The algorithms of the real cube root, the positive fourth root, the real fifth root and the real seventh root of a positive number

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In this paper we give the algorithms of the real cube root, the positive fourth root, the real fifth root and the real seventh root of a positive number. Each of the four algorithms starts with a positive number in decimal notation, then, for a non negative integer p, it writes $p+1$ integers g_i and it goes through $p+1$ steps in each of which it compares at most 10 pairs of integers and calculates two integers r_i and d_i, increasing (if necessary) p until $r_p = 0$.

The real cube root of a positive number

Let $x = \sum_{n=0}^{\infty} 10^{N-n} a_{N-n}$ be a positive real number, where $N \in \mathbb{Z}$, $a_{N-n} \in \{0, 1, ..., 9\}$ for all n and $a_N \neq 0$. By the division algorithm there exist unique integers q and r such that $N = 3q + r$ and $0 \leq r < 3$, then

$$x = \sum_{k=0}^{r} 10^{3q+k} a_{3q+k} + \sum_{i=1}^{\infty} \sum_{j=0}^{2} 10^{3(q-i)+j} a_{3(q-i)+j}.$$

Let p be a non negative integer and $g_0, ..., g_p$ be the integers

$$g_0 = \sum_{k=0}^{r} 10^k a_{3q+k},$$

and

$$g_s = \sum_{k=0}^{2} 10^k a_{3(q-s)+k},$$

for each s such that $0 < s \leq p$.
At the first step, find

$$y_0 = \max\{y \in \{0, 1, ..., 9\} : y^3 \leq g_0\},$$

and write

$$r_0 = g_0 - y_0^3$$

and

$$d_0 = 10^3 r_0 + g_1.$$

After find

$$y_1 = \max\{y \in \{0, 1, ..., 9\} : y(y^2 + 30(10y_0 + y)y_0) \leq d_0\},$$

and write

$$r_1 = d_0 - y_1(y_1^2 + 30(10y_0 + y_1)y_0)$$

and

$$d_1 = 10^3 r_1 + g_2.$$

At the s-th step, find

$$y_s = \max\{y \in \{0,1,...,9\} : y(y^2 + 30(\sum_{t=0}^{s-1} 10^{s-t} y_t + y)(\sum_{t=0}^{s-1} 10^{s-1-t} y_t)) \leq d_{s-1}\},$$

and write

$$r_s = d_{s-1} - y_s(y_s^2 + 30(\sum_{t=0}^{s} 10^{s-t} y_t)(\sum_{t=0}^{s-1} 10^{s-1-t} y_t))$$

and

$$d_s = 10^3 r_s + g_{s+1}.$$

If the decimal expansion of the real cube root is finite, increase (if necessary) p until $r_p = 0$. Finally , the real cube root z of x is

$$z = \sum_{s=0}^{p} 10^{q-s} y_s.$$

The positive fourth root of a positive number

Let $x = \sum_{n=0}^{\infty} 10^{N-n} a_{N-n}$ be a positive real number, where $N \in \mathbb{Z}$, $a_{N-n} \in \{0,1,...,9\}$ for all n and $a_N \neq 0$. By the division algorithm there exist unique integers q and r such that $N = 4q + r$ and $0 \leq r < 4$, then

$$x = \sum_{k=0}^{r} 10^{4q+k} a_{4q+k} + \sum_{i=1}^{\infty} \sum_{j=0}^{3} 10^{4(q-i)+j} a_{4(q-i)+j}.$$

Let p be a non negative integer and $g_0, ..., g_p$ be the integers

$$g_0 = \sum_{k=0}^{r} 10^k a_{4q+k},$$

and

$$g_s = \sum_{k=0}^{3} 10^k a_{4(q-s)+k},$$

for all $0 < s \leq p$.

At the first step, find

$$y_0 = \max\{y \in \{0, 1, ..., 9\} : y^4 \leq g_0\},$$

and write

$$r_0 = g_0 - y_0^4$$

and

$$d_0 = 10^4 r_0 + g_1.$$

After find

$$y_1 = \max\{y \in \{0, 1, ..., 9\} : y(y^3 + 10(40(10y_0 + y)y_0^2 + 2y(2y + 10y_0)y_0)) \leq d_0\},$$

and write

$$r_1 = d_0 - y_1(y_1^3 + 10(40(10y_0 + y_1)y_0^2 + 2y_1(2y_1 + 10y_0)y_0))$$

and

$$d_1 = 10^4 r_1 + g_2.$$

At the s-th step, find

$$y_s = \max\{y \in \{0, 1, ..., 9\} : y(y^3 + 10(40(\sum_{t=0}^{s-1} 10^{s-t} y_t + y)(\sum_{t=0}^{s-1} 10^{s-1-t} y_t)^2 + 2y(2y + \sum_{t=0}^{s-1} 10^{s-t} y_t)(\sum_{t=0}^{s-1} 10^{s-1-t} y_t))) \leq d_{s-1}\},$$

and write

$$r_s = d_{s-1} - y_s(y_s^3 + 10(40(\sum_{t=0}^{s} 10^{s-t} y_t)(\sum_{t=0}^{s-1} 10^{s-1-t} y_t)^2 + 2y_s(2y_s + \sum_{t=0}^{s-1} 10^{s-t} y_t)(\sum_{t=0}^{s-1} 10^{s-1-t} y_t))),$$

and

$$d_s = 10^4 r_s + g_{s+1}.$$

If the decimal expansion of the positive fourth root is finite, increase (if necessary) p until $r_p = 0$. Finally , the positive fourth root z of x is

$$z=\sum_{s=0}^{p}10^{q-s}y_s.$$

The real fifth root of a positive number

Let $x=\sum_{n=0}^{\infty}10^{N-n}a_{N-n}$ be a positive real number, where $N\in\mathbb{Z}$, $a_{N-n}\in\{0,1,...,9\}$ for all n and $a_N\neq 0$. By the division algorithm there exist unique integers q and r such that $N=5q+r$ and $0\leq r<5$, then

$$x=\sum_{k=0}^{r}10^{5q+k}a_{5q+k}+\sum_{i=1}^{\infty}\sum_{j=0}^{4}10^{5(q-i)+j}a_{5(q-i)+j}.$$

Let p be a non negative integer and $g_0,...,g_p$ be the integers

$$g_0=\sum_{k=0}^{r}10^k a_{5q+k},$$

and

$$g_s=\sum_{k=0}^{4}10^k a_{5(q-s)+k},$$

for all $0<s\leq p$.
At the first step, find

$$y_0=\max\{y\in\{0,1,...,9\}:y^5\leq g_0\},$$

and write

$$r_0=g_0-y_0^5$$

and

$$d_0=10^5r_0+g_1.$$

After find

$$y_1=\max\{y\in\{0,1,...,9\}:y(y^4+50(100(10y_0+y)y_0^3+(10y_0+y)^2y_0y))\leq d_0\},$$

and write

$$r_1=d_0-y_1(y_1^4+50(100(10y_0+y_1)y_0^3+(10y_0+y_1)^2y_0y_1))$$

and

$$d_1=10^5r_1+g_2.$$

At the s-th step, find

$$y_s = \max\{y \in \{0,1,...,9\} : y(y^4 + 50(100(\sum_{t=0}^{s-1} 10^{s-t} y_t + y)(\sum_{t=0}^{s-1} 10^{s-1-t} y_t)^3 + (\sum_{t=0}^{s-1} 10^{s-t} y_t + y)^2 (\sum_{t=0}^{s-1} 10^{s-1-t} y_t) y)) \leq d_{s-1}\},$$

and write

$$r_s = d_{s-1} - y_s(y_s^4 + 50(100(\sum_{t=0}^{s} 10^{s-t} y_t)(\sum_{t=0}^{s-1} 10^{s-1-t} y_t)^3 + (\sum_{t=0}^{s} 10^{s-t} y_t)^2 (\sum_{t=0}^{s-1} 10^{s-1-t} y_t) y_s))$$

and

$$d_s = 10^5 r_s + g_{s+1}.$$

If the decimal expansion of the real fifth root is finite, increase (if necessary) p until $r_p = 0$. Finally , the real fifth root z of x is

$$z = \sum_{s=0}^{p} 10^{q-s} y_s.$$

The real seventh root of a positive number

Let $x = \sum_{n=0}^{\infty} 10^{N-n} a_{N-n}$ be a positive real number, where $N \in \mathbb{Z}$, $a_{N-n} \in \{0,1,...,9\}$ for all n and $a_N \neq 0$. By the division algorithm there exist unique integers q and r such that $N = 7q + r$ and $0 \leq r < 7$, then

$$x = \sum_{k=0}^{r} 10^{7q+k} a_{7q+k} + \sum_{i=1}^{\infty} \sum_{j=0}^{6} 10^{7(q-i)+j} a_{7(q-i)+j}.$$

Let p be a non negative integer and $g_0, ..., g_p$ be the integers

$$g_0 = \sum_{k=0}^{r} 10^k a_{7q+k},$$

and

$$g_s = \sum_{k=0}^{6} 10^k a_{7(q-s)+k},$$

for all $0 < s \leq p$.

At the first step, find

$$y_0 = \max\{y \in \{0,1,...,9\} : y^7 \leq g_0\},$$

and write

$$r_0 = g_0 - y_0^7$$

and

$$d_0 = 10^7 r_0 + g_1.$$

After find

$$y_1 = \max\{y \in \{0, 1, ..., 9\} : \\ y(y^6 + 70(100(10y_0 + y)^3 y_0^3 + 100(10y_0 + y)y_0^3 y^2 + (10y_0 + y)^3 y_0 y^2)) \leq d_0\},$$

and write

$$r_1 = \\ d_0 - y_1(y_1^6 + 70(100(10y_0 + y_1)^3 y_0^3 + 100(10y_0 + y_1)y_0^3 y_1^2 + (10y_0 + y_1)^3 y_0 y_1^2))$$

and

$$d_1 = 10^7 r_1 + g_2.$$

At the s-th step, find

$$y_s = \max\{y \in \{0, 1, ..., 9\} : y(y^6 + 70(100(\sum_{t=0}^{s-1} 10^{s-t} y_t + y)^3 (\sum_{t=0}^{s-1} 10^{s-1-t} y_t)^3 + \\ 100(\sum_{t=0}^{s-1} 10^{s-t} y_t + y)(\sum_{t=0}^{s-1} 10^{s-1-t} y_t)^3 y^2 + (\sum_{t=0}^{s-1} 10^{s-t} y_t + y)^3 (\sum_{t=0}^{s-1} 10^{s-1-t} y_t) y^2)) \leq \\ d_{s-1}\},$$

and write

$$r_s = d_{s-1} - y_s(y_s^6 + 70(100(\sum_{t=0}^{s} 10^{s-t} y_t)^3 (\sum_{t=0}^{s-1} 10^{s-1-t} y_t)^3 + \\ 100(\sum_{t=0}^{s} 10^{s-t} y_t)(\sum_{t=0}^{s-1} 10^{s-1-t} y_t)^3 y_s^2 + (\sum_{t=0}^{s} 10^{s-t} y_t)^3 (\sum_{t=0}^{s-1} 10^{s-1-t} y_t) y_s^2))$$

and

$$d_s = 10^7 r_s + g_{s+1}.$$

If the decimal expansion of the real seventh root is finite, increase (if necessary) p until $r_p = 0$. Finally , the real seventh root z of x is

$$z = \sum_{s=0}^{p} 10^{q-s} y_s.$$

Time complexity of the four algorithms

The algorithms are of polynomial time complexity because for an input number (in decimal notation) with length $n = \eta p + r + 1$ with $\eta = 3, 4, 5, 7$, after writing the numbers g_i in time $O(p)$, in each of the following $p + 1$ steps it compares at most 10 pairs of numbers calculated in time $O(p^2)$ and it writes the two numbers r_i and d_i in time $O(p^2)$. Therefore, since $O(p^3) = O(n^3)$, the time complexity of the algorithm is $T(n) = O(n^3)$.