

The curvature tensor of the stationary accelerated frame in the gravity field

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ABSTRACT

In the general relativity theory, defines the accelerated frame that moves in \hat{t} -axis in the curved time-space. And calculates the curvature tensor of the stationary accelerated frame in the gravity field.

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1.Introduction

This theory's object is that defines the accelerated frame that moves in \hat{r} -axis in the curved time-space.

The Schwarzschild solution is

$$d\tau^2 = \left(1 - \frac{2GM}{rc^2}\right) dt^2 - \frac{1}{c^2} \left[\frac{dr^2}{1 - \frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \quad (1)$$

In this time, a moving matter's acceleration is a in the Schwarzschild time-space.

$$a = \frac{d}{dt} \left(\frac{u}{\sqrt{1 - \frac{2GM}{rc^2} - \frac{u^2}{c^2}}} \right), u = \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}} \frac{dr}{dt} \quad (2)$$

If $a_0 = a / \sqrt{1 - \frac{2GM}{rc^2}}$ is,

$$a_0 = \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}} \frac{d}{dt} \left(\frac{u}{\sqrt{1 - \frac{2GM}{rc^2} - \frac{u^2}{c^2}}} \right) = \frac{d}{d\hat{t}} \left(\frac{V}{\sqrt{1 - \frac{V^2}{c^2}}} \right),$$

$$V = \frac{d\hat{r}}{d\hat{t}} = \frac{dr}{dt} \frac{1}{\left(1 - \frac{2GM}{rc^2}\right)}, d\hat{t} = dt \sqrt{1 - \frac{2GM}{rc^2}}, d\hat{r} = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}}$$

$$a_0 \hat{t} = \frac{V}{\sqrt{1 - \frac{V^2}{c^2}}}, V = \frac{a_0 \hat{t}}{\sqrt{1 + \frac{a_0^2 \hat{t}^2}{c^2}}}, V \text{ is the } \hat{r}\text{-axis's velocity} \quad (3)$$

If $\frac{d\theta}{dt} = \frac{d\phi}{dt} = 0$, the solution is

$$\begin{aligned} d\tau^2 &= \left(1 - \frac{2GM}{rc^2}\right) dt^2 - \frac{1}{c^2} \frac{dr^2}{1 - \frac{2GM}{rc^2}} = d\hat{t}^2 - \frac{1}{c^2} d\hat{r}^2 = d\hat{t}^2 \left(1 - \frac{V^2}{c^2}\right) \\ &= \frac{d\hat{t}^2}{\sqrt{1 + \frac{a_0^2 \hat{t}^2}{c^2}}} \quad (4) \end{aligned}$$

In this time,

$$\tau = \int d\tau = \int \frac{d\hat{t}}{\sqrt{1 + \frac{a_0^2 \hat{t}^2}{c^2}}} = \frac{c}{a_0} \sinh^{-1}\left(\frac{a_0}{c} \hat{t}\right),$$

$$\hat{t} = \frac{c}{a_0} \sinh\left(\frac{a_0 \tau}{c}\right), \quad \hat{r} = \int V d\hat{t} = \int \frac{a_0 \hat{t} d\hat{t}}{\sqrt{1 + \frac{a_0^2 \hat{t}^2}{c^2}}} = \frac{c^2}{a_0} \sqrt{1 + \frac{a_0^2 \hat{t}^2}{c^2}}$$

$$= \frac{c^2}{a_0} \cosh\left(\frac{a_0 \tau}{c}\right)$$

$$\frac{d\hat{t}}{d\tau} = \cosh\left(\frac{a_0}{c} \tau\right), \quad \frac{1}{c} \frac{d\hat{r}}{d\tau} = \sinh\left(\frac{a_0}{c} \tau\right) \quad (5)$$

2. The tetrad in the curved time-space

The tetrad $e^{\hat{\alpha}}_{\hat{\mu}}$ is the unit vector defined by the following formula.

$$\eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}_{\hat{\mu}} e^{\hat{\beta}}_{\hat{\nu}} = g_{\hat{\mu}\hat{\nu}} \quad (6)$$

In this time, if a matter moves in \hat{r} -axis in the curved time-space,

$$\eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}_{\hat{\mu}}(\tau) e^{\hat{\beta}}_{\hat{\nu}}(\tau) = g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}, \quad g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} \quad (7)$$

Hence, Eq(6),Eq(7) is

$$\eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}_{\hat{0}}(\tau) e^{\hat{\beta}}_{\hat{0}}(\tau) = \eta_{\hat{0}\hat{0}} = -1 \quad (8)$$

$$d\tau^2 = -\frac{1}{c^2} \eta_{\hat{\alpha}\hat{\beta}} d\hat{x}^{\hat{\alpha}} d\hat{x}^{\hat{\beta}}$$

$$\rightarrow -1 = \eta_{\hat{\alpha}\hat{\beta}} \left(\frac{1}{c} \frac{d\hat{x}^{\hat{\alpha}}}{d\tau}\right) \left(\frac{1}{c} \frac{d\hat{x}^{\hat{\beta}}}{d\tau}\right) = \eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}_{\hat{0}}(\tau) e^{\hat{\beta}}_{\hat{0}}(\tau)$$

$$\hat{x}^{\alpha} = (c\hat{t}, \hat{r}, \hat{\theta}, \hat{\phi}) \quad (9)$$

According to Eq(5),Eq(9)

$$e^{\hat{\alpha}}_{\hat{0}}(\tau) = \frac{1}{c} \frac{d\hat{x}^{\hat{\alpha}}}{d\tau} = \left(\cosh\left(\frac{a_0 \tau}{c}\right), \sinh\left(\frac{a_0 \tau}{c}\right), 0, 0\right) \quad (10)$$

About $\hat{\theta}$ -axis's and $\hat{\phi}$ -axis's orientation

$$\eta_{\hat{2}\hat{2}} e^{\hat{2}}_{\hat{2}}(\tau) e^{\hat{2}}_{\hat{2}}(\tau) = \eta_{\hat{2}\hat{2}} = 1, \quad e^{\hat{\alpha}}_{\hat{2}}(\tau) = (0, 0, 1, 0)$$

$$\eta_{\hat{3}\hat{3}} e^{\hat{3}}_{\hat{3}}(\tau) e^{\hat{3}}_{\hat{3}}(\tau) = \eta_{\hat{3}\hat{3}} = 1, \quad e^{\hat{a}}_{\hat{3}}(\tau) = (0,0,0,1) \quad (11)$$

And the other vector $e^{\hat{a}}_{\hat{1}}(\tau)$ has to satisfy the tetrad condition, Eq (6),Eq(7)

$$e^{\hat{a}}_{\hat{1}}(\tau) = (\sinh(\frac{a_0 \tau}{c}), \cosh(\frac{a_0 \tau}{c}), 0, 0) \quad (12)$$

In this time,

$$\bar{e}_t^{\rho} = (1/\sqrt{1-\frac{2GM}{rc^2}}, 0, 0, 0), \bar{e}_r^{\rho} = (0, \sqrt{1-\frac{2GM}{rc^2}}, 0, 0)$$

$$\bar{e}_\theta^{\rho} = (0, 0, 1/r, 0) \quad , \quad \bar{e}_\phi^{\rho} = (0, 0, 0, 1/r \sin \theta)$$

$$g_{\rho\sigma} \bar{e}_{\hat{\alpha}}^{\rho} \bar{e}_{\hat{\beta}}^{\sigma} = \eta_{\hat{\alpha}\hat{\beta}} \quad (13)$$

$$\frac{a_0}{c} \hat{t} = \sinh(\frac{a_0}{c} \tau) = \frac{v/c}{\sqrt{1-v^2/c^2}}, \sqrt{1+\frac{a_0^2 \hat{t}^2}{c^2}} = \cosh(\frac{a_0}{c} \tau) = \frac{1}{\sqrt{1-v^2/c^2}} \quad (14)$$

Therefore, the Lorentz transformation $B^{\hat{\alpha}}_{\hat{\mu}}(v)$ is

$$B^{\hat{\alpha}}_{\hat{\mu}}(v) = \begin{pmatrix} 1 & v/c & 0 & 0 \\ \frac{v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= e^{\hat{\alpha}}_{\hat{\mu}}(\tau) = \begin{pmatrix} \cosh(\frac{a_0}{c} \tau) & \sinh(\frac{a_0}{c} \tau) & 0 & 0 \\ \sinh(\frac{a_0}{c} \tau) & \cosh(\frac{a_0}{c} \tau) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (15)$$

$$\bar{e}^{\hat{1}}_{\hat{\mu}}{}^{\rho} = B^{\hat{\alpha}}_{\hat{\mu}}(v) \bar{e}_{\hat{\alpha}}^{\rho} = e^{\hat{\alpha}}_{\hat{\mu}}(\tau) \bar{e}_{\hat{\alpha}}^{\rho} \quad (16)$$

Hence,

$$g_{\rho\sigma} \bar{e}_{\hat{\alpha}}^{\rho} \bar{e}_{\hat{\beta}}^{\sigma} = \eta_{\hat{\alpha}\hat{\beta}}$$

$$g_{\rho\sigma} B^{\hat{\alpha}}_{\hat{\mu}}(v) \bar{e}_{\hat{\alpha}}^{\rho} B^{\hat{\beta}}_{\hat{\nu}}(v) \bar{e}_{\hat{\beta}}^{\sigma} = g_{\rho\sigma} \bar{e}^{\hat{1}}_{\hat{\mu}}{}^{\rho} \bar{e}^{\hat{1}}_{\hat{\nu}}{}^{\sigma} = \eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}_{\hat{\mu}}(\tau) e^{\hat{\beta}}_{\hat{\nu}}(\tau) = \eta_{\hat{\mu}\hat{\nu}} \quad (17)$$

3. The accelerated frame in the curved time-space

About the accelerated frame $\hat{\xi}$ in the curved time-space,

$$\begin{aligned}
 d\tau^2 &= -\frac{1}{c^2} \eta_{\hat{\alpha}\hat{\beta}} d\hat{\chi}^\alpha d\hat{\chi}^\beta = -\frac{1}{c^2} \eta_{\hat{\alpha}\hat{\beta}} \frac{\partial \hat{\chi}^\alpha}{\partial \hat{\xi}^\mu} \frac{\partial \hat{\chi}^\beta}{\partial \hat{\xi}^\nu} d\hat{\xi}^\mu d\hat{\xi}^\nu \\
 &= -\frac{1}{c^2} \eta_{\hat{\alpha}\hat{\beta}} e^{\hat{\alpha}}_{\hat{\mu}} e^{\hat{\beta}}_{\hat{\nu}} d\hat{\xi}^\mu d\hat{\xi}^\nu \\
 &= -\frac{1}{c^2} g_{\hat{\mu}\hat{\nu}} d\hat{\xi}^\mu d\hat{\xi}^\nu
 \end{aligned} \tag{18}$$

$$e^{\hat{\alpha}}_{\hat{\mu}} = \frac{\partial \hat{\chi}^\alpha}{\partial \hat{\xi}^\mu}, \quad \frac{\partial e^{\hat{\alpha}}_{\hat{0}}}{\partial \hat{\xi}^1} = \frac{\partial^2 \hat{\chi}^\alpha}{\partial \hat{\xi}^0 \partial \hat{\xi}^1} = \frac{\partial e^{\hat{\alpha}}_{\hat{1}}}{\partial \hat{\xi}^0} \tag{19}$$

3-1.Case-1. Rindler-coordinate

In this time, in Eq(10),Eq(11),Eq(12), if uses $\hat{\xi}^0$ instead of τ ,

$$e^{\hat{\alpha}}_{\hat{0}}(\hat{\xi}^0) = \frac{1}{c} \frac{\partial \hat{\chi}^\alpha}{\partial \hat{\xi}^0} = \left(\left(1 + \frac{a_0 \hat{\xi}^1}{c^2}\right) \cosh\left(\frac{a_0 \hat{\xi}^0}{c}\right), \left(1 + \frac{a_0 \hat{\xi}^1}{c^2}\right) \sinh\left(\frac{a_0 \hat{\xi}^0}{c}\right), 0, 0 \right) \tag{20}$$

$$e^{\hat{\alpha}}_{\hat{1}}(\hat{\xi}^0) = \frac{\partial \hat{\chi}^\alpha}{\partial \hat{\xi}^1} = \left(\sinh\left(\frac{a_0 \hat{\xi}^0}{c}\right), \cosh\left(\frac{a_0 \hat{\xi}^0}{c}\right), 0, 0 \right) \tag{21}$$

$$e^{\hat{\alpha}}_{\hat{2}}(\hat{\xi}^0) = \frac{\partial \hat{\chi}^\alpha}{\partial \hat{\xi}^2} = (0, 0, 1, 0), \quad e^{\hat{\alpha}}_{\hat{3}}(\hat{\xi}^0) = \frac{\partial \hat{\chi}^\alpha}{\partial \hat{\xi}^3} = (0, 0, 1, 0) \tag{22}$$

$$d\hat{\chi}^\alpha = \frac{\partial \hat{\chi}^\alpha}{\partial \hat{\xi}^\mu} d\hat{\xi}^\mu = e^{\hat{\alpha}}_{\hat{0}}(\hat{\xi}^0) c d\hat{\xi}^0 + e^{\hat{\alpha}}_{\hat{1}}(\hat{\xi}^0) d\hat{\xi}^1 + e^{\hat{\alpha}}_{\hat{2}}(\hat{\xi}^0) d\hat{\xi}^2 + e^{\hat{\alpha}}_{\hat{3}}(\hat{\xi}^0) d\hat{\xi}^3 \tag{23}$$

Hence,

$$cd\hat{t} = cdt \sqrt{1 - \frac{2GM}{rc^2}} = \left(1 + \frac{a_0 \hat{\xi}^1}{c^2}\right) \cosh\left(\frac{a_0 \hat{\xi}^0}{c}\right) c d\hat{\xi}^0 + \sinh\left(\frac{a_0 \hat{\xi}^0}{c}\right) d\hat{\xi}^1$$

$$d\hat{r} = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} = \left(1 + \frac{a_0 \hat{\xi}^1}{c^2}\right) \sinh\left(\frac{a_0 \hat{\xi}^0}{c}\right) c d\hat{\xi}^0 + \cosh\left(\frac{a_0 \hat{\xi}^0}{c}\right) d\hat{\xi}^1$$

$$d\hat{\theta} = d\hat{\xi}^2, \quad d\hat{\phi} = d\hat{\xi}^3 \tag{24}$$

$$\begin{aligned}
d\tau^2 &= \left(1 - \frac{2GM}{rc^2}\right) dt^2 - \frac{1}{c^2} \left[\frac{dr^2}{1 - \frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \\
&= d\hat{t}^2 - \frac{1}{c^2} [d\hat{r}^2 + d\hat{\theta}^2 + d\hat{\phi}^2] \\
&= \left(1 + \frac{a_0 \hat{\xi}^1}{c^2}\right)^2 (d\hat{\xi}^0)^2 - \frac{1}{c^2} [(d\hat{\xi}^1)^2 + (d\hat{\xi}^2)^2 + (d\hat{\xi}^3)^2]
\end{aligned} \tag{25}$$

The coordinate transformation is

$$\begin{aligned}
c\hat{t} &= \left(\frac{c^2}{a_0} + \hat{\xi}^1\right) \sinh\left(\frac{a_0 \hat{\xi}^0}{c}\right), \quad \hat{r} = \left(\frac{c^2}{a_0} + \hat{\xi}^1\right) \cosh\left(\frac{a_0 \hat{\xi}^0}{c}\right) - \frac{c^2}{a_0} \\
\hat{\theta} &= \hat{\xi}^2, \quad \hat{\phi} = \hat{\xi}^3
\end{aligned} \tag{26}$$

The inverse-transformation is

$$\begin{aligned}
\hat{\xi}^0 &= \frac{c}{a_0} \tanh^{-1}\left(\frac{c\hat{t}}{\hat{r} + \frac{c^2}{a_0}}\right), \quad \hat{\xi}^1 = \sqrt{\left(\hat{r} + \frac{c^2}{a_0}\right)^2 - c^2 \hat{t}^2} - \frac{c^2}{a_0} \\
\hat{\xi}^2 &= \hat{\theta}, \quad \hat{\xi}^3 = \hat{\phi}
\end{aligned} \tag{27}$$

If calculates the curvature tensor $R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}(\hat{\xi})$,

$$\begin{aligned}
R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}(\hat{\xi}) &= \frac{\partial \hat{X}^\alpha}{\partial \hat{\xi}^\mu} \frac{\partial \hat{X}^\beta}{\partial \hat{\xi}^\nu} \frac{\partial \hat{X}^\gamma}{\partial \hat{\xi}^\rho} \frac{\partial \hat{X}^\delta}{\partial \hat{\xi}^\lambda} R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}(\hat{X}) \\
&= e^{\hat{\alpha}}_{\hat{\mu}}(\hat{\xi}^0) e^{\hat{\beta}}_{\hat{\nu}}(\hat{\xi}^0) e^{\hat{\gamma}}_{\hat{\rho}}(\hat{\xi}^0) e^{\hat{\delta}}_{\hat{\lambda}}(\hat{\xi}^0) R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}(\hat{X})
\end{aligned} \tag{28}$$

$$\begin{aligned}
R_{\hat{t}\hat{r}\hat{t}\hat{r}} &= -R_{\hat{r}\hat{t}\hat{r}\hat{t}} = R_{\hat{r}\hat{t}\hat{t}\hat{r}} = -R_{\hat{t}\hat{r}\hat{t}\hat{r}} = \frac{2GM}{r^3 c^2}, \\
R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} &= -R_{\hat{\theta}\hat{t}\hat{t}\hat{\theta}} = R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = -R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = -\frac{GM}{r^3 c^2} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = -R_{\hat{\phi}\hat{t}\hat{t}\hat{\phi}} = R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = -R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} \\
R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} &= -R_{\hat{\phi}\hat{\theta}\hat{\theta}\hat{\phi}} = R_{\hat{\theta}\hat{\phi}\hat{\phi}\hat{\theta}} = -R_{\hat{\phi}\hat{\theta}\hat{\phi}\hat{\theta}} = -\frac{2GM}{r^3 c^2} \\
R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} &= -R_{\hat{\theta}\hat{r}\hat{r}\hat{\theta}} = R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}} = -R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = \frac{GM}{r^3 c^2} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -R_{\hat{\phi}\hat{r}\hat{r}\hat{\phi}} = R_{\hat{\phi}\hat{r}\hat{\phi}\hat{r}} = -R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}}
\end{aligned} \tag{29}$$

Therefore,

$$e^{\hat{\alpha}_0}(\hat{\xi}^0) = \left(\left(1 + \frac{a_0 \hat{\xi}^1}{c^2}\right) \cosh\left(\frac{a_0 \hat{\xi}^0}{c}\right), \left(1 + \frac{a_0 \hat{\xi}^1}{c^2}\right) \sinh\left(\frac{a_0 \hat{\xi}^0}{c}\right), 0, 0 \right)$$

$$e^{\hat{\alpha}_1}(\hat{\xi}^0) = \left(\sinh\left(\frac{a_0 \hat{\xi}^0}{c}\right), \cosh\left(\frac{a_0 \hat{\xi}^0}{c}\right), 0, 0 \right)$$

$$e^{\hat{\alpha}_2}(\hat{\xi}^0) = (0, 0, 1, 0), e^{\hat{\alpha}_3}(\hat{\xi}^0) = (0, 0, 1, 0) \quad (30)$$

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}}(\hat{\xi}) = \frac{2GM}{r^3 c^2} \left(1 + \frac{a_0 \hat{\xi}^1}{c^2}\right)^2, R_{\hat{0}\hat{2}\hat{0}\hat{2}}(\hat{\xi}) = R_{\hat{0}\hat{3}\hat{0}\hat{3}}(\hat{\xi}) = -\frac{GM}{r^3 c^2} \left(1 + \frac{a_0 \hat{\xi}^1}{c^2}\right)^2$$

$$R_{\hat{2}\hat{3}\hat{2}\hat{3}}(\hat{\xi}) = -\frac{2GM}{r^3 c^2}, R_{\hat{1}\hat{2}\hat{1}\hat{2}}(\hat{\xi}) = R_{\hat{1}\hat{3}\hat{1}\hat{3}}(\hat{\xi}) = \frac{GM}{r^3 c^2} \quad (31)$$

Specially, if $t = 0$,

$$u = \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}} \frac{dr}{dt} = 0 \rightarrow V = \frac{d\hat{r}}{d\hat{t}} = \frac{a_0 \hat{t}}{\sqrt{1 + \frac{a_0^2 \hat{t}^2}{c^2}}} = \frac{dr}{dt} \frac{1}{\left(1 - \frac{2GM}{rc^2}\right)} = 0 \quad (32)$$

Therefore, if $t = \hat{t} = \hat{\xi}^0 = 0$, the theory treats the real situation.

$$\hat{\xi}^1 = \sqrt{\left(\hat{r} + \frac{c^2}{a_0}\right)^2 - c^2 \hat{t}^2} - \frac{c^2}{a_0} = \hat{r}$$

$$d\hat{r} = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} \rightarrow \hat{r} = \sqrt{r} \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln \left| \sqrt{r} + \sqrt{r - \frac{2GM}{c^2}} \right|$$

$$- \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln \left| \sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}} \right|$$

$$a_0 = \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}} \frac{d}{dt} \left(\frac{u}{\sqrt{1 - \frac{2GM}{rc^2} - \frac{u^2}{c^2}}} \right) = \frac{a}{\sqrt{1 - \frac{2GM}{rc^2}}}$$

$$a = \frac{d}{dt} \left(\frac{u}{\sqrt{1 - \frac{2GM}{rc^2} - \frac{u^2}{c^2}}} \right)$$

r_0 is the location of the stationary accelerated frame (33)

In this time, in the curved time-space, the curvature tensor $R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}(\hat{\xi})$ of the stationary accelerated frame is

$$\begin{aligned}
R_{\hat{0}\hat{1}\hat{0}\hat{1}}(\hat{\xi}) &= \frac{2GM}{r^3 c^2} \left(1 + \frac{a_0 \hat{\xi}^1}{c^2}\right)^2 = \frac{2GM}{r^3 c^2} \left(1 + \frac{a_0 \hat{r}}{c^2}\right)^2 \\
&= \frac{2GM}{r^3 c^2} \left[1 + \frac{a_0}{c^2} \left\{ \sqrt{r} \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln \left| \sqrt{r} + \sqrt{r - \frac{2GM}{c^2}} \right| \right. \right. \\
&\quad \left. \left. - \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln \left| \sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}} \right| \right\} \right]^2 \\
&= \frac{2GM}{r^3 c^2} \left[1 + \frac{1}{c^2} \frac{a}{\sqrt{1 - \frac{2GM}{rc^2}}} \left\{ \sqrt{r} \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln \left| \sqrt{r} + \sqrt{r - \frac{2GM}{c^2}} \right| \right. \right. \\
&\quad \left. \left. - \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln \left| \sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}} \right| \right\} \right]^2 \\
R_{\hat{0}\hat{2}\hat{0}\hat{2}}(\hat{\xi}) &= R_{\hat{0}\hat{3}\hat{0}\hat{3}}(\hat{\xi}) = -\frac{GM}{r^3 c^2} \left(1 + \frac{a_0 \hat{\xi}^1}{c^2}\right)^2 = -\frac{GM}{r^3 c^2} \left(1 + \frac{a_0 \hat{r}}{c^2}\right)^2 \\
&= -\frac{GM}{r^3 c^2} \left[1 + \frac{a_0}{c^2} \left\{ \sqrt{r} \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln \left| \sqrt{r} + \sqrt{r - \frac{2GM}{c^2}} \right| \right. \right. \\
&\quad \left. \left. - \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln \left| \sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}} \right| \right\} \right]^2 \\
&= -\frac{GM}{r^3 c^2} \left[1 + \frac{1}{c^2} \frac{a}{\sqrt{1 - \frac{2GM}{rc^2}}} \left\{ \sqrt{r} \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln \left| \sqrt{r} + \sqrt{r - \frac{2GM}{c^2}} \right| \right. \right. \\
&\quad \left. \left. - \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln \left| \sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}} \right| \right\} \right]^2 \\
R_{\hat{2}\hat{3}\hat{2}\hat{3}}(\hat{\xi}) &= -\frac{2GM}{r^3 c^2}, \quad R_{\hat{1}\hat{2}\hat{1}\hat{2}}(\hat{\xi}) = R_{\hat{1}\hat{3}\hat{1}\hat{3}}(\hat{\xi}) = \frac{GM}{r^3 c^2}
\end{aligned}$$

r_0 is the location of the stationary accelerated frame

(34)

3-2. Case-2. Marzke-Wheeler coordinate

In this time, in Eq(10),Eq(11),Eq(12), if uses $\hat{\xi}^0$ instead of τ and multiply $\exp(\frac{a_0}{c^2} \hat{\xi}^1)$

$$e^{\hat{\alpha}_0}(\hat{\xi}^0) = \frac{1}{c} \frac{\partial \hat{\chi}^\alpha}{\partial \hat{\xi}^0} = (\exp(\frac{a_0}{c^2} \hat{\xi}^1) \cosh(\frac{a_0 \hat{\xi}^0}{c}), \exp(\frac{a_0}{c^2} \hat{\xi}^1) \sinh(\frac{a_0 \hat{\xi}^0}{c}), 0, 0) \quad (35)$$

$$e^{\hat{\alpha}_1}(\hat{\xi}^0) = \frac{\partial \hat{\chi}^\alpha}{\partial \hat{\xi}^1} = (\exp(\frac{a_0}{c^2} \hat{\xi}^1) \sinh(\frac{a_0 \hat{\xi}^0}{c}), \exp(\frac{a_0}{c^2} \hat{\xi}^1) \cosh(\frac{a_0 \hat{\xi}^0}{c}), 0, 0)$$

(36)

$$e^{\hat{\alpha}_2}(\hat{\xi}^0) = \frac{\partial \hat{\chi}^\alpha}{\partial \hat{\xi}^2} = (0, 0, 1, 0), e^{\hat{\alpha}_3}(\hat{\xi}^0) = \frac{\partial \hat{\chi}^\alpha}{\partial \hat{\xi}^3} = (0, 0, 1, 0) \quad (37)$$

$$d\hat{\chi}^\alpha = \frac{\partial \hat{\chi}^\alpha}{\partial \hat{\xi}^\mu} d\hat{\xi}^\mu = e^{\hat{\alpha}_0}(\hat{\xi}^0) c d\hat{\xi}^0 + e^{\hat{\alpha}_1}(\hat{\xi}^0) d\hat{\xi}^1 + e^{\hat{\alpha}_2}(\hat{\xi}^0) d\hat{\xi}^2 + e^{\hat{\alpha}_3}(\hat{\xi}^0) d\hat{\xi}^3 \quad (38)$$

Hence,

$$cdf = c dt \sqrt{1 - \frac{2GM}{rc^2}} = \exp(\frac{a_0}{c^2} \hat{\xi}^1) \cosh(\frac{a_0 \hat{\xi}^0}{c}) c d\hat{\xi}^0 + \exp(\frac{a_0}{c^2} \hat{\xi}^1) \sinh(\frac{a_0 \hat{\xi}^0}{c}) d\hat{\xi}^1$$

$$d\hat{r} = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} = \exp(\frac{a_0}{c^2} \hat{\xi}^1) \sinh(\frac{a_0 \hat{\xi}^0}{c}) c d\hat{\xi}^0 + \exp(\frac{a_0}{c^2} \hat{\xi}^1) \cosh(\frac{a_0 \hat{\xi}^0}{c}) d\hat{\xi}^1$$

$$d\hat{\theta} = d\hat{\xi}^2, \quad d\hat{\phi} = d\hat{\xi}^3 \quad (39)$$

$$d\tau^2 = (1 - \frac{2GM}{rc^2}) dt^2 - \frac{1}{c^2} \left[\frac{dr^2}{1 - \frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

$$= d\hat{t}^2 - \frac{1}{c^2} [d\hat{r}^2 + d\hat{\theta}^2 + d\hat{\phi}^2]$$

$$= \exp(2 \frac{a_0 \hat{\xi}^1}{c^2}) (d\hat{\xi}^0)^2 - \frac{1}{c^2} [\exp(2 \frac{a_0 \hat{\xi}^1}{c^2}) (d\hat{\xi}^1)^2 + (d\hat{\xi}^2)^2 + (d\hat{\xi}^3)^2]$$

(40)

The coordinate transformation is

$$c\hat{t} = \frac{c^2}{a_0} \exp(\frac{a_0}{c^2} \hat{\xi}^1) \sinh(\frac{a_0 \hat{\xi}^0}{c}), \hat{r} = \frac{c^2}{a_0} \exp(\frac{a_0}{c^2} \hat{\xi}^1) \cosh(\frac{a_0 \hat{\xi}^0}{c}) - \frac{c^2}{a_0}$$

$$\hat{\theta} = \hat{\xi}^2, \hat{\phi} = \hat{\xi}^3 \quad (41)$$

The inverse-transformation is

$$\hat{\xi}^0 = \frac{c}{a_0} \tanh^{-1}\left(\frac{c\hat{t}}{\hat{r} + \frac{c^2}{a_0}}\right), \hat{\xi}^1 = \frac{c^2}{a_0} \ln \left| \frac{a_0}{c^2} \sqrt{\left(\hat{r} + \frac{c^2}{a_0}\right)^2 - c^2\hat{t}^2} \right|$$

$$\hat{\xi}^2 = \hat{\theta}, \hat{\xi}^3 = \hat{\phi} \quad (42)$$

If calculates the curvature tensor $R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}(\hat{\xi})$,

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}(\hat{\xi}) = \frac{\partial \hat{X}^\alpha}{\partial \hat{\xi}^\mu} \frac{\partial \hat{X}^\beta}{\partial \hat{\xi}^\nu} \frac{\partial \hat{X}^\gamma}{\partial \hat{\xi}^\rho} \frac{\partial \hat{X}^\delta}{\partial \hat{\xi}^\lambda} R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}(\hat{X})$$

$$= e^{\hat{\alpha}}_{\hat{\mu}}(\hat{\xi}^0) e^{\hat{\beta}}_{\hat{\nu}}(\hat{\xi}^0) e^{\hat{\gamma}}_{\hat{\rho}}(\hat{\xi}^0) e^{\hat{\delta}}_{\hat{\lambda}}(\hat{\xi}^0) R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}(\hat{X}) \quad (43)$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -R_{\hat{t}\hat{r}\hat{t}\hat{t}} = R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -R_{\hat{t}\hat{t}\hat{r}\hat{r}} = \frac{2GM}{r^3 c^2},$$

$$R_{\hat{t}\hat{t}\hat{\theta}\hat{\theta}} = -R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = -R_{\hat{\theta}\hat{t}\hat{t}\hat{\theta}} = -\frac{GM}{r^3 c^2} = R_{\hat{t}\hat{t}\hat{\phi}\hat{\phi}} = -R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = -R_{\hat{\phi}\hat{t}\hat{t}\hat{\phi}}$$

$$R_{\hat{\theta}\hat{\theta}\hat{\phi}\hat{\phi}} = -R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = -R_{\hat{\theta}\hat{\theta}\hat{\phi}\hat{\phi}} = -\frac{2GM}{r^3 c^2}$$

$$R_{\hat{r}\hat{t}\hat{\theta}\hat{\theta}} = -R_{\hat{r}\hat{\theta}\hat{t}\hat{t}} = R_{\hat{\theta}\hat{r}\hat{t}\hat{t}} = -R_{\hat{\theta}\hat{t}\hat{r}\hat{t}} = \frac{GM}{r^3 c^2} = R_{\hat{r}\hat{t}\hat{\phi}\hat{\phi}} = -R_{\hat{r}\hat{\phi}\hat{t}\hat{t}} = R_{\hat{\phi}\hat{r}\hat{t}\hat{t}} = -R_{\hat{\phi}\hat{t}\hat{r}\hat{t}}$$

(44)

Therefore,

$$e^{\hat{\alpha}}_{\hat{0}}(\hat{\xi}^0) = \left(\exp\left(\frac{a_0}{c^2} \hat{\xi}^1\right) \cosh\left(\frac{a_0 \hat{\xi}^0}{c}\right), \exp\left(\frac{a_0}{c^2} \hat{\xi}^1\right) \sinh\left(\frac{a_0 \hat{\xi}^0}{c}\right), 0, 0\right)$$

$$e^{\hat{\alpha}}_{\hat{1}}(\hat{\xi}^0) = \left(\exp\left(\frac{a_0}{c^2} \hat{\xi}^1\right) \sinh\left(\frac{a_0 \hat{\xi}^0}{c}\right), \exp\left(\frac{a_0}{c^2} \hat{\xi}^1\right) \cosh\left(\frac{a_0 \hat{\xi}^0}{c}\right), 0, 0\right)$$

$$e^{\hat{\alpha}}_{\hat{2}}(\hat{\xi}^0) = (0, 0, 1, 0), e^{\hat{\alpha}}_{\hat{3}}(\hat{\xi}^0) = (0, 0, 1, 0) \quad (45)$$

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}}(\hat{\xi}) = \frac{2GM}{r^3 c^2} \exp\left(2 \frac{a_0 \hat{\xi}^1}{c^2}\right), R_{\hat{0}\hat{2}\hat{0}\hat{2}}(\hat{\xi}) = R_{\hat{0}\hat{3}\hat{0}\hat{3}}(\hat{\xi}) = -\frac{GM}{r^3 c^2} \exp\left(2 \frac{a_0 \hat{\xi}^1}{c^2}\right)$$

$$R_{\hat{2}\hat{3}\hat{2}\hat{3}}(\hat{\xi}) = -\frac{2GM}{r^3 c^2}, R_{\hat{1}\hat{2}\hat{1}\hat{2}}(\hat{\xi}) = R_{\hat{1}\hat{3}\hat{1}\hat{3}}(\hat{\xi}) = \frac{GM}{r^3 c^2} \exp\left(2 \frac{a_0 \hat{\xi}^1}{c^2}\right) \quad (46)$$

Specially, if $t = 0$,

$$u = \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}} \frac{dr}{dt} = 0 \rightarrow V = \frac{d\hat{r}}{d\hat{t}} = \frac{a_0 \hat{t}}{\sqrt{1 + \frac{a_0^2 \hat{t}^2}{c^2}}} = \frac{dr}{dt} \frac{1}{(1 - \frac{2GM}{rc^2})} = 0 \quad (47)$$

Therefore, if $t = \hat{t} = \hat{\xi}^0 = 0$, the theory treats the real situation.

$$\hat{\xi}^1 = \frac{c^2}{a_0} \ln \left| \frac{a_0}{c^2} \sqrt{(\hat{r} + \frac{c^2}{a_0})^2 - c^2 \hat{t}^2} \right| = \frac{c^2}{a_0} \ln \left| 1 + \frac{a_0}{c^2} \hat{r} \right|$$

$$\exp\left(\frac{a_0}{c^2} \hat{\xi}^1\right) = 1 + \frac{a_0}{c^2} \hat{r}$$

$$d\hat{r} = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}} \rightarrow \hat{r} = \sqrt{r} \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln \left| \sqrt{r} + \sqrt{r - \frac{2GM}{c^2}} \right|$$

$$- \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln \left| \sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}} \right|$$

$$a_0 = \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}} \frac{d}{dt} \left(\frac{u}{\sqrt{1 - \frac{2GM}{rc^2} - \frac{u^2}{c^2}}} \right) = \frac{a}{\sqrt{1 - \frac{2GM}{rc^2}}}$$

$$a = \frac{d}{dt} \left(\frac{u}{\sqrt{1 - \frac{2GM}{rc^2} - \frac{u^2}{c^2}}} \right)$$

r_0 is the location of the stationary accelerated frame (48)

In this time, in the curved time-space, the curvature tensor $R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}(\hat{\xi})$ of the stationary accelerated frame is

$$R_{\hat{0}\hat{1}\hat{0}\hat{1}}(\hat{\xi}) = \frac{2GM}{r^3 c^2} \exp\left(4 \frac{a_0 \hat{\xi}^1}{c^2}\right) = \frac{2GM}{r^3 c^2} \left(1 + \frac{a_0}{c^2} \hat{r}\right)^4$$

$$= \frac{2GM}{r^3 c^2} \left[1 + \frac{a_0}{c^2} \left\{ \sqrt{r} \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln \left| \sqrt{r} + \sqrt{r - \frac{2GM}{c^2}} \right| \right. \right.$$

$$\left. \left. - \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln \left| \sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}} \right| \right\} \right]^4$$

$$= \frac{2GM}{r^3 c^2} \left[1 + \frac{1}{c^2} \frac{a}{\sqrt{1 - \frac{2GM}{rc^2}}} \left\{ \sqrt{r} \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln \left| \sqrt{r} + \sqrt{r - \frac{2GM}{c^2}} \right| \right. \right. \\ \left. \left. - \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln \left| \sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}} \right| \right\} \right]^4$$

$$R_{\hat{0}\hat{2}\hat{0}\hat{2}}(\hat{\xi}) = R_{\hat{0}\hat{3}\hat{0}\hat{3}}(\hat{\xi}) = -\frac{GM}{r^3 c^2} \exp\left(2 \frac{a_0 \hat{\xi}^1}{c^2}\right) = -\frac{GM}{r^3 c^2} \left(1 + \frac{a_0 \hat{r}}{c^2}\right)^2$$

$$= -\frac{GM}{r^3 c^2} \left[1 + \frac{a_0}{c^2} \left\{ \sqrt{r} \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln \left| \sqrt{r} + \sqrt{r - \frac{2GM}{c^2}} \right| \right. \right.$$

$$\left. \left. - \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln \left| \sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}} \right| \right\} \right]^2$$

$$= -\frac{GM}{r^3 c^2} \left[1 + \frac{1}{c^2} \frac{a}{\sqrt{1 - \frac{2GM}{rc^2}}} \left\{ \sqrt{r} \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln \left| \sqrt{r} + \sqrt{r - \frac{2GM}{c^2}} \right| \right. \right.$$

$$\left. \left. - \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln \left| \sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}} \right| \right\} \right]^2$$

$$R_{\hat{2}\hat{3}\hat{2}\hat{3}}(\hat{\xi}) = -\frac{2GM}{r^3 c^2},$$

$$R_{\hat{1}\hat{2}\hat{1}\hat{2}}(\hat{\xi}) = R_{\hat{1}\hat{3}\hat{1}\hat{3}}(\hat{\xi}) = \frac{GM}{r^3 c^2} \exp\left(2 \frac{a_0 \hat{\xi}^1}{c^2}\right) = \frac{GM}{r^3 c^2} \left(1 + \frac{a_0 \hat{r}}{c^2}\right)^2$$

$$= \frac{GM}{r^3 c^2} \left[1 + \frac{a_0}{c^2} \left\{ \sqrt{r} \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln \left| \sqrt{r} + \sqrt{r - \frac{2GM}{c^2}} \right| \right. \right.$$

$$\left. \left. - \sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln \left| \sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}} \right| \right\} \right]^2$$

$$= \frac{GM}{r^3 c^2} \left[1 + \frac{1}{c^2} \frac{a}{\sqrt{1 - \frac{2GM}{rc^2}}} \left\{ \sqrt{r} \sqrt{r - \frac{2GM}{c^2}} + \frac{2GM}{c^2} \ln \left| \sqrt{r} + \sqrt{r - \frac{2GM}{c^2}} \right| \right. \right.$$

$$-\sqrt{r_0} \sqrt{r_0 - \frac{2GM}{c^2}} - \frac{2GM}{c^2} \ln \left| \sqrt{r_0} + \sqrt{r_0 - \frac{2GM}{c^2}} \right| \Big]^2$$

r_0 is the location of the stationary accelerated frame (49)

4. Conclusion

In the general relativity theory, defines the accelerated frame that moves in \hat{r} -axis in the curved time-space. Specially, if $t = \hat{t} = \hat{\xi}^0 = 0$, this theory treats the curvature tensor of the stationary accelerated frame in the curved time-space in two-cases.

Reference

- [1]S.Weinberg, Gravitation and Cosmology (John wiley & Sons,Inc,1972)
- [2]P.Bergman, Introduction to the Theory of Relativity (Dover Pub. Co.,Inc., New York,1976),Chapter V
- [3]C.Misner, K,Thorne and J. Wheeler, Gravitation(W.H.Freedman & Co.,1973)
- [4]S.Hawking and G. Ellis,The Large Scale Structure of Space-Time(Cam-bridge University Press,1973)
- [5]R.Adler, M.Bazin and M.Schiffer, Introduction to General Relativity(McGraw-Hill,Inc.,1965)
- [6]M.Schwarzschild, Structure and Evolution of the Stars(Princeton University Press,1958;reprint,Dover,N.Y.1965),chapter II
- [7]S.Chandrasekhar, Mon,Not.Roy.Astron.Soc.95.207(1935)
- [8]C.Rhoades, "Investigations in the Physics of Neutron Stars", doctoral dissertation, Princeton University
- [9]J.Oppenheimer and H.Snyder, phys.Rev.56,455(1939)
- [10][Massimo Pauri](#), [Michele Vallisneri](#), "Marzke-Wheeler coordinates for accelerated observers in special relativity":Arxiv:gr-qc/0006095(2000)