The precession of the perihelion of Mercury explained by Celestial Mechanics of Laplace

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ABSTRACT – We calculate in this article an exact theoretical value obtained classically for the secular precession of the perihelion of Mercury, followed by the theory of Stockwell, based on planetary theory of Laplace, your *Mécanique Céleste*: found 5600'',84 of arc per century for the angular velocity of the longitude of the perihelion of Mercury, $d\varpi/dt$, adding to the precession of the equinoxes of the Earth relative to the beginning of the year 1850, as calculated by Stockwell.

Keywords: precession, perihelion, Mercury, anomaly, classical explanation, exact value, theoretical value, observed value, Le Verrier, Stockwell, Laplace, Newton, Weinberg, General Relativity, classical theory, newtonian theory, classical mechanics, celestial mechanics, *Mécanique Céleste*.

The best known anomaly of the motion of Mercury is the advance of the perihelion precession in relation to the classical theory, discovered by Le Verrier [1], anomaly that explained by General Relativity [2], [3]. We intend in this article to calculate this precession of the perihelion of Mercury following the Newtonian theory, the *Mécanique Céleste* of Laplace [4], and show that the theoretical value obtained is in excellent agreement with the observed value. This will lead to the conclusion that General Relativity does not explain the precession of the perihelion of Mercury, unlike the classical theory. First we calculate this precession based on the data of Stockwell [5].

John Nelson Stockwell (1832-1920) published in 1872 an excellent job on the secular variations of the orbital elements of the eight planets of the solar system [5].

Of the six orbital elements studied in Celestial Mechanics,

1) mean motion (means angular displacement in the considered period) (n)
2) average distance from the Sun (a)
3) eccentricity of the orbit (e)
4) inclination of the orbit (ϕ)
5) longitude of the perihelion (ϖ)
6) longitude of the node (θ)

the first two are considered constant, and the last four were objects of study in Stockwell [5], in order to determine their numerical values for each planet.

The reciprocal of the masses which Stockwell used to get to his results (masses relative to the mass of the Sun) are described in Table 1 below, obtained on page 5 of [5]. The numbers in parentheses in the first column correspond to the indexes in roman numerals commonly used in the equations of the planetary system, and here transformed into latin numbers.

Planet	**M_{planet} (kg)**	**$M_{satellites}$(kg)**	**$m^{-1} = M_S / (M_p+M_s)$**	**Stockwell**	**μ**
Mercury (0)	$3{,}3022 \times 10^{23}$	0	6 023 560,05	4 865 751	-0,1922
Venus (1)	$4{,}8685 \times 10^{24}$	0	408 565,27	390 000	-0,04544
Earth (2)	$5{,}9736 \times 10^{24}$	$7{,}349 \times 10^{22}$	328 935,07	368 689	0,1209
Mars (3)	$6{,}4174 \times 10^{23}$	$1{,}26 \times 10^{16}$	3 099 541,81	2 680 637	-0,1352
Jupiter (4)	$1{,}8986 \times 10^{27}$	$3{,}9701 \times 10^{23}$	1 047,48	1 047, 879	0,0003809
Saturn (5)	$5{,}6846 \times 10^{26}$	$1{,}4051 \times 10^{23}$	3 498,24	3 501,6	0,0009605
Uranus (6)	$8{,}6810 \times 10^{25}$	$9{,}1413 \times 10^{21}$	22 910,85	24 905	0,08704
Neptune (7)	$1{,}0243 \times 10^{26}$	$2{,}1489 \times 10^{22}$	19 415,04	18 780	-0,03271

Table 1 - Mass of planets (M_p) and satellites (M_s) of the solar system in kg and reciprocal of the sum in relation to the mass of the Sun ($M_S = 1{,}9891 \times 10^{30}$ kg).

μ is the mass parameter adjustment, satisfying

$$m_{corrected} = m_{preliminar}(1 + \mu), \tag{1}$$

where

$$m = \frac{M_{planet}}{M_{Sun}}, \tag{2}$$

i.e., the mass of the planet relative to the mass of the Sun.

Other invariable elements of the planets, and also required for calculation of variable elements, are shown in table 2 below:

Planet	Mean motion in one julian year (n)	Mean distance from the Sun (a) (U.A.)
Mercury (0)	5 381 016'',200 0	0,387 098 7
Venus (1)	2 106 641'',438 0	0,723 332 3
Earth (2)	1 295 977'',440 0	1,000 000 0
Mars (3)	689 050'',902 3	1,523 687 8
Jupiter (4)	109 256'',719 0	5,202 798 0
Saturn (5)	43 996'',127 0	9,538 852 0
Uranus (6)	15 424'',509 4	19,183 581 0
Neptune (7)	7 873'',993 0	30,033 860 0

Table 2 – Mean motion in one julian year and mean distance from the Sun to the planets of the solar system, according to Stockwell.

Current values of the elements listed in table 2 are given in tables 3 and 4 (obtained from *Wikipedia*). The mean motion was obtained by the formula

$$n = \frac{365{,}25 \times 360 \times 60 \times 60}{P}, \qquad (3)$$

where P is the orbital period in days.

Planet	Average orbital speed (km/s)	Orbital period (d)	Mean motion in one julian year ('')
Mercury (0)	47,87	87,969 1	5 381 025, 837 5
Venus (1)	35,02	224,701	2 106 639,489 8
Earth (2)	29,78	365,256 363 004	1 295 977, 422 8
Mars (3)	24,077	686,971	689 059, 654 6
Jupiter (4)	13,07	4 331,572	109 282,265 2
Saturn (5)	9,69	10 759,22	43 996,126 1
Uranus (6)	6,81	30 799,095	15 402,418 8
Neptune (7)	5,43	60 190,030	7 864,491 8

Table 3 - Speed and orbital period and mean motion of the planets of the solar system, current data.

Planet	Perihelion (U.A.)	Aphelion (U.A.)	Mean distance from the Sun (a) (U.A.)
Mercury (0)	0,307 499	0,466 697	0,387 098
Venus (1)	0,718	0,728	0,723
Earth (2)	0,983 291 34	1,016 713 88	1,000 002 61
Mars (3)	1,381 497	1,665 861	1,523 679
Jupiter (4)	4,950 429	5,458 104	5,204 267
Saturn (5)	9,048 076 35	10,115 958 04	9,582 017 20
Uranus (6)	18,375 518 63	20,083 305 26	19,229 411 95
Neptune (7)	29,766 070 95	30,441 252 06	30,103 661 51

Table 4 - Perihelion, aphelion and mean distances (major semi-axis) from the Sun to planets of the solar system, current data.

The values used by Stockwell for constant elements and our respective current values are approximately equal, but none of them is exactly the same, not even the average distance from Earth to the Sun. Mean motions of the Earth and Saturn used by Stockwell, however, are with excellent approximations, as did the average distances to the Sun of Mercury, Venus, Earth and Mars. Otherwise the approximations can be considered good or reasonable.

The values of the precession of the perihelion of Mercury obtained due to the influence of other planets, with and without adjustment masses, with and without satellites, are recorded in table 5 below, rounded to two decimal digits after the decimal point. Will be added to each of these values the precession of the equinoxes on Earth in relation to the apparent ecliptic, which calculation based on Stockwell (for the period 1850-1950) provides $\Delta\psi_1 =$ 5024'',749 831 ≈ 5024'',75. The actual calculation of the perihelion advance in relation to classical theory uses as reference the value of 5600'',73 [6].

Masses adjustment	Satellites mass	Precession ('')	Perihelion advance ('')
= 0	without satellites	548,69	27,29
≠ 0	without satellites	543,77	32,21
≠ 0	with satellites	544,93	31,05

Table 5 - Values of the secular precession of the perihelion of Mercury, based in Stockwell.

The values tabulated above correspond to the 100-year period from 1850 to 1950 (January 1), and is noted that the advance of the perihelion for the three cases is less than the value currently accepted [6]: (43.11 ± 0, 45)'', i.e., calculations based on Stockwell are closer to the observed values than the current [6], and are also better than the values of Le Verrier [1] and Newcomb [7].

The longitude $\varpi^{(i)}$ of the perihelion of a planet (i) of the solar system, taking into account only the mutual influence of the planets, and according to the Celestial Mechanics of Laplace [4], is obtained from the arctangent of the ratio between a sum of sines ($h^{(i)}$) and a sum of cosines ($l^{(i)}$), such that

$$\varpi^{(i)} = arctg\frac{h^{(i)}}{l^{(i)}}\,, \tag{4}$$

where

$$h^{(i)} = e^{(i)} sen\,\varpi^{(i)} \tag{5}$$
$$l^{(i)} = e^{(i)} cos\,\varpi^{(i)} \tag{6}$$
$$e^{2\,(i)} = h^{2\,(i)} + l^{2\,(i)} \tag{7}$$

with the index (0) referring to Mercury, (1) to Venus, (2) to the Earth, etc., and $e^{(i)}$ is the eccentricity of the orbit of the planet (i).

The solutions to the various h and l must meet the 16 linear ordinary differential equations system of first degree

$$\begin{cases} \frac{dh^{(i)}}{dt} = \left\{\sum_{k=0,k\neq i}^{7}(i,k)\right\} l^{(i)} - \sum_{k=0,k\neq i}^{7}[i,k]\, l^{(k)} \\ \frac{dl^{(i)}}{dt} = -\left\{\sum_{k=0,k\neq i}^{7}(i,k)\right\} h^{(i)} + \sum_{k=0,k\neq i}^{7}[i,k]\, h^{(k)} \end{cases} \tag{8}$$

for i equals 0 to 7, corresponding to the eight planets of the solar system (nothing prevents adding up to 8, including Pluto, as it also orbits around the sun and was considered planet, but its contribution would be negligible, and the contributions of other more distant bodies).

The following notation is used above:

$$(i,k) = -\frac{3m^{(k)}n^{(i)}a^{2(i)}a^{(k)}(a^{(i)},a^{(k)})'}{4(a^{2(k)}-a^{2(i)})^2} = \tag{9}$$
$$= -\frac{3m^{(k)}n^{(i)}\alpha^2 b_{-1/2}^{(1)}}{4(1-\alpha^2)^2},$$

where m is the mass of the planet relative to the Sun's mass, n is the mean motion, a is the mean distance from the Sun and

$$\alpha = \frac{a^{(i)}}{a^{(k)}}. \tag{10}$$

$(a,a'), (a,a')', (a,a')''$, etc. are the coefficients of the cosine series development of

$$(a^2 - 2aa'cos\theta + a'^2)^{1/2} = (a, a') + (a, a')'cos\theta + (a, a')''cos2\theta + \cdots + (a, a')^{(n)}cosn\theta + \cdots \quad (11)$$

and

$$(a, a')' = a'b^{(1)}_{-1/2}, \quad (12)$$

$$b^{(1)}_{-1/2} = -\frac{1}{3}\cdot(1-\alpha^2)^2\cdot 2\alpha\cdot \left\{\frac{3}{2} + \frac{3}{2}\cdot\frac{3\cdot 5}{2\cdot 4}\cdot\alpha^2 + \right. \left. + \frac{3\cdot 5}{2\cdot 4}\cdot\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\cdot\alpha^4 + \cdots\right\}. \quad (13)$$

We also use

$$[i, k] = -\frac{3m^{(k)}n^{(i)}\alpha\left\{(1+\alpha^2)b^{(1)}_{-1/2} + \frac{1}{2}\alpha b^{(0)}_{-1/2}\right\}}{2(1-\alpha^2)^2}, \quad (14)$$

with

$$b^{(0)}_{-1/2} = (1-\alpha^2)^2\cdot 2\cdot\{1 + \left(\frac{3}{2}\right)^2\cdot\alpha^2 + \left(\frac{3\cdot 5}{2\cdot 4}\right)^2\cdot\alpha^4 + + \left(\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\right)^2\cdot\alpha^6 + \cdots\}. \quad (15)$$

The values of $b^{(0)}_{-1/2}$ are positive and $b^{(1)}_{-1/2}$ are negative, which is easy to see, while (i, k) and [i, k] have the same sign equal to the sign of $n^{(i)}$.

As an example, Stockwell obtains the following values for the coefficients of the disturbance suffered by Mercury:

$$\begin{aligned} (0,1) &= (1 + \mu')\cdot 2'', 9986729 \\ (0,2) &= (1 + \mu'')\cdot 0'', 8617070 \\ (0,3) &= (1 + \mu''')\cdot 0'', 0279815 \\ (0,4) &= (1 + \mu^{IV})\cdot 1'', 6028375 \\ (0,5) &= (1 + \mu^{V})\cdot 0'', 0772642 \\ (0,6) &= (1 + \mu^{VI})\cdot 0'', 0013324 \\ (0,7) &= (1 + \mu^{VII})\cdot 0'', 0004603 \end{aligned} \quad (16)$$

and

$$[0,1] = (1 + \mu')\cdot 1'', 926868$$

$$[0,2] = (1 + \mu'') \cdot 0'',4087579$$
$$[0,3] = (1 + \mu''') \cdot 0'',008812816$$
$$[0,4] = (1 + \mu^{IV}) \cdot 0'',1489646 \qquad (17)$$
$$[0,5] = (1 + \mu^{V}) \cdot 0'',00391854$$
$$[0,6] = (1 + \mu^{VI}) \cdot 0'',0000336068$$
$$[0,7] = (1 + \mu^{VII}) \cdot 0'',00000741495.$$

The sum $\sum_{k=1}^{7}(0,k)$ is especially important because it represents the constant part of the angular velocity of the perihelion of Mercury on the time t in julian years, without taking into account part of this variable speed: the eccentricities of the planets and the cosines of the differences $(\varpi^{(k)} - \varpi^{(0)})$. In general we have (Méc Cél, p. 611, eq. [1126]):

$$\frac{d\varpi^{(i)}}{dt} = \sum_{k=0,k\neq i}^{7}(i,k) - \sum_{k=0,k\neq i}^{7}[i,k]\frac{e^{(k)}}{e^{(i)}}cos(\varpi^{(k)} - \varpi^{(i)}). \qquad (18)$$

In the specific case of Mercury, adding the values given in (16) and without mass adjustments, we obtain

$$\sum_{k=1}^{7}(0,k) = 5'',5702558. \qquad (19)$$

With the mass adjustments of table 1 we obtain

$$\sum_{k=1}^{7}(0,k) = 5'',5351790. \qquad (20)$$

Let us then, as a more accurate estimate of $\frac{d\varpi^{(0)}}{dt}$, calculate (18) for the year 1850, taking as reference the eccentricities and initial values of the various $\varpi^{(k)}$ given by Stockwell in your tables (pp. 187-195).

k	Planet	[0, k] / (1+ μ)	μ	e	ϖ
0	Mercury	-x-	-0,1922	0,2056180	75º 07’ 00’’,0
1	Venus	1’’,926868	-0,04544	0,0068420	129º 28’ 52’’,0
2	Earth	0’’,4087579	0,1209	0,0167712	100º 21’ 41’’,0
3	Mars	0’’,008812816	-0,1352	0,0931324	333º 17’ 47’’,8
4	Jupiter	0’’,1489646	0,0003809	0,0482388	11º 54’ 53’’,1
5	Saturn	0’’,00391854	0,0009605	0,0559956	90º 06’ 12’’,0
6	Uranus	0’’,0000336068	0,08704	0,0462149	170º 34’ 17’’,7
7	Neptune	0’’,00000741495	-0,03271	0,0091739	50º 16’ 38’’,6

Table 6 - Values for calculating $\sum_{k=1}^{7}[0,k]\frac{e^{(k)}}{e^{(0)}}cos(\varpi^{(k)} - \varpi^{(0)})$.

Making the calculation of the second sum we obtain

$$\sum_{k=1}^{7}[0,k]\frac{e^{(k)}}{e^{(0)}}cos\big(\varpi^{(k)}-\varpi^{(0)}\big)=0'',085547. \qquad (21)$$

Subtracting (21) from (20) we obtain for (18), relative to Mercury, the value

$$\frac{d\varpi^{(0)}}{dt}(t=0)=5'',449632, \qquad (22)$$

equivalent to 544'',96 arc per century, close to that obtained in Table 5 (544'',93), through a mean value of solutions of the system (8).

For a more accurate calculation of the value above, recalculating the coefficients (0, k) e [0, k] using the actual values of m, n, a, supposedly constant, found in tables 1, 3 and 4, respectively, we obtain the following results, as shown in table 7.

k	Planet	α	$b_{-1/2}^{(0)}$	$-b_{-1/2}^{(1)}$	(0, k)	[0, k]
1	Venus	0,535405	2,14610541	0,57451806	3,19697528	3,42334861
2	Earth	0,387097	2,07565165	0,40173925	1,02192663	0,79116803
3	Mars	0,254055	2,03240427	0,25817105	0,02479246	0,01259726
4	Jupiter	0,074381	2,00276722	0,07448380	1,60540264	0,23882212
5	Saturn	0,040398	2,00081610	0,04041487	0,07634219	0,00616819
6	Uranus	0,020131	2,00020262	0,02013256	0,00143829	0,00005791
7	Neptune	0,012859	2,00008268	0,01285937	0,00044213	0,00001137

Table 7 - Values of (0, k) e [0, k] for calculating $\frac{d\varpi^{(0)}}{dt}$ relative to the beginning of 1850.

Using the coefficients calculated above and the parameters of eccentricities and longitudes of the perihelion given in table 6, (20) is recalculated as

$$\sum_{k=1}^{7}(0,k)=5'',92731962 \qquad (23)$$

and for (21) we obtain

$$\sum_{k=1}^{7}[0,k]\frac{e^{(k)}}{e^{(0)}}cos\big(\varpi^{(k)}-\varpi^{(0)}\big)=0'',15466066. \qquad (24)$$

The final result for the angular velocity of the perihelion of Mercury is then, according to (18), the difference between (23) and (24), i.e.,

$$\frac{d\varpi^{(0)}}{dt}(t=0) = 5'', 77265895 \tag{25}$$

of arc per year, or approximately 577'', 27 of arc per century.

Adding (25) to 50'',23572 annual precession of the equinoxes calculated for 1850, according to Stockwell [5] (pg. 175), we arrive at 56'',00837895 of arc per year, or about 5600'',84 of arc per century, in accordance with the observed value of the secular motion of precession of the perihelion of Mercury, according to Weinberg [6]: 5600'',73 ± 0'',41.

Within the experimental precision, the theoretical value obtained by the theory of Stockwell, which is the Newtonian theory the same theory of Laplace, is in agreement with the observed value, then it is not true to say that the classical, Newtonian theory, is not able to explain the advance of secular precession of the perihelion of the planets, and Mercury in particular. Rather, the Newton's gravitation explains with surprising accuracy.

See that our calculations were based on the year 1850, because it is the reference time used by Stockwell. Most likely fixes for the most recent 1950, 2000, 2014, etc. will reach another overral value to this precession, but must proceed in accordance with its observed value of the epoch, their values not differing much from one second of arc per century. The precession of the equinoxes is the largest component in the calculation of the total value of the precession of the perihelion, so it must be the object of careful attention.

If for some reason our calculations were not so surprisingly coincident with the observational result, they would already be able to show the most important: the precession of the perihelion obtained with General Relativity, equal to 43'',03 of arc per century [6], is completely at odds with any hypothetical advance of this precession, because this movement (or deviation, difference) would be much lower, for example, the one obtained with the coefficients of Stockwell, about 31'',05 arc per century (table 5). However, the difference between theory and observations obtained here is, essentially, zero: theoretical value = value observed, within the measurement accuracy. I.e.: General Relativity does not explain the precession of the perihelion of Mercury.

We close this note clarifying that do not exactly reproduce the calculations of Stockwell, but we rely on it. Our calculations initially used their coefficients and data, we even used all the coefficients obtained for the solution of the system (8), given by sums of sines and cosines, but we used our mass corrections, the masses of the planets added to the masses of the satellites, and the calculations were made by computer programs in C language, using *double* variables. The average annual motion of the perihelion of Mercury really calculated by Stockwell is 5'',463803 (pg. xi, Introduction).

In Laplace was found (13) and (15) into infinite series, recalling the known series expansions of elliptic integrals, while in Stockwell these polynomials in α are converted to decimal numbers with up to 7 significant digits; $b_{-1/2}^{(0)}$ be a polynomial of degree 30 and $b_{-1/2}^{(1)}$ a polynomial of degree 31 in α, indicating clearly that the two series are indeed endless.

Laplace tells us that both series only converge for $\alpha < 1$, otherwise (and if $\alpha \neq 1$) we should calculate (k, i) and [k, i] instead of (i, k) and [i, k], using the following relations:

$$(i,k) = (k,i)\,\frac{m^{(k)}n^{(i)}a^{(i)}}{m^{(i)}n^{(k)}a^{(k)}} \tag{26}$$

and

$$[i,k] = [k,i]\,\frac{m^{(k)}n^{(i)}a^{(i)}}{m^{(i)}n^{(k)}a^{(k)}}. \tag{27}$$

Furthermore, the important equation (18) we also find in Laplace only, not in Stockwell. The system (8) becomes unnecessary when what we want is just to calculate the value of the instantaneous temporal variation of ϖ, instead of the exact value of ϖ at some time t, and have the values of the various $\varpi^{(k)}$ e $e^{(k)}$ previously tabulated, as the example shown here.

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