi_p(x) = \exp(\mathrm{i}px/\hbar). \tag{20}$$

2.4.3. Normalisability The Born interpretation demands that wave packets are normalisable. Although individually, the functions in (20) are not normalised; nonetheless, they do form a complete set of orthogonal functions from which we can construct a Fourier sum $\Psi(x)$ and its inverse $a(p)$ that are normalisable:

$$\Psi(x) = \int\limits_{p=-\infty}^{+\infty} a(p)\exp(+\mathrm{i}px/\hbar)\,\mathrm{d}p; \tag{21}$$

$$a(p) = \frac{1}{2\pi}\int\limits_{x=-\infty}^{+\infty} \Psi(x)\exp(-\mathrm{i}px/\hbar)\,\mathrm{d}x. \tag{22}$$

2.4.4. Probability Probability is then defined from statistical mathematics as $P_p = \left|\int_{x=-\infty}^{+\infty}\psi_p^*(x)\,\Psi(x)\,\mathrm{d}x\right|^2$ with:

$$\begin{aligned}
\int\limits_{x=-\infty}^{+\infty}\psi_p^*(x)\,\Psi(x)\,\mathrm{d}x &= \int\limits_{x=-\infty}^{+\infty}\left[\exp(\mathrm{i}px/\hbar)\right]^*\left[\int\limits_{p'=-\infty}^{+\infty} a(p')\exp(+\mathrm{i}p'x/\hbar)\,\mathrm{d}p'\right]\mathrm{d}x \\
&= \int\limits_{p'=-\infty}^{+\infty}\left[\int\limits_{x=-\infty}^{+\infty}\exp(-\mathrm{i}px/\hbar)\,a(p')\exp(+\mathrm{i}p'x/\hbar)\,\mathrm{d}x\right]\mathrm{d}p' \\
&= \int\limits_{p'=-\infty}^{+\infty} a(p')\left[\int\limits_{x=-\infty}^{+\infty}\exp\{\mathrm{i}x(p'-p)/\hbar\}\,\mathrm{d}x\right]\mathrm{d}p' \\
&= \int\limits_{p'=-\infty}^{+\infty} a(p')\,\delta(p'-p)\,\mathrm{d}p' \\
&= a(p)
\end{aligned}$$

---

3. Wave mechanics as a first-order theory under the Field Axioms

Assumption 1: Initialise proceedings by firstly adopting the Field Axioms.

3.1. The general eigenvalue proposition

Rewrite the general eigenvalue equation (3) from mathematical physics as a formula in first-order logic, thus:

$$\exists \mathbf{a} \exists a \exists \alpha \left(\mathbf{a} \left[\alpha \right] = a \left[\alpha \right] \right). \tag{23}$$

The use of quantifiers in this formula needs explanation. a is simply a bound variable and $\exists a$ naturally takes its normal meaning; but definitions of $\exists \alpha$ and $\exists \mathbf{a}$ need clarification because α and $\mathbf{a}$ are not proposed scalars, but are arrays of them. The example of the two dimensional case illustrates how they should be interpreted:

$$\exists \alpha \left(\left[\alpha \right] \right) \equiv \exists \alpha_1 \exists \alpha_2 \left(\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \right);$$

and

$$\exists \mathbf{a} \left(\mathbf{a} \right) \equiv \exists \mathrm{a}_{11} \exists \mathrm{a}_{12} \exists \mathrm{a}_{21} \exists \mathrm{a}_{22} \left(\begin{bmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} \\ \mathrm{a}_{21} & \mathrm{a}_{22} \end{bmatrix} \right),$$

where $\alpha_1, \alpha_2; \mathrm{a}_{11}\mathrm{a}_{12}\mathrm{a}_{21}\mathrm{a}_{22}$ are all *bound variables.* Also, vectors in (23) are members of a *general* vector space and denoted by square brackets like this: $[\alpha]$. The Dirac notation is dropped as the reader could infer that vectors are in Hilbert space and are orthogonal or orthonormal; none of these is implied. Indeed no scalar product is assumed at all. In cases where the operator $\mathbf{a}$ is diagonalisable and equivalent by similarity transformation to some diagonal operator we happen to know (23) is valid but this fact is put to one side for now.

3.2. Position eigenvalue proposition

Assumption 2: Assume validity under the Field Axioms of an eigenvalue proposition for the existence of position:

$$\exists \mathbf{x} \exists x \exists \psi \left(\mathbf{x} \left[\psi \right] = x \left[\psi \right] \right). \tag{24}$$

3.3. Existence of the homogeneity symmetry (Lie bracket)

In the manipulations that follow, quantifiers $\forall$ and $\exists$ do not necessarily commute and so their orders preserved.

3.3.1. Translation By the same arguments leading to (7), there is a translated version of (24), thus:

$$\exists \mathbf{x} \exists x' \exists \psi' \left(\mathbf{x} \left[\psi' \right] = x' \left[\psi' \right] \right) \tag{25}$$

which, when slightly rearranged is:

$$\exists \mathbf{x} \exists x' \exists \psi' \left(\mathbf{x} \left[\psi' \right] - x' \left[\psi' \right] = 0 \right). \tag{26}$$

Introducing the trivial eigenvalue equation:

$$\forall \epsilon \exists \psi' \left(\epsilon \mathbf{1} \left[\psi' \right] = \epsilon \left[\psi' \right] \right) \tag{27}$$

and rearranging similarly:

$$\forall\epsilon\exists\psi'\left(\epsilon\mathbf{1}\left[\psi'\right]-\epsilon\left[\psi'\right]=0\right). \tag{28}$$

Equating (26) and (28) gives:

$$\begin{aligned}&\exists\mathbf{x}\exists x'\exists\psi'\left(\mathbf{x}\left[\psi'\right]-x'\left[\psi'\right]\right)=\forall\epsilon\exists\psi'\left(\epsilon\mathbf{1}\left[\psi'\right]-\epsilon\left[\psi'\right]\right)\\&\Rightarrow\left(\exists\mathbf{x}\exists x'\left(\mathbf{x}\left[\psi'\right]-x'\left[\psi'\right]\right)=\forall\epsilon\left(\epsilon\mathbf{1}\left[\psi'\right]-\epsilon\left[\psi'\right]\right)\right)\exists\psi'.\end{aligned} \tag{29}$$

As in (8), translation $x\mapsto x'=(x-\epsilon)$ provides the substitution:

$$\forall x'\forall\epsilon\exists x\left(x'=x-\epsilon\right) \tag{30}$$

which when substituted into (29) yields:

$$\begin{aligned}&\left(\exists\mathbf{x}\forall\epsilon\exists x\left(\mathbf{x}\left[\psi'\right]-(x-\epsilon)\left[\psi'\right]\right)=\forall\epsilon\left(\epsilon\mathbf{1}\left[\psi'\right]-\epsilon\left[\psi'\right]\right)\right)\exists\psi'\\&\Rightarrow\exists\mathbf{x}\forall\epsilon\left(\exists x\left(\mathbf{x}\left[\psi'\right]-(x-\epsilon)\left[\psi'\right]\right)=\epsilon\mathbf{1}\left[\psi'\right]-\epsilon\left[\psi'\right]\right)\exists\psi'.\end{aligned} \tag{31}$$

Again comparing with (8) translation $\psi\equiv\psi_{(x)}\mapsto\psi'\equiv\psi'_{(x)}=\psi_{(x+\epsilon)}$ provides the substitution:

$$\forall\psi'\forall\epsilon\exists x\exists\psi\left(\psi'=\psi_{(x+\epsilon)}\right) \tag{32}$$

which when substituted into (31) yields:

$$\begin{aligned}&\exists\mathbf{x}\forall\epsilon\left(\exists x\left(\mathbf{x}\left[\psi_{(x+\epsilon)}\right]-(x-\epsilon)\left[\psi_{(x+\epsilon)}\right]\right)=\epsilon\mathbf{1}\left[\psi_{(x+\epsilon)}\right]-\epsilon\left[\psi_{(x+\epsilon)}\right]\right)\forall\epsilon\exists x\exists\psi\\&\Rightarrow\exists\mathbf{x}\forall\epsilon\left(\mathbf{x}\left[\psi_{(x+\epsilon)}\right]-(x-\epsilon)\left[\psi_{(x+\epsilon)}\right]=\epsilon\mathbf{1}\left[\psi_{(x+\epsilon)}\right]-\epsilon\left[\psi_{(x+\epsilon)}\right]\right)\forall\epsilon\exists x\exists\psi\end{aligned}$$

culminating in the propositional version of (9):

$$\exists\mathbf{x}\forall\epsilon\exists x\exists\psi\left(\left(\mathbf{x}+\epsilon\mathbf{1}\right)\left[\psi_{(x+\epsilon)}\right]=x\left[\psi_{(x+\epsilon)}\right]\right). \tag{33}$$

We now put (33) on hold and turn attention not to a unitary transformation but to similarity.

3.3.2. Similarity transformation Corresponding to the unitary transformation in (12), now form the similarity transformation of $\mathbf{x}$.

These manipulations result in (40), the proposed existence for a sequence of operators $\mathsf{S}_{(\epsilon\mathbb{Q})}$ for all *finite* translations ϵ. Following on from that, (40) is shown to derive (42) the corresponding relation for *infinitesimal* translations. This step to the infinitesimal is a conventional step yielding an algebra preferable to (40) because it is linear. Incorporation of scaling produces (43) which most students of quantum mechanics would recognise as closely resembling the canonical commutation relation (16).

We start with the position eigenvalue proposition (24):

$$\begin{aligned}&\exists\mathbf{x}\exists x\,\exists\psi\left(\mathbf{x}\left[\psi\right]=x\left[\psi\right]\right)\\&\Rightarrow\exists\mathbf{x}\left(\mathbf{x}\left(\left[\psi\right]\right)\exists\psi=\exists x\left(x\right)\left(\left[\psi\right]\right)\exists\psi\right).\end{aligned} \tag{34}$$

For any vector ψ in a space of dimension n, and any transformation $\mathsf{S}\in\mathsf{GL}\left(n,\mathbb{F}\right)$, the general linear group, there exists another vector ψ' thus:

$$\forall\psi\,\forall\mathsf{S}\exists\psi'\left(\mathsf{S}^{-1}\left[\psi'\right]=\left[\psi\right]\right). \tag{35}$$

Substituting this into (34) to form the similarity transformation:

$$\begin{aligned}
&\exists \mathbf{x} \left(\mathbf{x} \left(\mathsf{S}^{-1} [\psi'] \right) \forall \mathsf{S} \exists \psi' = \exists x \,(x) \left(\mathsf{S}^{-1} [\psi'] \right) \forall \mathsf{S} \exists \psi' \right) \\
&\Rightarrow \exists \mathbf{x} \left(\mathsf{S}\mathbf{x}\mathsf{S}^{-1} [\psi'] \right) \forall \mathsf{S} \exists \psi' = \exists x \,(x) \left(\mathsf{S}\mathsf{S}^{-1} [\psi'] \right) \forall \mathsf{S} \exists \psi' \\
&\Rightarrow \exists \mathbf{x} \left(\mathsf{S}\mathbf{x}\mathsf{S}^{-1} [\psi'] = \exists x \,(x) [\psi'] \right) \forall \mathsf{S} \exists \psi' \\
&\Rightarrow \exists \mathbf{x} \forall \mathsf{S} \left(\mathsf{S}\mathbf{x}\mathsf{S}^{-1} ([\psi']) \exists \psi' = \exists x \,(x) ([\psi']) \exists \psi' \right). \qquad (36)
\end{aligned}$$

3.3.3. One-parameter subgroup Again using (32) and substituting into (36) gives:

$$\begin{aligned}
&\exists \mathbf{x} \forall \mathsf{S} \left(\mathsf{S}\mathbf{x}\mathsf{S}^{-1} \left(\left[\psi_{(x+\epsilon)} \right] \right) \forall \epsilon \exists x \exists \psi = \exists x \,(x) \left(\left[\psi_{(x+\epsilon)} \right] \right) \forall \epsilon \exists x \exists \psi \right) \\
&\Rightarrow \exists \mathbf{x} \forall \mathsf{S} \forall \epsilon \exists x \exists \psi \left(\mathsf{S}\mathbf{x}\mathsf{S}^{-1} \left[\psi_{(x+\epsilon)} \right] = x \left[\psi_{(x+\epsilon)} \right] \right). \qquad (37)
\end{aligned}$$

But, as was mentioned, the operators S are members of the General Linear Group $\mathsf{GL}\,(n; \mathbb{F})$. There is therefore a one-parameter subgroup [19] of S, such that $\mathsf{S}_{(\epsilon)} \subset \mathsf{S}$. That is:

$$\exists \mathsf{S} \,(\mathsf{S}) = \exists \mathsf{S} \forall \epsilon \left(\mathsf{S}_{(\epsilon)} \right). \qquad (38)$$

Substituting (38) into (37) gives:

$$\exists \mathbf{x} \exists \mathsf{S} \forall \epsilon \exists x \exists \psi \left(\mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}_{(\epsilon)}^{-1} \left[\psi_{(x+\epsilon)} \right] = x \left[\psi_{(x+\epsilon)} \right] \right). \qquad (39)$$

3.3.4. Comparison of translation with similarity transformations

Assumption 3: Comparing translated equation (33) and the similarity transformed (39) we hypothesise that instances of ψ and x existing in one of these coincide with instances of ψ and x in the other, and that the following equivalence holds:

$$\exists \mathbf{x} \exists \mathsf{S} \forall \epsilon \left(\mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}_{(\epsilon)}^{-1} = \mathbf{x} + \epsilon \mathbf{1} \right). \qquad (40)$$

3.3.5. Exponential representation Now $\mathsf{S}_{(\epsilon)}$ is a one-parameter subgroup of $\mathsf{GL}\,(n; \mathbb{F})$. Hence there exists a unique linear operator $\mathbf{g}$ [19] such that:

$$\forall \mathsf{S} \exists \mathbf{g} \forall \epsilon \left(\begin{array}{l} \mathsf{S}_{(\epsilon)} = \mathrm{e}^{\epsilon \mathbf{g}} \\ \mathsf{S}_{(\epsilon)}^{-1} = \mathsf{S}_{(-\epsilon)} = \mathrm{e}^{-\epsilon \mathbf{g}} \end{array} \right). \qquad (41)$$

3.3.6. Infinitesimal translation Substitution of (41) into (40) gives:

$$\begin{aligned}
&\exists \mathbf{x} \exists \mathbf{g} \forall \epsilon \left(\exp(+\epsilon \mathbf{g}) \,\mathbf{x} \exp(-\epsilon \mathbf{g}) = \mathbf{x} + \epsilon \mathbf{1} \right) \\
&\Rightarrow \exists \mathbf{x} \exists \mathbf{g} \forall \epsilon \left(\left[\mathbf{1} + \epsilon \mathbf{g} + \mathcal{O}\left(\epsilon^2\right) \right] \mathbf{x} \left[\mathbf{1} - \epsilon \mathbf{g} + \mathcal{O}\left(\epsilon^2\right) \right] = \mathbf{x} + \epsilon \mathbf{1} \right) \\
&\Rightarrow \exists \mathbf{x} \exists \mathbf{g} \forall \epsilon \left(\left[\mathbf{x} + \epsilon \mathbf{g}\mathbf{x} + \mathcal{O}\left(\epsilon^2\right) \right] \left[\mathbf{1} - \epsilon \mathbf{g} + \mathcal{O}\left(\epsilon^2\right) \right] = \mathbf{x} + \epsilon \mathbf{1} \right) \\
&\Rightarrow \exists \mathbf{x} \exists \mathbf{g} \forall \epsilon \left(\mathbf{x} + \epsilon \mathbf{g}\mathbf{x} - \epsilon \mathbf{x}\mathbf{g} + \mathcal{O}\left(\epsilon^2\right) = \mathbf{x} + \epsilon \mathbf{1} \right) \\
&\Rightarrow \exists \mathbf{x} \exists \mathbf{g} \forall \epsilon \left(\mathbf{x} + \epsilon \,[\mathbf{g}, \mathbf{x}] + \mathcal{O}\left(\epsilon^2\right) = \mathbf{x} + \epsilon \mathbf{1} \right) \\
&\Rightarrow \exists \mathbf{x} \exists \mathbf{g} \forall \epsilon \left([\mathbf{g}, \mathbf{x}] = \mathbf{1} - \mathcal{O}(\epsilon) \right)
\end{aligned}$$

At the limit, as $\epsilon \to 0$:

$$\exists \mathbf{x} \exists \mathbf{g} \left([\mathbf{g}, \mathbf{x}] = \mathbf{1} \right) \qquad (42)$$

3.3.7. Arbitrariness of scale The sentence in (42) is the Lie bracket encapsulating the homogeneity of space. However, it describes this homogeneity at a single absolute scale; yet space is homogeneous irrespective of scale. And so (42) is not the general description for homogeneity at arbitrary scale yet this arbitrariness is a freedom which wave mechanics should incorporate. And so, consider the scaling of (42) by any scalar ξ.

$$\forall\xi\exists\mathbf{x}\exists\mathbf{g}\left(\xi\left[\mathbf{g},\mathbf{x}\right]=\xi\mathbf{1}\right).$$

Defining $\mathbf{p}$ as follows:

$$\forall\xi\exists\mathbf{g}\exists\mathbf{p}\left(\mathbf{p}=\xi\mathbf{g}\right)$$

results in:

$$\forall\xi\exists\mathbf{x}\exists\mathbf{p}\left(\left[\mathbf{p},\mathbf{x}\right]=\xi\mathbf{1}\right) \tag{43}$$

This is a most important result; (43) strategically takes the place of the canonical commutation relation in standard theory. It is the Lie bracket for the algebra of the general homogeneity of space. It is axiomatic to wave mechanics and is the bracket for which representations are written in Section 3.4.

3.4. Representation

We wish to know of particular instances of $\mathbf{p}$ and $\mathbf{x}$ that satisfy (43). Bearing in mind that (43) was derived from (33) and (39), we see that both relations act on the vector $\left[\psi_{(x+\epsilon)}\right]$. Remembering that (43) is valid for the limiting case when $\epsilon\to 0$, we expect instances of $\mathbf{p}$ and $\mathbf{x}$, satisfy (43), which act on position eigenvectors $\left[\psi_{(x)}\right]$. However, (43) is antisymmetric in $\mathbf{p}$ and $\mathbf{x}$, so if we had begun at (24), with an eigenvalue proposition for $\mathbf{p}$ instead of one for $\mathbf{x}$, we would have arrived at this present point in the text with momentum eigenvectors $\left[\phi_{(p)}\right]$. Therefore, we can also expect instances of $\mathbf{p}$ and $\mathbf{x}$, satisfying (43), which act on vectors $\left[\phi_{(p)}\right]$. In these respects, two representations are sought, each isomorphically satisfying (43).

Finite polynomials only Arguments below employ functions ψ and ϕ; and in the context of a first-order theory under the Field Axioms, rather than one of applied mathematics, it is important to understand what classes of function can be included in the discussion. All functions referred to are finite polynomials. These are valid under the Field Axioms. Transcendental functions such as the exponential exist undecidably [12]. This is because the exponential behaves differently in different fields. While $\exp:\mathbb{R}\to\mathbb{R}$ and $\exp:\mathbb{C}\to\mathbb{C}$; in contrast $\exp:\mathbb{Q}\not\to\mathbb{Q}$. In my argument I approximate the exponential by a finite polynomial with accuracy that can be extended to any desired degree.

3.4.1. The position space representation Observe the identity:

$$\forall\xi\forall\psi\forall x\left(\left[\left(\xi\frac{\mathrm{d}}{\mathrm{d}x}\right)x-x\left(\xi\frac{\mathrm{d}}{\mathrm{d}x}\right)\right]\psi\left(x\right)=\xi\psi\left(x\right)\right). \tag{44}$$

(44) isomorphically satisfies (43), indicating particular free values for **p**,**x** and **1**, as is confirmed in the algebra resulting in (48).

Assumption 4:

$$\forall\xi\left(\mathbf{p}=\xi\frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{45}$$

$$\mathbf{x}=x, \tag{46}$$

$$\forall x\left(\mathbf{1}\left[\psi_{(x)}\right]=1\psi\left(x\right)\right). \tag{47}$$

Substituting (45) into (44):

$$\forall\xi\forall\psi\forall x\left(\left[\mathbf{p}x-x\mathbf{p}\right]\psi\left(x\right)=\xi\psi\left(x\right)\right)$$

Now substituting (46) and (47):

$$\begin{aligned}
&\forall\xi\forall\psi\forall x\left(\left[\mathbf{px}-\mathbf{xp}\right]\mathbf{1}\left[\psi_{(x)}\right]=\xi\mathbf{1}\left[\psi_{(x)}\right]\right)\\
&\Rightarrow\forall\xi\forall\psi\forall x\left(\left[\mathbf{p},\mathbf{x}\right]\left[\psi_{(x)}\right]=\xi\mathbf{1}\left[\psi_{(x)}\right]\right)\\
&\Rightarrow\forall\xi\left(\left[\mathbf{p},\mathbf{x}\right]=\xi\mathbf{1}\right)\\
&\Rightarrow\forall\xi\exists\mathbf{p}\left(\left[\mathbf{p},\mathbf{x}\right]=\xi\mathbf{1}\right)\\
&\Rightarrow\forall\xi\exists\mathbf{x}\exists\mathbf{p}\left(\mathbf{p},\mathbf{x}=\xi\mathbf{1}\right).
\end{aligned} \tag{48}$$

3.4.2. The momentum space representation Now observe this second identity:

$$\forall\xi\forall\phi\forall p\left(\left[\left(-\xi\frac{\mathrm{d}}{\mathrm{d}p}\right)p-p\left(-\xi\frac{\mathrm{d}}{\mathrm{d}p}\right)\right]\phi\left(p\right)=-\xi\phi\left(p\right)\right). \tag{49}$$

(49) isomorphically satisfies (43), indicating particular free values for **p**,**x** and **1**, as is confirmed in the algebra resulting in (53).

Assumption 5:

$$\mathbf{p}=p, \tag{50}$$

$$\forall\xi\left(\mathbf{x}=-\xi\frac{\mathrm{d}}{\mathrm{d}p}\right), \tag{51}$$

$$\forall p\left(\mathbf{1}\left[\phi_{(p)}\right]=1\phi\left(p\right)\right). \tag{52}$$

Substituting (51) into (49):

$$\forall\xi\forall\phi\forall p\left(\left[\left(\mathbf{x}\right)p-p\left(\mathbf{x}\right)\right]\phi\left(p\right)=-\xi\phi\left(p\right)\right)$$

Now substituting (50) and (52):

$$\begin{aligned}
&\forall\xi\forall\phi\forall p\left(\left[\mathbf{xp}-\mathbf{px}\right]\mathbf{1}\left[\phi_{(p)}\right]=-\xi\mathbf{1}\left[\phi_{(p)}\right]\right)\\
&\Rightarrow\forall\xi\forall\phi\forall p\left(\left[\mathbf{x},\mathbf{p}\right]\left[\phi_{(p)}\right]=-\xi\mathbf{1}\left[\phi_{(p)}\right]\right)\\
&\Rightarrow\forall\xi\left(\left[\mathbf{p},\mathbf{x}\right]=\xi\mathbf{1}\right)\\
&\Rightarrow\forall\xi\exists\mathbf{p}\left(\left[\mathbf{p},\mathbf{x}\right]=\xi\mathbf{1}\right)\\
&\Rightarrow\forall\xi\exists\mathbf{x}\exists\mathbf{p}\left(\left[\mathbf{p},\mathbf{x}\right]=\xi\mathbf{1}\right).
\end{aligned} \tag{53}$$

3.5. The Free Particle

The two representations above, in Sections 3.4.1 and 3.4.2 equally represent (43). Moreover, they are equally valid, logically. We suppose, therefore, they must reconcile. But it will transpire that they do not, unless spaces of vectors $\left[\psi_{(x)}\right]$ and $\left[\phi_{(p)}\right]$ are each assumed orthogonal: an assumption, logically independent of Axioms, which entrains mathematical undecidability into the theory. This assumption of orthogonality is postponed to the point in the theory where it becomes irresistible.

We proceed by developing formulae for the wave mechanics of the free particle, for each of the two separate representations. Section 3.5.2 develops momentum solutions in position space, and Section 3.5.4 develops position solutions in momentum space.

3.5.1. Planck's constant With $\mathbf{x}$ and $\mathbf{p}$ interpreted as position and momentum operators respectively, independent of their values, ξ must contain a constant factor with units of action that balances dimensionality. Thus:

$$\exists\hbar\forall\eta\exists\xi\left(\xi=\eta\hbar\right). \tag{54}$$

In (54), the $\exists\hbar$ quantifier precedes $\forall\eta$ and in doing so specifies constancy of $\hbar$. Validity of (54) implies $\hbar$ must exist for all fields and so must be rational [12]. Substitution of $\hbar$ with the free variable $\hbar^{\mathbb{Q}}$ eliminates the $\exists\hbar$ quantifier.

$$\forall\eta\exists\xi\left(\xi=\eta\hbar^{\mathbb{Q}}\right). \tag{55}$$

3.5.2. Momentum eigensolutions in position-space

Assumption 6: Under the Field Axioms, assume validity of the eigenvalue formula proposing the existence of momentum:

$$\exists\mathbf{p}\exists\psi\exists p\exists x\left(\mathbf{p}\left[\psi_{p,(x)}\right]=p\left[\psi_{p,(x)}\right]\right). \tag{56}$$

Substituting free variable $p^{\mathbb{Q}}$ for the bound variable p, eliminating the $\exists p$ quantifier.

$$\exists\mathbf{p}\exists\psi\exists x\left(\mathbf{p}\left[\psi_{p^{\mathbb{Q}},(x)}\right]=p^{\mathbb{Q}}\left[\psi_{p^{\mathbb{Q}},(x)}\right]\right). \tag{57}$$

If (56) is valid, model theory guarantees the validity of (57) providing this formula is true irrespective of the field in which $p^{\mathbb{Q}}$ resides. Instances of $p^{\mathbb{Q}}$ must therefore be rational since that is only field contained by all fields [12]. The $\mathbb{Q}$ superscript denotes the rational status of $p^{\mathbb{Q}}$.

Now write (57) in terms of the representation (45), (46) and (47) producing the proposed differential equation:

$$\forall\xi\exists\psi\exists x\left(\xi\frac{\mathrm{d}}{\mathrm{d}x}\psi_{p^{\mathbb{Q}}}\left(x\right)=p^{\mathbb{Q}}\psi_{p^{\mathbb{Q}}}\left(x\right)\right) \tag{58}$$

Substituting (55) to expose the constant factor $\hbar^{\mathbb{Q}}$, correct dimensionality:

$$\forall\eta\exists\psi\exists x\left(\eta\hbar^{\mathbb{Q}}\frac{\mathrm{d}}{\mathrm{d}x}\psi_{p^{\mathbb{Q}}}\left(x\right)=p^{\mathbb{Q}}\psi_{p^{\mathbb{Q}}}\left(x\right)\right). \tag{59}$$

3.5.3. Momentum in position-space: the general superposition

Assumption 7: Assume the eigensolutions:

$$\forall\eta\forall x\exists\psi\left(\psi_{p^{\mathbb{Q}}}\left(x\right)=\exp\left(\frac{p^{\mathbb{Q}}x}{\eta\hbar^{\mathbb{Q}}}\right)\right). \tag{60}$$

These satisfy (59) for all values of x and therefore bear the quantifier $\forall x$.

Differential equation (59) evidently has a solution $\psi_{p^{\mathbb{Q}}}\left(x\right)$ for each eigenvalue $p^{\mathbb{Q}}$, and since this differential equation is *linear*, every linear combination of these solutions is also a solution. A simple example solution is the sum:

$$\Psi\left(x\right)=a_1\psi_{p_1^{\mathbb{Q}}}\left(x\right)+a_2\psi_{p_2^{\mathbb{Q}}}\left(x\right). \tag{61}$$

This shows just two amplitudes, a_1 and a_2. Actually, there is an infinite number of other amplitudes not shown: all with implicit value, zero. Those shown in (61) should properly be regarded as the only non-zero amplitudes of a continuum. These amplitudes form a function on the continuous domain $p^{\mathbb{Q}}$. Now since any and every linear combination of eigensolutions is a solution, it is natural to expect that every combination is permissible, suggesting $a\left(p^{\mathbb{Q}}\right)$ can be any function at all. Proceeding with this notion, we write proposition (62) which embodies this expectation. Note the quantifier $\forall a$:

$$\forall x\forall a\exists\Psi\exists\psi\left(\Psi\left(x\right)=\int_{\mathbb{Q}}a\left(p^{\mathbb{Q}}\right)\psi\left(p^{\mathbb{Q}},x\right)\mathrm{d}p^{\mathbb{Q}}\right), \tag{62}$$

Explicitly,

$$\forall\eta\forall x\forall a\exists\Psi\left(\Psi\left(x\right)\mathrel{\mathring{=}}\int_{\mathbb{Q}}a\left(p^{\mathbb{Q}}\right)\exp\left(\frac{p^{\mathbb{Q}}x}{\eta\hbar^{\mathbb{Q}}}\right)\mathrm{d}p^{\mathbb{Q}}\right). \tag{63}$$

This sum bares the equality $\mathring{=}$ because the integral of the exponential function over a rational domain is not exact. The Riemann integral can be constructed on a rational domain, providing that each point in the domain maps to a value of the function. This is not the case for the transcendental functions such as the exponential. However, it is the case for all finite polynomials whose coefficients are rational. Hence the exponential can be regarded as the limiting case of a finite polynomial whose accuracy can be extended to any desired degree. The analysis proceeds on that basis.

3.5.4. Position eigensolutions in momentum-space

Assumption 8: Under the Field Axioms, assume validity of the eigenvalue formula proposing the existence of position:

$$\exists\mathbf{x}\exists\phi\exists x\exists p\left(\mathbf{x}\left[\phi_{x,(p)}\right]=x\left[\phi_{x,(p)}\right]\right). \tag{64}$$

Substituting free variable $x^{\mathbb{Q}}$ for the bound variable x, eliminating the $\exists x$ quantifier.

$$\exists\mathbf{x}\exists\phi\exists p\left(\mathbf{x}\left[\phi_{x,(p)}\right]=x^{\mathbb{Q}}\left[\phi_{x,(p)}\right]\right). \tag{65}$$

If (64) is valid, model theory guarantees the validity of (65) providing this formula is true irrespective of the field in which $x^{\mathbb{Q}}$ resides. Instances of $x^{\mathbb{Q}}$ must therefore be rational since that is only field contained by all fields [12]. The $\mathbb{Q}$ superscript denotes the rational status of $x^{\mathbb{Q}}$.

Now write (65) in terms of the representation (50), (51) and (52) producing the proposed differential equation:

$$\forall\xi\exists\phi\exists p\left(\xi\frac{\mathrm{d}}{\mathrm{d}p}\phi_{x^{\mathbb{Q}}}(p)=x^{\mathbb{Q}}\phi_{x^{\mathbb{Q}}}(p)\right) \tag{66}$$

Substituting (55) to expose the constant factor $\hbar^{\mathbb{Q}}$, correct dimensionality:

$$\forall\eta\exists\phi\exists p\left(-\eta\hbar^{\mathbb{Q}}\frac{\mathrm{d}}{\mathrm{d}p}\phi_{x^{\mathbb{Q}}}(p)=x^{\mathbb{Q}}\phi_{x^{\mathbb{Q}}}(p)\right). \tag{67}$$

3.5.5. Position in momentum-space: the general superposition

Assumption 9: Assume the eigensolutions:

$$\forall\eta\forall p\exists\phi\left(\phi_{x^{\mathbb{Q}}}(p)=\exp\left(\frac{px^{\mathbb{Q}}}{-\eta\hbar^{\mathbb{Q}}}\right)\right). \tag{68}$$

These satisfy (59) for all values of p and therefore bear the quantifier $\forall p$.

Like the general superposition solutions (63) for the momentum formula (59), position formula (67) has general superpositions:

$$\forall\eta\forall p\forall b\exists\Phi\left(\Phi(p)\doteq\int_{\mathbb{Q}}b\left(x^{\mathbb{Q}}\right)\exp\left(\frac{px^{\mathbb{Q}}}{-\eta\hbar^{\mathbb{Q}}}\right)\mathrm{d}x^{\mathbb{Q}}\right). \tag{69}$$

3.6. Validity of general superposition

Let us scrutinise the validity of proposed superpositions, (63) and (69). Although *any* and *every* linear combination of the respective eigensolutions are themselves solutions of their differential equations, *not all* combinations validly satisfy the quantifiers in these two propositions. In (63), lack of restriction on amplitudes $a\left(p^{\mathbb{Q}}\right)$ results in a failure to satisfy the $\exists\Psi$ quantifier. This is because not all sums satisfy the Field Axioms: some functions $a\left(p^{\mathbb{Q}}\right)$ furnish values of Ψ that do not exist. There are examples, such as $a\left(p^{\mathbb{Q}}\right)=1$, for which the integral is infinite; but infinity is not a value that exists under the Field Axioms. In fact, this integral exists only if $\int_{\mathbb{Q}}a\left(p^{\mathbb{Q}}\right)\mathrm{d}p^{\mathbb{Q}}$ converges to a finite limit. Equivalent arguments apply to (69). For this reason, as they stand, (63) and (69) are logically invalid.

Particles The logical invalidity of (63) and (69) prevents the existence of non-particle wave packets. This places the fundamental reason for particles in Nature on validity under the Field Axioms. And explains the origin of the rule in Wave Mechanics which says “only square integrable solutions are acceptable”.

3.6.1. Valid superpositions Weaker but logically valid superpositions may proposed by modifying (63) and (69). In these the quantifiers $\forall a$ and $\forall b$ are replaced by $\exists a$ and $\exists b$:

$$\forall\eta\forall x\exists a\exists\Psi\left(\Psi\left(x\right)\doteq\int_{\mathbb{Q}}a\left(p^{\mathbb{Q}}\right)\exp\left(\frac{p^{\mathbb{Q}}x}{\eta\hbar^{\mathbb{Q}}}\right)\mathrm{d}p^{\mathbb{Q}}\right); \tag{70}$$

$$\forall\eta\forall p\exists b\exists\Phi\left(\Phi\left(p\right)\doteq\int_{\mathbb{Q}}b\left(x^{\mathbb{Q}}\right)\exp\left(\frac{px^{\mathbb{Q}}}{-\eta\hbar^{\mathbb{Q}}}\right)\mathrm{d}x^{\mathbb{Q}}\right). \tag{71}$$

These allow the escape of instances of $a\left(p^{\mathbb{Q}}\right)$ and $b\left(x^{\mathbb{Q}}\right)$ that produce the infinite integrals.

3.6.2. Towards Fourier At this point in the story, it is instructive to realise the close relationship (70) and (71) have with the Fourier transform and its inverse: for that is where the analysis is heading. Fourier transform pairs are particular instances of these propositions.

Noting that (70) and (71) are equally valid propositions describing conditions within the same system, it is reasonable to suppose they may be regarded as a pair within a mutual logical environment where each might possibly affect the other. The analysis proceeds with this notion in mind. Proposition (70) confirms the existence of logically valid pairs of functions: $\Psi\left(x\right)$ and $a\left(p^{\mathbb{Q}}\right)$. The possibility now arises: given any particular one of the pairs: $\Psi\left(x\right)$ and $a\left(p^{\mathbb{Q}}\right)$; does a particular $\Phi\left(p^{\mathbb{Q}}\right)$ exist that matches, and may substitute this particular $a\left(p^{\mathbb{Q}}\right)$? The antisymmetrical possibility also applies in the pair: $\Phi\left(p\right)$ and $b\left(x^{\mathbb{Q}}\right)$, and the substitution candidate: $\Psi\left(x^{\mathbb{Q}}\right)$. Notice that such substitutions would initiate a system of feedback.

Assumption 10: Hypothesise

$$\forall a\exists\Phi\left(a\left(p^{\mathbb{Q}}\right)=\Phi\left(p^{\mathbb{Q}}\right)\right);$$
$$\forall b\exists\Psi\left(b\left(p^{\mathbb{Q}}\right)=\Psi\left(p^{\mathbb{Q}}\right)\right).$$

In order for this to proceed bound variables p and x must be freed and assigned values $p := p^{\mathbb{Q}}$ and $x := x^{\mathbb{Q}}$; thus:

$$\forall\eta\exists a\exists\Psi\left(\Psi\left(x^{\mathbb{Q}}\right)\doteq\int_{\mathbb{Q}}a\left(p^{\mathbb{Q}}\right)\exp\left(\frac{p^{\mathbb{Q}}x^{\mathbb{Q}}}{\eta\hbar^{\mathbb{Q}}}\right)\mathrm{d}p^{\mathbb{Q}}\right) \tag{72}$$

$$\forall\eta\exists b\exists\Phi\left(\Phi\left(p^{\mathbb{Q}}\right)\doteq\int_{\mathbb{Q}}b\left(x^{\mathbb{Q}}\right)\exp\left(\frac{p^{\mathbb{Q}}x^{\mathbb{Q}}}{-\eta^{\mathbb{F}}\hbar^{\mathbb{Q}}}\right)\mathrm{d}x^{\mathbb{Q}}\right) \tag{73}$$

Now make substitutions: $a\left(p^{\mathbb{Q}}\right) := \Phi\left(p^{\mathbb{Q}}\right)$ and $b\left(x^{\mathbb{Q}}\right) := \Psi\left(x^{\mathbb{Q}}\right)$, thus:

$$\forall\eta\exists\Psi\exists\Phi\left(\begin{array}{c}\Psi\left(x^{\mathbb{Q}}\right)\doteq\int_{\mathbb{Q}}\Phi\left(p^{\mathbb{Q}}\right)\exp\left(\frac{p^{\mathbb{Q}}x^{\mathbb{Q}}}{\eta\hbar^{\mathbb{Q}}}\right)\mathrm{d}p^{\mathbb{Q}}\\ \\ \Phi\left(p^{\mathbb{Q}}\right)\doteq\int_{\mathbb{Q}}\Psi\left(x^{\mathbb{Q}}\right)\exp\left(\frac{p^{\mathbb{Q}}x^{\mathbb{Q}}}{-\eta\hbar^{\mathbb{Q}}}\right)\mathrm{d}x^{\mathbb{Q}}\end{array}\right) \tag{74}$$

3.6.3. Consistency? We now check (74) for consistency. Whereas the Fourier transform inverts to its inverse; an attempt to invert one of these formula to obtain the other, instead reveals they are not in general mutually consistent. Rewriting the top formula of (74) with a fresh dummy variable $p'^{\mathbb{Q}}$:

$$\Psi\left(x^{\mathbb{Q}}\right) \doteq \int_{\mathbb{Q}} \Phi\left(p'^{\mathbb{Q}}\right) \exp\left(\frac{p'^{\mathbb{Q}} x^{\mathbb{Q}}}{\eta \hbar^{\mathbb{Q}}}\right) \mathrm{d}p'^{\mathbb{Q}}. \tag{75}$$

In this, a primed $p'^{\mathbb{Q}}$ is chosen to distinguish from instances of $p^{\mathbb{Q}}$ that occur in the integration that comes next. Multiplying both sides of (75) by the negative exponential and integrating with respect to $x^{\mathbb{Q}}$ gives:

$$\begin{aligned}
&\int_{\mathbb{Q}} \Psi\left(x^{\mathbb{Q}}\right) \exp\left(\frac{p^{\mathbb{Q}} x^{\mathbb{Q}}}{-\eta \hbar^{\mathbb{Q}}}\right) \mathrm{d}x^{\mathbb{Q}} \\
&\qquad \doteq \int_{\mathbb{Q}} \left[\int_{\mathbb{Q}} \Phi\left(p'^{\mathbb{Q}}\right) \exp\left(\frac{p'^{\mathbb{Q}} x^{\mathbb{Q}}}{\eta \hbar^{\mathbb{Q}}}\right) \mathrm{d}p'^{\mathbb{Q}}\right] \exp\left(\frac{p^{\mathbb{Q}} x^{\mathbb{Q}}}{-\eta \hbar^{\mathbb{Q}}}\right) \mathrm{d}x^{\mathbb{Q}} && (76)\\
&\qquad \doteq \int_{\mathbb{Q}} \left[\int_{\mathbb{Q}} \Phi\left(p'^{\mathbb{Q}}\right) \exp\left(\frac{p'^{\mathbb{Q}} x^{\mathbb{Q}}}{\eta \hbar^{\mathbb{Q}}}\right) \exp\left(\frac{p^{\mathbb{Q}} x^{\mathbb{Q}}}{-\eta \hbar^{\mathbb{Q}}}\right) \mathrm{d}p'^{\mathbb{Q}}\right] \mathrm{d}x^{\mathbb{Q}} \\
&\qquad \doteq \int_{\mathbb{Q}} \left[\int_{\mathbb{Q}} \Phi\left(p'^{\mathbb{Q}}\right) \exp\left(\frac{p'^{\mathbb{Q}} x^{\mathbb{Q}}}{\eta \hbar^{\mathbb{Q}}}\right) \exp\left(\frac{p^{\mathbb{Q}} x^{\mathbb{Q}}}{-\eta \hbar^{\mathbb{Q}}}\right) \mathrm{d}x^{\mathbb{Q}}\right] \mathrm{d}p'^{\mathbb{Q}} \\
&\qquad \doteq \int_{\mathbb{Q}} \left[\int_{\mathbb{Q}} \Phi\left(p'^{\mathbb{Q}}\right) \exp\left(\frac{x^{\mathbb{Q}}}{\eta \hbar^{\mathbb{Q}}}\left(p'^{\mathbb{Q}} - p^{\mathbb{Q}}\right)\right) \mathrm{d}x^{\mathbb{Q}}\right] \mathrm{d}p'^{\mathbb{Q}} \\
&\qquad \doteq \int_{\mathbb{Q}} \Phi\left(p'^{\mathbb{Q}}\right) \left[\int_{\mathbb{Q}} \exp\left(\frac{x^{\mathbb{Q}}}{\eta \hbar^{\mathbb{Q}}}\left(p'^{\mathbb{Q}} - p^{\mathbb{Q}}\right)\right) \mathrm{d}x^{\mathbb{Q}}\right] \mathrm{d}p'^{\mathbb{Q}} && (77)
\end{aligned}$$

This means that (74) implies the proposition:

$$\forall\eta\exists\Psi\exists\Phi \left(\begin{array}{l}
\int_{\mathbb{Q}} \Psi\left(x^{\mathbb{Q}}\right) \exp\left(\frac{p^{\mathbb{Q}} x^{\mathbb{Q}}}{-\eta \hbar^{\mathbb{Q}}}\right) \mathrm{d}x^{\mathbb{Q}} \\
\qquad \doteq \int_{\mathbb{Q}} \Phi\left(p'^{\mathbb{Q}}\right) \left[\int_{\mathbb{Q}} \exp\left(\frac{x^{\mathbb{Q}}}{\eta \hbar^{\mathbb{Q}}}\left(p'^{\mathbb{Q}} - p^{\mathbb{Q}}\right)\right) \mathrm{d}x^{\mathbb{Q}}\right] \mathrm{d}p'^{\mathbb{Q}} \\
\\
\Phi\left(p^{\mathbb{Q}}\right) \doteq \int_{\mathbb{Q}} \Psi\left(x^{\mathbb{Q}}\right) \exp\left(\frac{p^{\mathbb{Q}} x^{\mathbb{Q}}}{-\eta \hbar^{\mathbb{Q}}}\right) \mathrm{d}x^{\mathbb{Q}}
\end{array}\right) \tag{78}$$

Viewed as a single proposition, (78) can never be said to be logically valid. The integral $\int_{\mathbb{Q}} \exp\left(\frac{x^{\mathbb{Q}}}{\eta \hbar^{\mathbb{Q}}}\left(p'^{\mathbb{Q}} - p^{\mathbb{Q}}\right)\right) \mathrm{d}x^{\mathbb{Q}}$, has unbounded value and does not converge to any value of scalar; and so does not exist for all η, hence the $\forall\eta$ quantifier in (78) does not hold. On the other hand, if (78) is viewed as a pair of separate formulae sharing identical instances of $\Psi\left(x^{\mathbb{Q}}\right)$ and $\Phi\left(p^{\mathbb{Q}}\right)$, then these two formulae of the pair are inconsistent because the top formulae denies the existence of the integral $\int_{\mathbb{Q}} \Psi\left(x^{\mathbb{Q}}\right) \exp\left(\frac{p^{\mathbb{Q}} x^{\mathbb{Q}}}{-\eta \hbar^{\mathbb{Q}}}\right) \mathrm{d}x^{\mathbb{Q}}$ while the bottom formula confirms it.

3.6.4. Undecidability relieves the inconsistency There is a proposition, weaker than (78), where the $\exists\eta$ quantifier replaces $\forall\eta$. Thus:

$$\exists\eta\exists\Psi\exists\Phi \left(\begin{array}{l}
\int_{\mathbb{Q}} \Psi\left(x^{\mathbb{Q}}\right) \exp\left(\frac{p^{\mathbb{Q}} x^{\mathbb{Q}}}{-\eta \hbar^{\mathbb{Q}}}\right) \mathrm{d}x^{\mathbb{Q}} \\
\qquad \doteq \int_{\mathbb{Q}} \Phi\left(p'^{\mathbb{Q}}\right) \left[\int_{\mathbb{Q}} \exp\left(\frac{x^{\mathbb{Q}}}{\eta \hbar^{\mathbb{Q}}}\left(p'^{\mathbb{Q}} - p^{\mathbb{Q}}\right)\right) \mathrm{d}x^{\mathbb{Q}}\right] \mathrm{d}p'^{\mathbb{Q}} \\
\\
\Phi\left(p^{\mathbb{Q}}\right) \doteq \int_{\mathbb{Q}} \Psi\left(x^{\mathbb{Q}}\right) \exp\left(\frac{p^{\mathbb{Q}} x^{\mathbb{Q}}}{-\eta \hbar^{\mathbb{Q}}}\right) \mathrm{d}x^{\mathbb{Q}}
\end{array}\right) \tag{79}$$

This proposition is also inconsistent. However, it is optionally undecidable, conditional on the following assumption.

Assumption 11: Independent of the Field Axioms, assume the undecidable scalar η:

$$\exists\eta\left(\eta\eta=-1\right). \tag{80}$$

This sentence is true in the field $\mathbb{C}$ but not in others and is therefore undecidable [12]. Furthermore, any proposition relying on (80) also assumes this undecidability. And so, taking (79) assign $p^{\mathbb{Q}}=p'^{\mathbb{Q}}$; then free bound variable η, and adopting (80), assign $\eta:=\mathrm{i}\equiv\sqrt{-1}$. The resulting proposition is:

$$\exists\Psi\exists\Phi\left(\begin{array}{c}\int_{\mathbb{Q}}\Psi\left(x^{\mathbb{Q}}\right)\exp\left(\frac{p^{\mathbb{Q}}x^{\mathbb{Q}}}{-\mathrm{i}\hbar^{\mathbb{Q}}}\right)\mathrm{d}x^{\mathbb{Q}}\circeq\Phi\left(p^{\mathbb{Q}}\right)\\ \\ \Phi\left(p^{\mathbb{Q}}\right)\circeq\int_{\mathbb{Q}}\Psi\left(x^{\mathbb{Q}}\right)\exp\left(\frac{p^{\mathbb{Q}}x^{\mathbb{Q}}}{-\mathrm{i}\hbar^{\mathbb{Q}}}\right)\mathrm{d}x^{\mathbb{Q}}\end{array}\right) \tag{81}$$

3.6.5. Orthogonality (79) is a particular instance of the pair of superpositions (70) and (71) which is consistent but undecidable. Consistency ensues from orthogonality in the function space, permitted by the undecidable existence of the independent scalar assumed in (80). (78) is resolved with the integral, known as the Dirac delta function:

$$\int_{\mathbb{Q}}\exp\left(\frac{x^{\mathbb{Q}}}{\mathrm{i}\hbar^{\mathbb{Q}}}\left(p^{\mathbb{Q}}-p'^{\mathbb{Q}}\right)\right)\mathrm{d}x^{\mathbb{Q}}\circeq\left\{\begin{array}{ccc}1 & ; & p^{\mathbb{Q}}=p'^{\mathbb{Q}}\\ 0 & ; & p^{\mathbb{Q}}\neq p'^{\mathbb{Q}}\end{array}\right. . \tag{82}$$

Procedures in Section 3.6.3 inadvertently introduce an inner product within the vector space of exponential functions. This is where (75) is multiplied through by the negative exponential and integrated with respect to x, yielding (76). At that point in the story, that space has no orthogonal functions, nor indeed any scalar product at all: not until their existence is permitted by (82) which in turn assumes the indeterminate sentence (80).

The work of Baylis, Huschilt and Jiansu Wei [1] is extended in [13] to show that propositions asserting the existence of orthogonal spaces are always accompanied by (80). This is the case for all spaces of orthogonal polynomials even when they do not show the imaginary unit explicitly. The case studied in this paper has been the free particle, but analogous treatment of the harmonic oscillator energies would illustrate similar dependency of indeterminacy on orthogonal functions.

3.7. Logically valid existence of probability

Combinations of undecidable propositions can be logically valid [12]. Although (81) is undecidable, it derives the logically valid existence of distributions of rational scalars $P^{\mathbb{Q}}\left(p^{\mathbb{Q}}\right)$. Taking the bottom formula of (81) gives the undecidable proposition:

$$\exists\Phi\exists\Psi\left(\Phi\left(p^{\mathbb{Q}}\right)\circeq\int_{\mathbb{Q}}\Psi\left(x^{\mathbb{Q}}\right)\exp\left(\frac{p^{\mathbb{Q}}x^{\mathbb{Q}}}{-\mathrm{i}\hbar^{\mathbb{Q}}}\right)\mathrm{d}x^{\mathbb{Q}}\right).$$

This means that the existence of any particular $\Phi\left(p^{\mathbb{Q}}\right)$ cannot be confirmed, but more importantly, it cannot be denied. But for all instances of $\Phi\left(p^{\mathbb{Q}}\right)$ there exists another conjugate instance: $\Phi^*\left(p^{\mathbb{Q}}\right)$ which also cannot be denied. And so the real product:

$$P\left(p^{\mathbb{Q}}\right)=\Phi^*\left(p^{\mathbb{Q}}\right)\Phi\left(p^{\mathbb{Q}}\right) \tag{83}$$

cannot be denied either. Amongst the plethora of the instances: $\Phi\left(p^{\mathbb{Q}}\right)$ and $\Phi^*\left(p^{\mathbb{Q}}\right)$, some will accidentally be those that form the subset for which values P are rational:

$$P^{\mathbb{Q}}\left(p^{\mathbb{Q}}\right)=\Phi^*\left(p^{\mathbb{Q}}\right)\Phi\left(p^{\mathbb{Q}}\right);$$

and we may write the logically valid formula:

$$\exists P\left(P\left(p^{\mathbb{Q}}\right)=\Phi^*\left(p^{\mathbb{Q}}\right)\Phi\left(p^{\mathbb{Q}}\right)\right) \tag{84}$$

Validity of this formula is interpreted as cause in Nature for values of probability $P^{\mathbb{Q}}$.

Pythagorean addition of amplitudes Addition of rational scalars yields other rational scalars. And so addition of rational scalars yields other logically valid scalars:

$$\forall P_1 \forall P_2 \exists P\left(P=P_1+P_2\right).$$

Hence sums of amplitudes are logically valid:

$$\forall \Psi_1 \forall \Psi_2 \exists \Psi\left(|\Psi|^2=|\Psi_1|^2+|\Psi_2|^2\right).$$

This is seen as cause in Nature for Pythagorean sums of amplitudes.

4. Conclusions

Gödelian undecidability

An inconsistent theory, replaced by one which is undecidable is strongly reminiscent of Gödel. This transition takes place just where a theory under the Field Axioms assumes orthogonality. It is an utterly astonishing revelation to find that Elemér Rosinger has shown that orthogonality is definable in a self referent manner [34]. With the three concepts of inconsistency, mathematical undecidability and self reference, all associated with one particular feature in the theory, it is difficult to resist the notion that quantum physics is inherently Gödelian.

References

[1] Baylis WE, Huschilt J and Jiansu Wei 1991 *Why i?* American Journal of Physics 1992 60/9 p 788-797

[2] Baym G 1969 *Lectures on Quantum Mechanics* W A Benjamin Inc.

[3] Boolos GS and Jeffrey RC 1974 *Computability and Logic* Third Edition 1989 Cambridge University Press UK

[4] Born M and Jordan P 1925 Z f Phys......Heisenberg W 1925 *Quantum-Theoretical Re-interpretation of Kinematic and Mechanical Relations* [Book: Sources of Quantum Mechanics by van Der Waerden (Zurich), copyright 1967 North holland Publishing Co. Amsterdam] Dover Publications Inc. New york 1968 p 261-276

[5] Brown A et al Logic Option Course Team 1981 *M335 Studies in Pure Mathematics: Logic Units 5-8* The Open University Milton Keynes UK
[6] Cameron PJ 1999 *Sets, Logic and Categories* Springer-Verlag London Limited 1999
[7] Chaitin GJ 1982 *Gödel's Theorem and Information* International Journal of Physics 22 p 941-954
[8] Cornwell JF 1997 *Group Theory in Physics, An Introduction* Academic Press Ltd London
[9] Dicke R H and Wittke J P 1960 *Introduction to Quantum Mechanics* Addison-Wesley Publishing Company Inc.
[10] Dirac P A M 1930 *THE PRINCIPLES OF QUANTUM MECHANICS* Fourth ed. 1958 Oxford University Press
[11] Durrant A (Course Team Chairman) et al 1986 *SM355 Quantum Mechanics* The Open University Milton Keynes UK
[12] Faulkner S 2011 *Indeterminate scalars under the Field Axioms: foundation for Reichenbach's quantum logic* http://www.vixra.org/pdf/1101.0045v1.pdf
[13] Faulkner S 2011 *The general dependence of orthogonality on the square root of minus one* To be written
[14] Feynman R P 1985 *QED: The Strange Theory of Light and Matter* Princeton University Press
[15] Foschini L 2005 *On the logic of quantum physics and the concept of time* Institute TeSRE - CNR Via Gobetti 101 I-40129 Bologna Italy
[16] Franzén T 2005 *Gödel's Theorem: An Incomplete Guide to its Use and Abuse* AK Peters Ltd
[17] Goldstein H, Poole C and Safco J 2002 *Classical Mechanics, Third Edition* Addison Wesley
[18] Haack S 1978 *Philosophy of Logics* Cambridge University Press UK
[19] Hall B C 2003 *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction* Springer-Verlag New York Inc.
[20] Halmos PR 1996 *Linear Algebra problem Book* Dolciani Mathematical Expositions. The Mathematical Association of America
[21] Heisenberg W 1925 *Quantum-Theoretical Re-interpretation of Kinematic and Mechanical Relations* [Book: Sources of Quantum Mechanics by van Der Waerden (Zurich) copyright 1967 North holland Publishing Co. Amsterdam] Dover Publications Inc. New york 1968 p 261-276
[22] Isham C J 1995 *Lectures On Quantum Theory: Mathematical and Structural Foundations* Imperial College Press London
[23] Kaku M 1993 *Quantum Field Theory: A Modern Introduction* Oxford University Press
[24] Łukasiewicz J and Tarski A *op. cit.* p 1 The first publication by Łukasiewicz of his ideas was made in the Polish journal *Ruch Filozoficzny*, V (Lwow, 1920), 169-170
[25] Marker D 2002 *Model Theory An Introduction* Springer-Verlag
[26] Nering E D 1963 *Linear Algebra and Matrix Theory* second edition 1970 John Wiley & Sons Inc.
[27] Paterek T et al 2008 *Mathematical undecidability and quantum randomness* arXiv:0811.4542v1 [quant-ph] 27 Nov 2008
[28] Peskin M E & Schroeder D V 1993 *An Introduction to Quantum Field Theory* Perseus Books
[29] Putnam H 1957 *'Three-valued logic'* Philosophical Studies **VIII**/5 3-80
[30] S357 Course Team chair: Mackintosh R 1997 *Space, Time and Cosmology Unit 3* The Open University Milton Keynes UK
[31] Reichenbach H 1944 *Philosophic Foundations of Quantum Mechanics* University of California Press Berkeley California
[32] Robinett R W 1997 *Quantum Mechanics: Classical Results, Modern Systems, and Visualised Examples* Oxford University Press Oxford UK
[33] Rosenberg J 2003 *A Selective History of the Stone-von Neumann Theorem*
[34] Rosinger E E, van Zyl G 2009 *Self-Referential Definition of Orthogonality* http://arxiv.org/pdf/0904.0082
[35] Ryder L H 1985 *Quantum Field Theory Second Edition* Cambridge University Press UK
[36] Sakurai J J 1994 *Modern Quantum Mechanics Revised Edition* Addison-Wesley Publishing Company Inc.

[37] Svozil K 2005 *Undecidability everywhere?* Institut fur Theoreische Physik University of Technology Vienna

[38] Torretti R 1999 *The Philosophy of Physics* Cambridge University Press UK

[39] The Course Team 1979 *M203: Introduction to Pure Mathematics Third edition 1987* Unit LA5 The Open University Milton Keynes United Kingdom

[40] Woit P 2003 Lecture Notes: *Topics in Representation Theory: The Heisenberg Algebra* Dept. Mathematics Columbia University NY

[41] Yang CN *Square root of minus one, complex phases and Erwin Schrödinger* [Book: Schrödinger, Centenary Celebration of a Polymath. Ed. C W Kilmister Cambridge University Press 1987] p 53-64.