Integrability of Maxwell's Equations, Part II

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To my daughter

Abstract. In this article I pick up with [4] as well as [3] and show that the mathematical relations of quantum mechanics derive from classical electrodynamics, albeit without the use of the principle of indeterminacy.

1. Integrability Revisited

In [4] I showed that $\mathbb{R}^4$ decomposes into five different sets, the light cone itself, which is a closed hypersurface, and four open regions, namely: the forward and backward time-like regions, and the space-like regions of positive and negative parity. Further, it has been shown Maxwell's equations in Lorentz gauge can be (locally) integrated in each of the four open regions. However, whereas the time-like regions are convex sets and therefore simply connected, the two space-like regions are not simply connected, and on these regions the integration along a closed path are no more guaranteed to yield a zero result. Therefore, whereas the integral in the time-like regions is unique up to an additive constant, on the space-like regions this is not so: The obstacle are closed paths in the space-like regions around the origin, which does belong to the light cone, and which therefore cannot be contracted to a single point within these regions.

Still, we can integrate, both sources and the electromagnetic field $A = (A^0, \cdots, A^3)$: Let $\Omega \cup \{0\} \subset \mathbb{R}^4$ be one of these space-like regions including the origin, and let $\gamma : [0,1] \to \Omega$ be a continuous closed path around the origin, i.e.: $\gamma(0) = \gamma(1)$, and let $j = (j^0, \cdots, j^3)$ be the 4-tuple of charge density and flux which is assumed to be a continuously differentiable function of space and time as in [4], for which charge conservation holds:

$$\partial j^0/\partial x^0 + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0.$$

Let $\int j d\gamma = k_0 \in \mathbb{R} \subset \mathbb{C}$, and let k_0 be unequal zero. Now, note that this integral reverses sign upon parity inversion, so this functional is a pseudovector, which we can write by means of the exterial (or Grassmann algebra) as

$k_0\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3$ where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ denote the spatial unit vectors at the origin (see: [1]). The result then is that on Ω the the integral of $j = (j^0, \cdots, j^3)$ is given by

$$\Phi(x) = \int_a^x j d\gamma + k_0\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 + \lambda,$$

where $a \in \Omega\backslash\{0\}$, and $k_0, \lambda \in \mathbb{C}$ are arbitrary constants, and the similar result holds for the integration of the vector field $A = (A_0, \cdots, A_3)$ on Ω. In other words, $k_0\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3$ is an arbitrary constant of integration, which can be chosen to be zero.

Now, let me show that the discussion around the term $k_0\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3$ just arises, because of the unconvenient restriction of space-time to be real: The Minkowski metrics leads to algebraic equations which mathematically always are discussed within $\mathbb{C}$ rather than $\mathbb{R}$, because these equations can be always solved in $\mathbb{C}$, but not always in $\mathbb{R}$. So let's embed $\mathbb{R}^4$ into $\mathbb{C}^4$. Then we can diffeomorphically displace time on the space-like region with positive parity by $i\epsilon$ with $\epsilon > 0$, and on the negative parity space-like region by $-i\epsilon$. So, in both regions, closed loops around the origin diffeomorhically contract to points, $(\pm i\epsilon, 0, 0, 0)$, where the determinant of the Lorentz metrics is strictly unequal zero, in other words: integration along these loops gives zero! In other words, in the complex, both positive and negative pariity space-like regions are simply connected: In there it is the union of the two space-like regions, which is no more simply connected.

So, even in the complex, crossing from positive to negative parity space-like regions is not possible (due to the lack of injectivity of the Lorentz metrics). That means that the Lorentz metric itself enforces chiral symmetry breaking. In other words, as long as the Lorentz metrics holds, chriral symmetry is explicitly broken.

2. Gauge Invariance Revisited

Leaving out the constants of integration, the Maxwell equations in the Lorentz gauge can be integrated on each of the four regions above, yielding a scalar wave equation $\Box F(x) = \Phi(x)$, where $F(x) = \int_a^x A d\gamma$, $\Phi(x) = \int_b^x j d\gamma$, $a, b \in \mathbb{R}^4$ are elements of either region, and $A = (A^0, \cdots, A^3)$ and $j = (j^0, \cdots, j^3)$ are the 4-tuples of electromagnetic field and charge density flux, respectively (see: [4]). As discussed in [4], the problem is that this scalar equation must be relativistically invariant, and if so, then $(A^0, \cdots, A^3)$ and $j = (j^0, \cdots, j^3)$ do no more transform as Minkowski 4-vectors. In order to become 4-vectors, we need to add additional energy from one Lorentz boost to the other, and then A and j will be Minkowski 4-vectors, however both sides of the equation $\Box F(x) = \Phi(x)$ will change from boost to boost. Formally, on the left hand side, this is resolved by basing the Maxwell equations not on A, but on $(\nabla A^0, \boldsymbol{\nabla} \times \boldsymbol{A})$. But look: The right hand side faces the same problem: We'd have to add curls to j^0 and divergences of $\boldsymbol{j}$ and base calculation on∇j^0 and $\nabla \times \boldsymbol{j}$ in order to get both sides of the equations straight. So, what does that mean?

There are two answers to this question: a technical and a physical one: Technically, it means that $A = (A^0, \cdots, A^3)$ and $j(x) = (j^0, \cdots, j^3(x)$ are being added differential 4-forms $f(x)(\boldsymbol{e}_0 \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3)(x)$, where f is a continuously differentiable function (in either space-like/timelike region and $\boldsymbol{e}_0 \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3($ denotes the exterior product of the 4 space and time unit vectors at $x \in \mathbb{R}^4$, whereas $(A^1, \cdots, A^3)$. Since $d(f(x)\boldsymbol{e}_0 \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3) = 0$, the 4-form is integrable. So, $A(x)$ and $j(x)$ are added the tuples $(\int_a^x f(x)dx_0\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3, \cdots, \int_a^x f(x)dx_3\boldsymbol{e}_0 \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2)$.
Physically, the addition of $f(x)\boldsymbol{e}_0 \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3$ to j and A means adding (signed) energy, and, since mass is equivalent to energy, it means adding (signed) mass. Taken verbatim, a charge that accelerates would curl up charges of either sign, and that was to be equivalent to a mass gain. So there would be inertia with the charges themselves, and through this, the electromagnetic field could transmit inertia, thus also ensuring that A and j become relativistic 4-vectors.

The fact that we are allowed to add any mass of either sign and the fact that electromagnetic energy always is taken to be the square root of its absolute square, signals that some important restrictions are still missing. Obviously, we want any good scalar equation of electromagnetic fields to be relativistically invariant in all its involved quantities. According to [3], we can get at it the following way:

Because j^μ are scalar, j can also be integrated w.r.t. the differential form $ds = \gamma^0 dx^0 + \cdots + \gamma^3 dx^3$, where $\gamma^0, \cdots, \gamma^3$ are the Dirac matrices (see [3]). With this, $\int_a^x j d\gamma$ becomes a 4×4 matrix $\mathcal{S}j$, and its differential yields j again. We then suppose that the j^μ, apart from being smooth, have a compact support within Ω, where Ω denotes either of the four regions (forward/backward time-like and positive/negative parity space-like regions as above). (Generally, the j^μ always are locally limits of a sequence of such functions $j_k^\mu, (k \in \mathbb{N})$, and a relation that holds locally for the j_k^μ will also hold for the j^μ.) We then have by partial integration:

$$\int_\Omega \overline{j^\mu(x)} j_\mu(x) d^4x = \int_\Omega |\rho^2(x)| - \|\rho\boldsymbol{v}(x)\|^2 = \rho_0^2(x) d^4x = \int_\Omega \overline{\Phi(x)} \Box \Phi(x) d^4x,$$

where $\Phi = \int_a^x j d\gamma = \int_\gamma j^0 \gamma^0 (dx(s)^0/ds = ds + \cdots + j^3\gamma^3(dx^3(dx(s)/ds)ds$, so that the electromagnetic potential A^μ becomes a quantity that derives from the interaction of two different fluxes:

Let χ^μ be another smooth charge flux with compact support in Ω, and let $\Psi(x) = \int_a^x \chi d\gamma$. Then $\int_\Omega \overline{\Psi(x)}\Phi(x) d^4x = \int_\Omega \chi^\mu(x) A_\mu(x) d^4x$, i.e.:

$$A^\mu(x) = \mathcal{S}^2 j^\mu, \qquad (0 \le \mu \le 3),$$

where $\mathcal{S}$ is the operator

$$\mathcal{S} : j \to \int_a^x j d\gamma = \int_a^x (j^0(t)\gamma^0 dx_0(s) + \cdots j^3(t)\gamma^3 dx_3(s))ds.$$

The integrability of the electric fluxes now also allows to base the calculus on the action integrals Φ instead of the fluxes j themselves:

We can extract the unit (of action) from $\Phi = \int_a^x j d\gamma$, which makes Φ dimensionless, and give the differential operators the dimension of action instead, defining $P^0 = E := i\hbar\partial^0$ and $P^k := -i\hbar\partial^k, \quad (1 \leq k \leq 3)$ to be the energy and momentum operators respectively, acting on Φ.

This directly equates the flux j to the energy momentum density $(P^0\Phi, \cdots, P^3\Phi)$, which now is a 4-vector, and it equates $j_0^2 - \|\boldsymbol{j}\|$ to be the square of a rest mass density:

$$\int \overline{\Phi(x)}(P_0^2 - \|\boldsymbol{P}\|^2)\Phi(x)d^4x = \int (j_0^2(x) - \|\boldsymbol{j}(x)\|^2)d^4x.$$

In particular, it is seen that within classical electrodynamics, there are two conserved quantities at work: the net charge and the the absolute value of the charges, which is the energy or mass of the charged particles.

Also, note that basing dynamical calculus on the action states Φ instead of the fluxes j runs up to shift the concern from the particles to their (action) fields which goes with no loss of information: dynamical invariants of the particles become generalized eigenvectors of the energy momentum operators and vice versa. I illustrate that in the following:

3. The Free Field

Let $q(t) = (q^0, q^0dx^1/dt, q^0dx^1/dt, q^0dx^3/dt)$ be a charged particle moving freely in space time. Then its action function Φ_{free} - omitting the Dirac matrices $\gamma^0, \cdots, \gamma^3$ - is given by

$$\Phi_{free}(x) = a + q^0x^0 + q^0v^1x^1 + \cdots + q^0v^3x^3,$$

where $a = (a_0, \cdots, a_3)$, $(q^0x^0, q^0v^1x^1, \cdots, q^0v^3x^3) \in \mathbb{R}^4$, are constants of motion. In here, the distributions

$$q^0E\delta(y^0 - q^0x^0)\delta(y^1)\delta(y^2)\delta(y^3), \cdots\ q^0v^3\delta(y^0)\delta(y^1)\delta(y^2)\delta(y^3 - q^0v^3x^3)$$

are the generalized eigenvectors for the energy momentum operators $P^0, \cdots, P^3$, and $q^0, q^0v^1, \cdots, q^0v^3$ their eigenvalues.

(Note: Through this there is a slight shift of paradigm: Instead of Φ_{free} representing a system with fixed and constant combinations of distinct values $q^0v^0, \cdots, q^0v^3$ of a point charge only, by passing over to generalized eigenvectors, one allows Φ to be the action of a continuous flux of charges,in which the singular points are represented in terms of the Dirac distributions $\delta(x^0, \boldsymbol{x})$.)

With $P = (i\hbar\partial^0, -i\hbar\partial^1, \cdots - i\hbar\partial 3)$ as above, Φ obeys in each of the four space-like and time-like regions Ω of definitions:

$$\int_a^x P\Phi \cdot d\gamma = 0$$

for all smooth paths γ in Ω connecting $a, x \in \Omega$, where

$$x \cdot y := x^0 y^0 - x^1 y^1 - \cdots - x^3 y^3.$$

This allows us to define the free field to be given by an action function Φ for which $\int_a^x P\Phi \cdot d\gamma = 0$ holds (in each of the four regions Ω of definition). Now, that holds if and only if Φ is the Fourier inverse of a distribution $\hat{\Phi}$ which vanishes outside the light cone, i.e.: $supp\ \hat{\Phi} \subset \Gamma\mathcal{C}$, or, equivalently: $\Box\Phi = 0$ (see: [3]).

In the non-relativistic limit, the space-like regions drop out, and on the time-like ones, the wave operator is positive, so that $\Box^{1/2}$ is a well-defined selfadjoint operator within the timelike regions. Therefore, as $c \to \infty$, the equation converges to $i\hbar\partial^0\Phi = -\frac{\hbar^2}{2|q^0|}\nabla^2\Phi$, which is the free Schrödinger equation for the mass $m = |q^0|$.

4. External Electromagnetic Field

The dynamics of charged particles in an external electromagnetic field comes half way between free theory and the interaction of two electromagnetic particles: In it, the interacting particle is disregarded, and only the interaction of its field with the particle is considered.

This means that the action $F\Phi := \int_a^x A\Phi \cdot d\gamma$ of the external electromagnetic field $A = (A^0, \cdots, A^3)$ adds to the the action Φ of the free theory, replacing $\int_a^x P\Phi \cdot d\gamma = 0$ by

$$\int_a^x P\Phi \cdot d\gamma = \int_a^x A\Phi \cdot d\gamma.$$

So, the we get the equation of motion of charged psrticles in the external electromagnetic field by the replacement of P by $P - A$, hence the nonrelativistic limit of that is given by

$$i\hbar(\partial^0 - A^0)\Phi = -\frac{\hbar^2}{2|q^0|}(\nabla - \boldsymbol{A})^2\Phi.$$

This is the Schrödinger equation of an electrical particle state Φ in an external electromagnetic field (see: [2], Vol. III, Sec. 21-2).

5. Wrapping Up

So, we derived Schrödinger's equation (of a particle within an external electromagnetic field) from classical electrodynamics, an equation that cannot be derived from within quantum mechanics itself, and therefore had to be postulated. Along with the derivation, most, if not all results of non-relativistic quantum mechanics are now accessible from within classical electrodynamics, including black body radiation and the spectral values of the Hydrogen atom.

One might ask, how the Planck constant $\hbar$ comes in to the electrodynamics: Citing [4], within the Maxwell equations, we have two factors of electronic charge in the terms of either sign, and taking the time derivative gives an additional factor $1/c$, where c is the speed of light (which in our units is set equal to 1). The resulting constant q_e^2/c has the dimension of action, so it's got to be proportional to $\hbar$, which has the same dimension. The factor of proportionality is the dimensionless fine-structure constant. The point now is that q_e^2/c is in electrical units, whereas $\hbar$ is in mechanical units in terms of mass m[kg]. As was seen above, the absolute value of charge appears to be equivalent to mass which would make the electron's charge a good candidate for the definition of mass, either. But that was not known then, when Planck discovered $\hbar$ from studying the black body radiation. So again, $\hbar$ can be explained without the uncertainty principle.

References

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