

THE GENERALIZED CONTINUUM HYPOTHESIS

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In this paper we prove the generalized continuum hypothesis by categorical logic which is as the theory of categories a first-order theory and so a logically complete theory, proving that the initial ordinals and the transfinite cardinals are isomorphic universal algebras.

For isomorphic universal algebras are isomorphic categories, isomorphic categories are isomorphic structures, isomorphic structures are isomorphic theories by the fundamental theorem of mathematical logic, isomorphic categories are isomorphic theories by the fundamental theorem of categorical logic, and so isomorphic universal algebras are isomorphic theories.

We use the definitions of theory, isomorphism of categories and functor and the first-order axioms of the theory of categories.

Theorem "Generalized Continuum Hypothesis" *For every transfinite cardinal number α , there is no cardinal number between α and 2^α .*

Proof. Let **Card** be the class of transfinite cardinals and let **Ord** be class of initial ordinals. According to the definition of a category every structure of a formal language is a category, in particular, a preorder, which lays the foundation of categorical logic that is consequently as the theory of categories a first-order theory and so a logically complete theory.

Since **Card** and **Ord** are structures of a second-order logic formal language both are categories being as such well-ordered semirings. We prove that **Card** and **Ord** are isomorphic universal algebras proving that these are isomorphic categories acted upon by the exponential functor, proving that there exists a full and faithful functor $T: \mathbf{Card} \rightarrow \mathbf{Ord}$ such that each initial ordinal β is isomorphic to an initial ordinal $T\beta$ for a transfinite cardinal α .

Let $T: \mathbf{Card} \rightarrow \mathbf{Ord}$ be the function of categories which assigns to each transfinite cardinal α the initial ordinal $T\alpha$ of its equipotence class and to every arrow $f: \alpha \rightarrow \alpha'$ in **Card** the arrow $Tf: T\alpha \rightarrow T\alpha'$ in **Ord** for a dyad of transfinite cardinals α and α' . The function of categories T is well-defined because is well-defined on objects for each transfinite cardinal α lies in a unique equipotence class and because is well-defined on arrows for each arrow f in **Card** is a dyad of objects α and α' in **Card** for which $f: \alpha \rightarrow \alpha'$ is an arrow in **Card** by the first-order axioms of the theory of categories, to each dyad of transfinite cardinals α and α' there is a unique dyad of initial ordinals $T\alpha$ and $T\alpha'$ for T is a well-defined function of categories on objects, and each dyad of initial ordinals $T\alpha$ and $T\alpha'$ is a unique arrow $Tf: T\alpha \rightarrow T\alpha'$ by the first-order axioms of the theory of categories and for **Ord** is a preorder.

The function of categories T is a functor because preserves preorders, that is, preserves identities and composable dyads of arrows, for $T1_\alpha = 1_{T\alpha}$ and $T(f \circ g) = Tf \circ Tg$ for every identity 1_α and for every composable dyad of arrows f and g in **Card**. For each identity 1_α in **Card** is a transfinite cardinal α and every initial ordinal $T\alpha$ is an identity $1_{T\alpha}$ in **Ord** by the first-order axioms of the theory of categories. And for $T(f \circ g): T\alpha \rightarrow T\alpha''$ is an arrow in **Ord** for each arrow $f \circ g: \alpha \rightarrow \alpha''$ in **Card** because T is a function of categories, each arrow $f \circ g: \alpha \rightarrow \alpha''$ in **Card** is a dyad of composable arrows $f: \alpha \rightarrow \alpha'$ and $g: \alpha' \rightarrow \alpha''$ in **Card**, each dyad of composable arrows $f: \alpha \rightarrow \alpha'$ and $g: \alpha' \rightarrow \alpha''$ in **Card** is a triad of transfinite cardinals α , α' and α'' , to each triad of transfinite cardinals α , α' and α'' there is a triad of initial ordinals $T\alpha$, $T\alpha'$ and $T\alpha''$ for T is a function of categories, each triad of initial ordinals $T\alpha$, $T\alpha'$ and $T\alpha''$ is a dyad of composable arrows $Tf: T\alpha \rightarrow T\alpha'$ and $Tg: T\alpha' \rightarrow T\alpha''$ by the first-order axioms of the theory of categories, and so $Tf \circ Tg: T\alpha \rightarrow T\alpha''$ is an arrow in **Ord** unique for the dyad $T\alpha$ and $T\alpha''$ for **Ord** is a preorder, that is, $T(f \circ g) = Tf \circ Tg$.

The functor T is full because to each dyad of transfinite cardinals α and α' in **Card** and to each arrow $g: T\alpha \rightarrow T\alpha'$ in **Ord** there is an arrow $f: \alpha \rightarrow \alpha'$ in **Card** such that $g = Tf$ for T is a function of categories on preorders. The functor T is faithful because to each dyad of transfinite cardinals α and α' and to each dyad of arrows $f_1, f_2: \alpha \rightarrow \alpha'$ in **Card** the equality $Tf_1 = Tf_2$ implies $f_1 = f_2$ for T is a function of categories on preorders. And each initial ordinal β is isomorphic to the initial ordinal $T|\beta|$ because is isomorphic to its transfinite cardinal $|\beta|$ and every transfinite cardinal $|\beta|$ is isomorphic to the initial ordinal $T|\beta|$ by definition of T , so $\beta \cong T|\beta|$, therefore **Card** \cong **Ord**.

Thus, since the theories of isomorphic categories are isomorphic by the fundamental theorem of categorical logic for isomorphic categories are isomorphic structures and isomorphic structures are isomorphic theories by the fundamental theorem of mathematical logic, the theories of **Card** and **Ord** are isomorphic, and so, since there is no initial ordinal between ω and ω^ω by the theorem on the comparability of ordinals and $\aleph_0^{\aleph_0} = 2^{\aleph_0}$ by the fundamental theorem of transfinite cardinal arithmetic, the isomorphism between **Card** and **Ord** proves that there exists no transfinite cardinal between the initial transfinite cardinal numbers \aleph_0 and 2^{\aleph_0} , henceforth there exists no transfinite cardinal between any transfinite cardinal numbers α and 2^α . Consequently, there exist no inaccessible cardinals. In fact, the class **Card** of transfinite cardinals is isomorphic to ω because the function f of ω to **Card** which assigns to each ordinal $\alpha \in \omega$ the α -th transfinite cardinal $f\alpha$ is an order-preserving isomorphism that is unique by transfinite construction. ■

The theorem in universal algebra

Thus the theorem not only prove that the universal algebras of initial ordinals and transfinite cardinals are isomorphic, nor that the large category of transfinite cardinals is countable nondiscrete closed complete and cocomplete, with arrows the polynomial maps and the exponential maps, but also that the universal algebra of transfinite cardinals is acted upon by the covariant exponential functor universal algebra.

The theorem in categorical logic

In categorical logic the theorem not only proves that the theories of the universal algebras of transfinite cardinals and the initial ordinals are isomorphic, but also as first-order theories are well-orders isomorphic to ω so having transfinite cardinal \aleph_0 , that higher-order theories have cardinals greater than or equal to the cardinal of the continuum for they are partial orders isomorphic to products of first-order theories.

The theorem in topos theory

In topos theory the theorem proves that the category of transfinite cardinals **Card** is a topos which is isomorphic to the topos of initial ordinals **Ord** with topos of sheaves **Sets**^{**Card***} the category of contravariant functors on the category **Card** to the category **Sets** which assign to every transfinite cardinal β its set of transfinite cardinal functions on β in **Card*** the dual category of **Card** which turn out to be the continuous transfinite cardinal functions on the topology of transfinite cardinals.