Odd Perfect Number = 36k+9

By Isaac Mor - 19 july 2014

if n is a perfect number then $\boxed{\sigma(n)=2n}$ and $\boxed{n=P^{k-1}M}$ where:

$k>1$
P is a Prime Number
M is an Odd Natural
M is not multiple of P ($M \neq P\cdot A$)

$$\sigma(n)=2n=2\cdot P^{k-1}M$$

$$\sigma(n)=\sigma(P^{k-1}M)=\sigma(P^{k-1})\cdot\sigma(M)=\left(\frac{P^k-1}{P-1}\right)\cdot\sigma(M)$$

$$2\cdot P^{k-1}M=\left(\frac{P^k-1}{P-1}\right)\cdot\sigma(M)$$

$$\sigma(M)=2\cdot M\cdot(P-1)\cdot\left(\frac{P^{k-1}}{P^k-1}\right)$$

option #1: $P^k-1\,|\,P^{k-1}$
option #2: $P^k-1\,|\,2\cdot M\cdot(P-1)$

but option #1 is not possible! so we left only with option #2 Meaning that $P^k-1\,|\,2\cdot M\cdot(P-1)$

$N\cdot(P^k-1)=2\cdot M\cdot(P-1)$ where N is natural

$$M=\left(\frac{N(P^k-1)}{2(P-1)}\right)=\left(\frac{N}{2}\right)\cdot\left(\frac{P^k-1}{p-1}\right)$$

$$n=P^{k-1}M=P^{k-1}\cdot\left(P^k-1\right)\cdot\left(\frac{N}{2}\frac{1}{p-1}\right)$$

now we will check for an odd perfect number: $n=P^{\alpha}Q^2$

$P^{\alpha}=P^{k-1}$
$Q^2=M$

$P\equiv\alpha\equiv1(\mathrm{mod}4)$
$\alpha=k-1=4\lambda+1$

$$n=P^{\alpha}Q^2=P^{k-1}M=P^{k-1}\cdot\left(P^k-1\right)\cdot\left(\frac{N}{2}\frac{1}{p-1}\right)$$

$$n=P^{\alpha}Q^2=\frac{P^{\alpha}\left(P^{\alpha+1}-1\right)}{(P-1)}\cdot\left(\frac{N}{2}\right)$$

$$n = P^{\alpha}Q^2 = \left(\frac{N}{2}\right) \cdot P^{\alpha} \cdot \left(1 + P + P^2 + P^3 + \ldots + P^{\alpha}\right)$$

α is an odd number

$$\sigma(P^{\alpha}) = (1 + P^1) + (P^2 + P^3) + \ldots + (P^{\alpha-1} + P^{\alpha})$$
$$\sigma(P^{\alpha}) = (1 + Odd) + (Odd + Odd) + \ldots + (Odd + Odd)$$
$$\sigma(P^{\alpha}) = (Even) + (Even) + \ldots + (Even)$$

$$\boxed{\sigma(P^{\alpha}) = Even}$$

$$n = P^{\alpha}Q^2 = P^{\alpha} \cdot \left(\frac{\sigma(P^{\alpha})}{2}\right) \cdot N$$

$$\boxed{n = P^{\alpha}Q^2 = P^{\alpha} \cdot \left(\frac{1 + P + P^2 + P^3 + \ldots + P^{\alpha}}{2}\right) \cdot N}$$

N must be an odd number too!

$$N \cdot (P^k - 1) = 2 \cdot M \cdot (P - 1)$$

$$odd \cdot (P^k - 1) = 2 \cdot odd \cdot even$$

$$4 \mid (P^k - 1)$$

$$\alpha = k - 1 = 4\lambda + 1$$

$$4 \mid (P^{4\lambda+2} - 1)$$

$$P^{4\lambda+2} - 1 = 4t$$

$$\boxed{(P^{2\lambda+1} + 1)(P^{2\lambda+1} - 1) = 4t}$$

$$n = P^{\alpha}Q^2 = P^{\alpha} \cdot \left(\frac{P^{\alpha+1} - 1}{2(P - 1)}\right) \cdot N$$

$$n = P^{\alpha}Q^2 = P^{\alpha} \cdot \left(\frac{P^{4\lambda+2} - 1}{2(P - 1)}\right) \cdot N$$

$$n = P^{\alpha}Q^2 = P^{4\lambda+1} \cdot \left(\frac{P^{2\lambda+1} + 1}{2}\right) \cdot \left(\frac{P^{2\lambda+1} - 1}{P - 1}\right) \cdot N$$

$$\boxed{n = P^{4\lambda+1} \cdot \left(\frac{P^{2\lambda+1} + 1}{2}\right) \cdot \left(1 + P + P^2 + P^3 + \ldots + P^{2\lambda}\right) \cdot N}$$

now we will show that odd perfect number can only be in the form: $36x+9$

if (P is Prime) then $P^2 - 1 (\mathrm{mod}\, 24)$

$P^2 - 1 (\mathrm{mod}\, 24) \quad \Rightarrow \quad 24 | P^2 - 1 \quad \Rightarrow \quad 24 | (P-1)(P+1)$

Proof

Since P is odd, both $(P-1)$ and $(P+1)$ are even and therefore divisible by two.
Furthermor, since $P \equiv \pm 1 (\mathrm{mod}\, 4)$, either $(P-1)$ or $(P+1)$ is divisible by 4.
Finally, as is true for all prime numbers, $P \equiv \pm 1 (\mathrm{mod}\, 3)$, and therefore, either $(P-1)$ or $(P+1)$ is divisible by 4, the other is divisibly by 2, and one is divisible by 3. $(4)\cdot(2)\cdot(3) = 24$ and therefore $P^2 - 1 (\mathrm{mod}\, 24)$

$P \equiv \alpha \equiv 1 (\mathrm{mod}\, 4) \quad \Rightarrow \quad P - 1 = 4w$

α is an odd number $\quad \Rightarrow \quad P^{\alpha+1} \equiv 1 (\mathrm{mod}\, 24) \quad \Rightarrow \quad P^{\alpha+1} - 1 = 24t$

$$1 + P + P^2 + P^3 + \ldots + P^{\alpha} = \frac{(P^{\alpha+1} - 1)}{(P-1)} = \frac{(24t)}{(4w)} = \frac{6t}{w}$$

α is an odd number $\quad \Rightarrow \quad 1 + P + P^2 + P^3 + \ldots + P^{\alpha} = Even$

$$2r = 1 + P + P^2 + P^3 + \ldots + P^{\alpha} = \frac{(P^{\alpha+1} - 1)}{(P-1)} = \frac{(24t)}{(4w)} = \frac{6t}{w}$$

$$r = \frac{(P^{\alpha+1} - 1)}{2(P-1)} = \frac{(24t)}{2(4w)} = 3\frac{t}{w}$$

$$3 | \left(\frac{P^{\alpha+1} - 1}{2(P-1)} \right)$$

$$3d = \left(\frac{P^{\alpha+1} - 1}{2(P-1)} \right)$$

$$n = P^{\alpha} Q^2 = P^{\alpha} \cdot \left(\frac{P^{\alpha+1} - 1}{2(P-1)} \right) \cdot N$$

$$Q^2 = \left(\frac{P^{\alpha+1} - 1}{2(P-1)} \right) \cdot N = (3d) \cdot N = 3 \cdot (dN)$$

$3 | Q^2 \quad \Rightarrow$ there is at least one more multiplication of 3 $\quad \Rightarrow \quad 9 | Q^2 \quad \Rightarrow \quad 9 | n$

In 1953, Jacques Touchard proved that an odd perfect number must be of the form 12k + 1 or 36k + 9.
(Judy A. Holdener discovered a simpler proof of the theorem of Touchard in 2002)

if I am right then I (isaac mor lol) just showed that an odd perfect number must be of the form 36k+9
(19 july 2014)