Disposing Classical Field Theory, Part II

Hans Detlef Hüttenbach

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Abstract

It is shown that the symmetry group of a neutral, energy and momentum conserving particle theory is isomorphic to $SU(3) \times SU(2) \times U(1)$.

1 Introduction

In its covariant form, Maxwell's equations read

$$\Box A^\mu = j^\mu, (0 \le \mu \le 3),$$

where the A^μ, j^μ are functions of time $t = x^0$ and space coordinates x^1, x^2, x^3, and $c \equiv 1$ is understood. Furthermore, $A = (A^0, \cdots, A^3)$ is a 4-vector, if not subjected to a gauge (see below). Now, $\Box$ is a relativistic invariant, so $\Box A$ is a 4-vector, hence j is a 4-vector. Then $|j|^2 = j^* j = \bar{j}^\mu j_\mu = |j_0|^2 - |j_1|^2 - \cdots |j_3|^2$ is Lorentz invariant, and its square root $j = j^0\gamma^0 + \cdots + j^3\gamma^3$ in terms of Dirac matrices $\gamma^\mu, (0 \le \mu \le 3)$, is a Lorentz invariant (and unique up to unitary transformation for each point $x = (t, \mathbf{x})$ in space-time). The representation of the Dirac matrices are given in the Dirac basis throughout:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \text{ and } \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

In particular, let $\rho_0(t, \mathbf{x})$ be a stationary charge density of electrons. Boosting that to a speed $\mathbf{v}$, i.e. under a Lorentz transfomation $\Lambda : (t, \mathbf{x}) \mapsto (t', \mathbf{x}')$ to a system which moves with speed $-\mathbf{v}$ relative to the first, ρ_0 transforms to $\rho\gamma^0 + \rho\gamma^1 v^1 + \cdots + \rho\gamma^3 v^3$, and, by Lorentz invariance of ρ^2, we can choose the unitary representation such that $\rho_0 = \rho\gamma^0 + \rho\gamma^1 v^1 \cdots \rho\gamma^3 v^3$. Now I notice that the flux $\rho(\gamma \cdot \mathbf{v})$ is the source of the magnetic fields, and as such it is both, neutral and unequal zero.

Solving the above equation for ρ then gives ρ in terms of ρ_0:

$$\rho = \rho_0 \big(\gamma_0 + \gamma \cdot \mathbf{v}\big)^{-1}.$$

The charge therefore is not a relativistic invariant: it changes under boosts by the addition of some non-trivial, neutral matter, depending on the boost velocity. Still, a gauge transformation of the fields manages to eliminate this additive neutral matter in agreement with Maxwell's equations. It is easy to see why: Electrodynamics adds positive and negative charges up, so that only net charges and net charge fluxes contribute. The original Maxwell equations (in terms of elecrical and magnetic fields) therefore are invariant to the addition of energy from matter with zero net charge and zero net flux. That is alright for non-relativistic considerations. But with special relativity, not the charge, but the square of the charge enters the calculations first place: relativity asks for energy content, for which the subtraction of arbitrary neutral energy is not a symmetry.

That does not mean that electrodynamics was no gauge theory: the contrary is true: the above subtraction of mass is still a (local) gauge, namely a boost of $-\mathbf{v}$, but it is not a gauge invariance, when we take energy into account; not the scalar charge is Lorentz invariant in an inertial system, it's the rest charge that is.

All in all, we've seen that charge is a part of a 4×4 tensor q, and its geometric mean, $|q| := (q^2)^{1/2} := (q^*q)^{1/2}$ transforms like inert mass. Therefore it is mass, so charges are inert, and so they have a weight. Hence, only by refraining from the postulate that positive and negative charges add as positive and negative numbers, it is possible to reconcile mechanics with electrodynamics and even to include mechanics as a part of electrodynamics. That was grossly what [1] was about. However, in that article, I left an unanswered remark:

2 Problem Statement

By mapping the scalar charge and flux into the Clifford algebra $Cl(1,3)$, the equations in there are only unique up to unitary transformations $\omega \in U(4)$, $U(4)$ being the group of all unitary 4×4-matrices. That makes $U(4)$ the overall symmetry group for that theory. I'll show in a moment that the mass/charge tensor can be split into the sum of a pure charge and a neutral part and that the group of tranformations of these charges is isomorhic to $U(2)$. That decomposes $U(4)$ into the product of two subgroups: one for the charge transformations, the other one for transformations which are completely charge invariant.

Then, if there was only the constraint that matter was completely built out of positive and negative charges, only a fraction of $U(4)$ was needed. So, the problem is to figure out, how well the standard model $SU(3) \times SU(2) \times U(1)$ fits into $U(4)$.

3 Details

First, let's factor the charge tensor out: We need to find a charge inversion operator, $\mathcal{C}$, say, which then will be unique modulo $U(4)$. Writing 1_2 for the 2×2 unit matrix, $\mathcal{C} := -\gamma^1\gamma^2\gamma^3 = \left(\begin{smallmatrix} 0 & 1_2 \\ 1_2 & 0 \end{smallmatrix}\right)$ does that job, since it anticommutes with γ^0 and commutes with the γ^j. That allows splitting an arbitrary charge/mass tensor q into its net charge $(1/2)\big(q - \mathcal{C}q\mathcal{C}\big)$ which I'll call pure charge, and its neutral constituent $(1/2)\big(q + \mathcal{C}q\mathcal{C}\big)$.

Now, because charge is being conserved, it must be conserved under Lorentz transformations, and the unitary group of charge transformations is to be a normal subroup G of $U(4)$, hence $U(4) = (U(4)/G) \times G$.

Let's figure out what that is: $d^2s := dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ is the Lorentz invariant square of a differential. So, $D := \gamma^0 dx^0 + \cdots + \gamma^3 dx^3$ is the Lorentz invariant differential. Given $j = (j^0, \cdots, j^3)$, charge conservation then means that infinitesimally small, global Lorentz transformations must not change that quantity. That is:

$$Dj = \partial j^0/(\partial\gamma^0 x^0)\gamma^0 dx^0 + \cdots + \partial j^3/(\partial\gamma^3 x^3)\gamma^3 dx^3 = 0,$$

which gives

$$\partial j^0/\partial x^0 + \cdots + \partial j^3/\partial x^3 = 0.$$

This is two things at once: it is the standard definition of charge conservation, and it is the Lorentz gauge condition.

Taking into regard that the γ^μ all are invariant w.r.t. transformation of the of the first and second as well as third and fourth component of $\mathbb{C}^4$, the group of unitary charge transformations therefore is isomorphic to $U(2)$, and it is not just $SU(2)$, because it also contains charge inversion. And that group finally factors into a complex and unitary combination of two isomorphic groups: a group $SU(2)$ of pure unit charges, corresponding to a second group $SU(2)$ of neutral unit charges/masses: $\chi = \cos(\phi)q_c + i\sin(\phi)q_n)$, where $0 \le \phi \le 2\pi$ and $q_n \in SU(2)$ is the neutral counterpart of the charged unit charge $q_c \in SU(2)$.

Whereas in quantum theory the group $SU(2)$ is associated with the notion of spin, in here it is of geometric origin and necessary for charge conservation.

Wrapping up, if electrodynamics covering charged and neutral states was the only theory we had, then on the physical side, all matter will be constructable from electrons and positrons, which we know to be untrue, and on the side of mathematics all those charged and uncharged states will be contained in $U(2)$. That would leave the group $U(4)/U(2)$ of charge invariant symmetries open to be filled up by something else.

So, the straightforward question is: can the symmetry group of the neutrons fill up that gap?

Apart from charge, also have energy and momentum are conserved. So, let's playing the same trick on energy and momentum, beginning with energy: $\mathcal{T} := \gamma^5 := \gamma^0 \cdots \gamma^3$ is the energy-inversion on $U(4)/U(2)$. With γ^0 being the representative of positive unit energy, $-\gamma^0$ is its inverse, the line through γ^0 and its inverse forms a two dimensional vector space with the time axis, and the group of all energy transformations unitarily operates on this vector space, so that this group is isomorphic to $U(2)$. And again this group is isomorphic to $SU(2) \times U(1)$. $SU(2)$ is associated with isospin, and $U(1)$ defines the hypercharge.

For momentum/charge flux, however, replaying the trick fails: The momentum $(p^1, \cdots, p^3)$ maps into $Cl(1,3)$ as $p^1\gamma^0 + \cdots p^3\gamma^3$, where $\mathcal{P} := \gamma^0$ is the sought parity inversion. But $\gamma^0 = \gamma^5\gamma^1\gamma^2\gamma^3 = -\mathcal{TC}$. So, $\mathcal{P} mod(U(2) \times U(2))$ just equals the identity in $U(4)/(U(2) \times U(2))$, and only $SU(3)$ remains as symmetry group, which is isomorphic to $U(4)/(U(2) \times U(2))$ as is seen by counting their dimensions. What at first sight looks to be a break of parity, turns out to be complete symmetry with negative parity being captured by the other two symmetry subgroups.

Wrapping up, the symmetry group for a neutral particle theory which which is conserving energy and momentum in $U(4)$ is exactly the standard model group $SU(3) \times SU(2) \times U(1)$, where $SU(3)$ captures the quark colour states, relating to momentum conservation, $SU(2)$ relates to energy conservation as isospin, and $U(1)$ captures hypercharge. So, as a gauge theory, standard model fits into the electromagnetic equations, which in particular would give particles in that model an inert and gravitational mass.

References

[1] Huettenbach, H. D. *Disposing Classical Field Theory.* http://vixra.org/pdf/1209.0027v1.pdf, 2012.