The accelerated frame in the curved time-space in the general relativity theory

Sangwha-Yi

Department of Math , Taejon University 300-716

ABSTRACT

In the general relativity theory, defines the accelerated frame that moves in $\hat{r}$-axis in the curved time-space. And calculates the curvature tensor of the accelerated frame in the curved time-space.

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The curved time-space,

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e-mail address:sangwha1**@nate.com**

Tel:051-624-3953

I.Introduction

This theory's object is that defines the accelerated frame that moves in $\hat{r}$ -axis in the curved time-space.

The Schwarzschild solution is

$$d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}[\frac{dr^2}{1-\frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2] \quad (1)$$

In this time, a moving matter's acceleration is a in the Schwarzschild time-space.

$$a = \frac{d}{dt}(\frac{u}{\sqrt{1-\frac{2GM}{rc^2}-\frac{u^2}{c^2}}}), u = \frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}\frac{dr}{dt} \quad (2)$$

If $a_0 = a/\sqrt{1-\frac{2GM}{rc^2}}$ is,

$$a_0 = \frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}\frac{d}{dt}(\frac{u}{\sqrt{1-\frac{2GM}{rc^2}-\frac{u^2}{c^2}}}) = \frac{d}{d\hat{t}}(\frac{V}{\sqrt{1-\frac{V^2}{c^2}}}),$$

$$V = \frac{d\hat{r}}{d\hat{t}} = \frac{dr}{dt}\frac{1}{(1-\frac{2GM}{rc^2})}, \quad d\hat{t} = dt\sqrt{1-\frac{2GM}{rc^2}}, \quad d\hat{r} = \frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}}$$

$$a_0\hat{t} = \frac{V}{\sqrt{1-\frac{V^2}{c^2}}}, \quad V = \frac{a_0\hat{t}}{\sqrt{1+\frac{a_0^2\hat{t}^2}{c^2}}},$$ V is the $\hat{r}$ -axis's velocity (3)

If $\frac{d\theta}{dt} = \frac{d\phi}{dt} = 0$, the solution is

$$d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}\frac{dr^2}{1-\frac{2GM}{rc^2}} = d\hat{t}^2 - \frac{1}{c^2}d\hat{r}^2 = d\hat{t}^2(1-\frac{V^2}{c^2})$$

$$= \frac{d\hat{t}^2}{\sqrt{1+\frac{a_0^2\hat{t}^2}{c^2}}} \quad (4)$$

In this time,

$$\tau = \int d\tau = \int \frac{d\hat{t}}{\sqrt{1+\frac{a_0{}^2\hat{t}^2}{c^2}}} = \frac{c}{a_0}\sinh^{-1}(\frac{a_0}{c}\hat{t}),$$

$$\hat{t} = \frac{c}{a_0}\sinh(\frac{a_0\tau}{c}), \quad \hat{r} = \int V d\hat{t} = \int \frac{a_0\hat{t}d\hat{t}}{\sqrt{1+\frac{a_0{}^2\hat{t}^2}{c^2}}} = \frac{c^2}{a_0}\sqrt{1+\frac{a_0{}^2\hat{t}^2}{c^2}}$$

$$= \frac{c^2}{a_0}\cosh(\frac{a_0\tau}{c})$$

$$\frac{d\hat{t}}{d\tau} = \cosh(\frac{a_0}{c}\tau), \quad \frac{1}{c}\frac{d\hat{r}}{d\tau} = \sinh(\frac{a_0}{c}\tau) \tag{5}$$

II. The tetrad in the curved time-space

The tetrad $e^{\hat{\alpha}}{}_{\hat{\mu}}$ is the unit vector defined by the following formula.

$$\eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}{}_{\hat{\mu}} e^{\hat{\beta}}{}_{\hat{\upsilon}} = g_{\hat{\mu}\hat{\upsilon}} \tag{6}$$

In this time, if a matter moves in $\hat{r}$ -axis in the curved time-space,

$$\eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}{}_{\hat{\mu}}(\tau) e^{\hat{\beta}}{}_{\hat{\upsilon}}(\tau) = g_{\hat{\mu}\hat{\upsilon}} = \eta_{\hat{\mu}\hat{\upsilon}}, \quad g_{\hat{\mu}\hat{\upsilon}} = \eta_{\hat{\mu}\hat{\upsilon}} \tag{7}$$

Hence, Eq(6),Eq(7) is

$$\eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}{}_{\hat{0}}(\tau) e^{\hat{\beta}}{}_{\hat{0}}(\tau) = \eta_{\hat{0}\hat{0}} = -1 \tag{8}$$

$$d\tau^2 = -\frac{1}{c^2}\eta_{\hat{\alpha}\hat{\beta}} d\hat{x}^{\alpha} d\hat{x}^{\beta}$$

$$\rightarrow \; -1 = \eta_{\hat{\alpha}\hat{\beta}}(\frac{1}{c}\frac{d\hat{x}^{\alpha}}{d\tau})(\frac{1}{c}\frac{d\hat{x}^{\beta}}{d\tau}) = \eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}{}_{\hat{0}}(\tau) e^{\hat{\beta}}{}_{\hat{0}}(\tau) \tag{9}$$

According to Eq(5),Eq(9)

$$e^{\hat{\alpha}}{}_{\hat{0}}(\tau) = \frac{1}{c}\frac{d\hat{x}^{\alpha}}{d\tau} = (\cosh(\frac{a_0\tau}{c}), \sinh(\frac{a_0\tau}{c}), 0, 0) \tag{10}$$

About $\hat{x}^2$ -axis's and $\hat{x}^3$ -axis's orientation

$$\eta_{\hat{2}\hat{2}} e^{\hat{2}}{}_{\hat{2}}(\tau) e^{\hat{2}}{}_{\hat{2}}(\tau) = \eta_{\hat{2}\hat{2}} = 1, \qquad e^{\hat{\alpha}}{}_{\hat{2}}(\tau) = (0,0,1,0)$$

$$\eta_{\hat{3}\hat{3}} e^{\hat{3}}{}_{\hat{3}}(\tau) e^{\hat{3}}{}_{\hat{3}}(\tau) = \eta_{\hat{3}\hat{3}} = 1, \quad e^{\hat{\alpha}}{}_{\hat{3}}(\tau) = (0,0,0,1) \tag{11}$$

And the other vector $e^{\hat{\alpha}}{}_{\hat{1}}(\tau)$ has to satisfy the tetrad condition, Eq (6),Eq(7)

$$e^{\hat{\alpha}}{}_{\hat{1}}(\tau) = (\sinh(\frac{a_0\tau}{c}), \cosh(\frac{a_0\tau}{c}), 0, 0) \qquad (12)$$

In this time,

$$\bar{e}_{\hat{t}}{}^{\rho} = (1/\sqrt{1-\frac{2GM}{rc^2}}, 0, 0, 0), \bar{e}_{\hat{r}}{}^{\rho} = (0, \sqrt{1-\frac{2GM}{rc^2}}, 0, 0)$$

$$\bar{e}_{\hat{\theta}}{}^{\rho} = (0, 0, 1/r, 0) \quad , \quad \bar{e}_{\hat{\phi}}{}^{\rho} = (0, 0, 0, 1/r\sin\theta)$$

$$g_{\rho\sigma}\bar{e}_{\hat{\alpha}}{}^{\rho}\bar{e}_{\hat{\beta}}{}^{\sigma} = \eta_{\hat{\alpha}\hat{\beta}} \qquad (13)$$

$$\frac{a_0}{c}\hat{t} = \sinh(\frac{a_0}{c}\tau) = \frac{v/c}{\sqrt{1-v^2/c^2}}, \sqrt{1+\frac{a_0{}^2\hat{t}^2}{c^2}} = \cosh(\frac{a_0}{c}\tau) = \frac{1}{\sqrt{1-v^2/c^2}} \quad (14)$$

Therefore, the Lorentz transformation $B^{\hat{\alpha}}{}_{\hat{\mu}}(v)$ is

$$B^{\hat{\alpha}}{}_{\hat{\mu}}(v) = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{v/c}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ \frac{v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= e^{\hat{\alpha}}{}_{\hat{\mu}}(\tau) = \begin{pmatrix} \cosh(\frac{a_0}{c}\tau) & \sinh(\frac{a_0}{c}\tau) & 0 & 0 \\ \sinh(\frac{a_0}{c}\tau) & \cosh(\frac{a_0}{c}\tau) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (15)$$

$$\bar{e}'_{\hat{\mu}}{}^{\rho} = B^{\hat{\alpha}}{}_{\hat{\mu}}(v)\bar{e}_{\hat{\alpha}}{}^{\rho} = e^{\hat{\alpha}}{}_{\hat{\mu}}(\tau)\bar{e}_{\hat{\alpha}}{}^{\rho} \qquad (16)$$

Hence,

$$g_{\rho\sigma}\bar{e}_{\hat{\alpha}}{}^{\rho}\bar{e}_{\hat{\beta}}{}^{\sigma} = \eta_{\hat{\alpha}\hat{\beta}}$$

$$g_{\rho\sigma}B^{\hat{\alpha}}{}_{\hat{\mu}}(v)\bar{e}_{\hat{\alpha}}{}^{\rho}B^{\hat{\beta}}{}_{\hat{\upsilon}}(v)\bar{e}_{\hat{\beta}}{}^{\sigma} = g_{\rho\sigma}\bar{e}'_{\hat{\mu}}{}^{\rho}\,\bar{e}'_{\hat{\upsilon}}{}^{\sigma} = \eta_{\hat{\alpha}\hat{\beta}}e^{\hat{\alpha}}{}_{\hat{\mu}}(\tau)e^{\hat{\beta}}{}_{\hat{\upsilon}}(\tau) = \eta_{\hat{\mu}\hat{\upsilon}} \qquad (17)$$

III. The accelerated frame in the curved time-space

About the accelerated frame $\hat{\xi}$ in the curved time-space,

$$d\tau^2 = -\frac{1}{c^2}\eta_{\hat{\alpha}\hat{\beta}} d\hat{x}^{\alpha} d\hat{x}^{\beta} = -\frac{1}{c^2}\eta_{\hat{\alpha}\hat{\beta}} \frac{\partial \hat{x}^{\alpha}}{\partial \hat{\xi}^{\mu}} \frac{\partial \hat{x}^{\beta}}{\partial \hat{\xi}^{\upsilon}} d\hat{\xi}^{\mu} d\hat{\xi}^{\upsilon}$$

$$= -\frac{1}{c^2}\eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}{}_{\hat{\mu}} e^{\hat{\beta}}{}_{\hat{\upsilon}} d\hat{\xi}^{\mu} d\hat{\xi}^{\upsilon}$$

$$= -\frac{1}{c^2} g_{\hat{\mu}\hat{\upsilon}} d\hat{\xi}^{\mu} d\hat{\xi}^{\upsilon} \tag{18}$$

$$e^{\hat{\alpha}}{}_{\hat{\mu}} = \frac{\partial \hat{x}^{\alpha}}{\partial \hat{\xi}^{\mu}} \quad , \quad \frac{\partial e^{\hat{\alpha}}{}_{\hat{0}}}{\partial \hat{\xi}^{1}} = \frac{\partial^2 \hat{x}^{\alpha}}{c\partial \hat{\xi}^{0} \partial \hat{\xi}^{1}} = \frac{\partial e^{\hat{\alpha}}{}_{\hat{1}}}{c\partial \hat{\xi}^{0}} \tag{19}$$

In this time, in Eq(10),Eq(11),Eq(12), if uses $\hat{\xi}^0$ instead of τ ,

$$e^{\hat{\alpha}}{}_{\hat{0}}(\hat{\xi}^0) = \frac{1}{c}\frac{\partial \hat{x}^{\alpha}}{\partial \hat{\xi}^{0}} = ((1+\frac{a_0\hat{\xi}^1}{c^2})\cosh(\frac{a_0\hat{\xi}^0}{c}),(1+\frac{a_0\hat{\xi}^1}{c^2})\sinh(\frac{a_0\hat{\xi}^0}{c}),0,0) \tag{20}$$

$$e^{\hat{\alpha}}{}_{\hat{1}}(\hat{\xi}^0) = \frac{\partial \hat{x}^{\alpha}}{\partial \hat{\xi}^{1}} = (\sinh(\frac{a_0\hat{\xi}^0}{c}),\cosh(\frac{a_0\hat{\xi}^0}{c}),0,0) \tag{21}$$

$$e^{\hat{\alpha}}{}_{\hat{2}}(\hat{\xi}^0) = \frac{\partial \hat{x}^{\alpha}}{\partial \hat{\xi}^{2}} = (0,0,1,0), e^{\hat{\alpha}}{}_{\hat{3}}(\hat{\xi}^0) = \frac{\partial \hat{x}^{\alpha}}{\partial \hat{\xi}^{3}} = (0,0,1,0) \tag{22}$$

$$d\hat{x}^{\alpha} = \frac{\partial \hat{x}^{\alpha}}{\partial \hat{\xi}^{\mu}} d\hat{\xi}^{\mu} = e^{\hat{\alpha}}{}_{\hat{0}}(\hat{\xi}^0) c d\hat{\xi}^0 + e^{\hat{\alpha}}{}_{\hat{1}}(\hat{\xi}^0) d\hat{\xi}^1 + e^{\hat{\alpha}}{}_{\hat{2}}(\hat{\xi}^0) d\hat{\xi}^2 + e^{\hat{\alpha}}{}_{\hat{3}}(\hat{\xi}^0) d\hat{\xi}^3 \tag{23}$$

Hence,

$$cd\hat{t} = cdt\sqrt{1-\frac{2GM}{rc^2}} = (1+\frac{a_0\hat{\xi}^1}{c^2})\cosh(\frac{a_0\hat{\xi}^0}{c}) c d\hat{\xi}^0 + \sinh(\frac{a_0\hat{\xi}^0}{c}) d\hat{\xi}^1$$

$$d\hat{r} = \frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}} = (1+\frac{a_0\hat{\xi}^1}{c^2})\sinh(\frac{a_0\hat{\xi}^0}{c}) c d\hat{\xi}^0 + \cosh(\frac{a_0\hat{\xi}^0}{c}) d\hat{\xi}^1$$

$$d\hat{x}^2 = d\hat{\xi}^2 \quad , \quad d\hat{x}^3 = d\hat{\xi}^3 \tag{24}$$

$$d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2}[\frac{dr^2}{1-\frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2]$$

$$= d\hat{t}^2 - \frac{1}{c^2}[d\hat{r}^2 + (d\hat{x}^2)^2 + (d\hat{x}^3)^2]$$

$$= (1 + \frac{a_0\hat{\xi}^1}{c^2})^2 (d\hat{\xi}^0)^2 - \frac{1}{c^2}[(d\hat{\xi}^1)^2 + (d\hat{\xi}^2)^2 + (d\hat{\xi}^3)^2] \quad (25)$$

The coordinate transformation is

$$c\hat{t} = (\frac{c^2}{a_0} + \hat{\xi}^1)\sinh(\frac{a_0\hat{\xi}^0}{c}), \hat{r} = (\frac{c^2}{a_0} + \hat{\xi}^1)\cosh(\frac{a_0\hat{\xi}^0}{c}) - \frac{c^2}{a_0}$$

$$\hat{x}^2 = \hat{\xi}^2, \hat{x}^3 = \hat{\xi}^3 \quad (26)$$

The inverse-transformation is

$$\hat{\xi}^0 = \frac{c}{a_0}\tanh^{-1}(\frac{c\hat{t}}{\hat{r} + \frac{c^2}{a_0}}),\ \ \hat{\xi}^1 = \sqrt{(\hat{r} + \frac{c^2}{a_0})^2 - c^2\hat{t}^2} - \frac{c^2}{a_0}$$

$$\hat{\xi}^2 = \hat{x}^2, \hat{\xi}^3 = \hat{x}^3 \quad (27)$$

If calculates the curvature tensor $R_{\hat{\mu}\hat{\upsilon}\hat{\rho}\hat{\lambda}}(\hat{\xi})$,

$$R_{\hat{\mu}\hat{\upsilon}\hat{\rho}\hat{\lambda}}(\hat{\xi}) = \frac{\partial\hat{x}^\alpha}{\partial\hat{\xi}^\mu}\frac{\partial\hat{x}^\beta}{\partial\hat{\xi}^\upsilon}\frac{\partial\hat{x}^\gamma}{\partial\hat{\xi}^\rho}\frac{\partial\hat{x}^\delta}{\partial\hat{\xi}^\lambda} R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}(\hat{X})$$

$$= e^{\hat{\alpha}}{}_{\hat{\mu}}(\hat{\xi}^0) e^{\hat{\beta}}{}_{\hat{\upsilon}}(\hat{\xi}^0) e^{\hat{\gamma}}{}_{\hat{\rho}}(\hat{\xi}^0) e^{\hat{\delta}}{}_{\hat{\lambda}}(\hat{\xi}^0) R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}(\hat{X}) \quad (28)$$

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -R_{\hat{t}\hat{r}\hat{r}\hat{t}} = R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -R_{\hat{r}\hat{t}\hat{t}\hat{r}} = \frac{2GM}{r^3c^2},$$

$$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = -R_{\hat{t}\hat{\theta}\hat{\theta}\hat{t}} = R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = -R_{\hat{\theta}\hat{t}\hat{t}\hat{\theta}} = -\frac{GM}{r^3c^2} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = -R_{\hat{t}\hat{\phi}\hat{\phi}\hat{t}} = R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = -R_{\hat{\phi}\hat{t}\hat{t}\hat{\phi}}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = -R_{\hat{\theta}\hat{\phi}\hat{\phi}\hat{\theta}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = -R_{\hat{\theta}\hat{\phi}\hat{\phi}\hat{\theta}} = -\frac{2GM}{r^3c^2}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -R_{\hat{r}\hat{\theta}\hat{\theta}\hat{r}} = R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}} = -R_{\hat{\theta}\hat{r}\hat{r}\hat{\theta}} = \frac{GM}{r^3c^2} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -R_{\hat{r}\hat{\phi}\hat{\phi}\hat{r}} = R_{\hat{\phi}\hat{r}\hat{\phi}\hat{r}} = -R_{\hat{\phi}\hat{r}\hat{r}\hat{\phi}}$$

(29)

Therefore,

$$e^{\hat{\alpha}}{}_{\hat{0}}(\hat{\xi}^0) = ((1 + \frac{a_0\hat{\xi}^1}{c^2})\cosh(\frac{a_0\hat{\xi}^0}{c}), (1 + \frac{a_0\hat{\xi}^1}{c^2})\sinh(\frac{a_0\hat{\xi}^0}{c}), 0, 0)$$

$$e^{\hat{\alpha}}{}_{\hat{1}}(\hat{\xi}^0) = (\sinh(\frac{a_0\hat{\xi}^0}{c}), \cosh(\frac{a_0\hat{\xi}^0}{c}), 0, 0)$$

$$e^{\hat{\alpha}}{}_{\hat{2}}(\hat{\xi}^0) = (0,0,1,0), e^{\hat{\alpha}}{}_{\hat{3}}(\hat{\xi}^0) = (0,0,1,0) \quad (30)$$

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}}(\hat{\xi}) = \frac{2GM}{r^3c^2}(1+\frac{a_0\hat{\xi}^1}{c^2})^2, \; R_{\hat{0}\hat{2}\hat{0}\hat{2}}(\hat{\xi}) = R_{\hat{0}\hat{3}\hat{0}\hat{3}}(\hat{\xi}) = -\frac{GM}{r^3c^2}(1+\frac{a_0\hat{\xi}^1}{c^2})^2$$

$$R_{\hat{2}\hat{3}\hat{2}\hat{3}}(\hat{\xi}) = -\frac{2GM}{r^3c^2}, \; R_{\hat{1}\hat{2}\hat{1}\hat{2}}(\hat{\xi}) = R_{\hat{1}\hat{3}\hat{1}\hat{3}}(\hat{\xi}) = \frac{GM}{r^3c^2} \quad (31)$$

IV. Conclusion

In the general relativity theory, defines the accelerated frame that moves in $\hat{r}$ -axis in the curved time-space.

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