SET OF ALL PAIRS OF TWIN PRIME NUMBERS IS INFINITE

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Abstract

In this paper we formulate an intuitive Hypothesis about a new aspect of a well known method called “Sieve of Eratosthenes” and then prove that set of natural numbers $N = \{1, 2, \ldots\}$ contains infinite number of pairs of twin primes.

Introduction. Following established tradition, as $p_i, i \in N$, we shall denote $i$-th prime ($p_1 = 2$). $N^+$ shall designate numerical series $N \cup \{0\}$. $r(a \div b)$, where $b \neq 0$, shall designate the remainder of division of $a$ by $b$, assuming $0 \leq r(a \div b) < b$. Set of possible remainders from division by $b$, i.e. $\{0, 1, \ldots, b - 1\}$, will be denoted as $R(b)$. If $A$ is a finite set of numbers, then $|A|$ shall mean the count of integers in $A$. 
If \( i \in \mathbb{N} \) and \( S_i \subseteq R(p_i) \), then

\[
M_i(S_i) = \{ x \in \mathbb{N} \mid r\left(\frac{x}{p_i}\right) \in S_i \}.
\]

It is clear that if \( S_i = R(p_i) \) then \( M_i(S_i) = \mathbb{N} \), and if \( S_i \subset R(p_i) \) then sets \( M_i(S_i) \) and \( \mathbb{N} \setminus M_i(S_i) \) are infinite.

\section{The Hypothesis}

If \( i \in \mathbb{N} \), \( S_i \subset R(p_i) \) and \( |S_i| = 1 \), then increasing sequence of all elements of the set

\[
M_i(S_i) = \{ x \in \mathbb{N} \mid r\left(\frac{x}{p_i}\right) \in S_i \}
\]

will be an arithmetic progression with common difference of \( p_i \). Set of \( M_i(S_i) \), where \( 1 < |S_i| < p_i \), which combines corresponding arithmetic progressions, will be called a group arithmetic progression with common difference of \( p_i \).

It is clear that for every \( i \in \mathbb{N} \), if \( S_i \subset R(p_i) \), then

\[
\lim_{q \to \infty} \frac{|M_i(S_i) \cap \{ x \in \mathbb{N} \mid x \leq q \}|}{q} = \frac{|S_i|}{p_i}.
\]

Let \( n \in \mathbb{N} \) and \( S_i \subset R(p_i) \) for every \( i \leq n \). Let \( D_n \) is a set such that

\[
D_n = \bigcap_{i=1}^{n} M_i(S_i) = \{ x \in \mathbb{N} \mid r\left(\frac{x}{p_1}\right) \in S_1, \ldots, r\left(\frac{x}{p_n}\right) \in S_n \},
\]

and \( D_n(q) \), where \( q \in \mathbb{N} \) and \( q > p_n \), is a set such that

\[
D_n(q) = D_n \cap \{ x \in \mathbb{N} \mid p_n < x \leq q \}.
\]
Since all arithmetic progressions included in the group are disjoint, then for every $1 < i \leq n$

$$D_i = \bigcup_{k \in S_i} \{D_{i-1} \cap M_i(\{k\})\},$$

which, in view of mutual primeness of $p_i$ with $p_1, \ldots, p_{i-1}$, makes it quite probable that

$$\lim_{q \to \infty} |D_i(q)| = \lim_{q \to \infty} |D_{i-1}(q)| \cdot \frac{|S_i|}{p_i}.$$

Since all common differences $p_1, \ldots, p_n$ of arithmetic (possibly group) progressions included into $D_n$ are mutually prime, it is therefore very likely that

$$\lim_{q \to \infty} |D_n(q)| = \prod_{i=1}^{n} \frac{|S_i|}{p_i}.$$

**Hypothesis.**

$$|D_n(q)| \approx (q - p_n) \cdot \prod_{i=1}^{n} \frac{|S_i|}{p_i}.$$

(Approximation $\approx$ is due to finiteness of the set $\{x \in \mathbb{N} \mid p_n < x \leq q\}$.)

Let us consider two extreme cases - the minimum and the maximum sets $D_n$:

**Case 1.**

Let for any $i \leq n$ set $S_i = \{0\}$, then

$$D_n = \bigcap_{i=1}^{n} M_i(S_i) = \{x \in \mathbb{N} \mid r\left(\frac{x}{p_1}\right) = 0, \ldots, r\left(\frac{x}{p_n}\right) = 0\}.$$

It is sufficient to mention the well known arithmetic fact, that the intersection of two arithmetic progressions with mutually prime common differences $p_x, p_y$ is always an arithmetic progression with common difference of $p_x \times p_y$.

Therefore, this part of the Hypothesis, having the form
|\(D_n(q)| \approx (q - p_n) \cdot \frac{1}{p_1 \times \ldots \times p_n}\), is provable in Peano Arithmetic.

**Case 2.**

Let \(S_1 = R(p_1) \setminus \{0\}, \ldots, S_n = R(p_n) \setminus \{0\}\), then

\[
D(n) = \bigcap_{i=1}^{n} M_i(S_i) = \{x \in \mathbb{N} | r(x, p_1) \neq 0, \ldots, r(x, p_n) \neq 0\}
\]

and by **Hypothesis**

\[
|D_n(q)| \approx (q - p_n) \cdot \prod_{i=1}^{n} \frac{|S_i|}{p_i}
\]

It is clear that set \(D_n\) is a set of numbers that are mutually prime with \(p_1, p_2, \ldots, p_n\). It is also clear that \(D_n(p_n^2)\) is a set of approximately all primes that are greater than \(p_n\) and less than \(p_n^2\). If we follow the trend \(Td_n\) of changes of \(|D_n(p_n^2)|\) with increasing \(n\), we will notice that

\[
Td_n = \frac{|D_{n+1}(p_{n+1}^2)|}{|D_n(p_n^2)|} = \frac{p_{n+1}^2 - p_{n+1} \cdot (p_n + c - 1)}{p_n^2 - p_n} \cdot \frac{p_n + c}{p_{n+1}}
\]

substituting \(p_{n+1} = p_n + c\):

\[
Td_n = \frac{(p_n + c)^2 - (p_n + c) \cdot (p_n + c - 1)}{p_n^2 - p_n} \cdot \frac{p_n + c}{p_{n+1}} = \frac{p_n^2 + 2p_n \cdot c + c^2 - p_n - c \cdot (p_n + c - 1)}{p_n^2 - p_n} \cdot \frac{p_n + c}{p_{n+1}}.
\]

Since \(c \geq 2\) for any \(n > 1\), therefore \(Td_n > 1\) for any \(n > 1\). Taking into account the fact that \(|D_2(3^2)| > 0\), this means that there are infinitely many prime numbers in \(\mathbb{N}\) (which is proved in Peano Arithmetic).

Consistency with the axioms of Peano Arithmetic of the two extreme cases of **Hypothesis** suggests that all intermediate cases of the **Hypothesis** are consistent with axioms of Peano Arithmetic.
2 Twin prime numbers

It is well known that all prime numbers greater than 3 are numbers of the form $6m - 1$ or $6m + 1$. Therefore, every pair of twin prime numbers “surrounds” some integer which is a multiple of six, which becomes the “middle” of the pair. Taking this fact into account, let us define set $D_n, n > 2$ as follows:

$$D_n = \{ x \in N \mid r(\frac{x}{p_1}) = r(\frac{x}{p_2}) = 0 \& \bigwedge_{i=3}^{n} (r(\frac{x}{p_i}) \neq 1 \& r(\frac{x}{p_i}) \neq p_i - 1) \}.$$

It is clear that set $D_n$ contains only numbers $6m, m > 1$; moreover, the numbers $6m - 1$ and $6m + 1$ are mutually prime with $p_1, \ldots, p_n$. It is also clear that set $D_n(p_n^2)$ is a set of approximately all the “middles” of pairs of twin prime numbers laying between $p_n$ and $p_n^2$. If we follow the trend $Td_n$ of changes of $|D_n(p_n^2)|$ with increasing $n$ ($n \geq 3$), we will notice that ($c \geq 2$)

$$Td_n = \frac{|D_{n+1}(p_{n+1}^2)|}{|D_n(p_n^2)|} = \frac{(p_n + c)^2 - (p_n + c)}{p_n^2 - p_n} \cdot \frac{p_n + c - 2}{p_n + c} > 1.$$

Taken together with the apparent $|D_3(p_3^2)| > 0$, this fact means that there are infinitely many pairs of twin primes in the set of natural numbers $N$.

References
