

A Complete Proof of BEAL Conjecture

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Abstract

In 1997, Andrew Beal [1] announced the following conjecture : *Let A, B, C, m, n , and l be positive integers with $m, n, l > 2$. If $A^m + B^n = C^l$ then A, B , and C have a common factor.* We begin to construct the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$ with p, q integers depending of A^m, B^n and C^l . We resolve $x^3 - px + q = 0$ and we obtain the three roots x_1, x_2, x_3 as functions of p, q and a parameter θ . Since $A^m, B^n, -C^l$ are the only roots of $x^3 - px + q = 0$, we discuss the conditions that x_1, x_2, x_3 are integers. A numerical example is given.

Keywords: Prime numbers, divisibility, roots of polynomials of third degree.

O my Lord! Increase me further in knowledge.

(Holy Quran, Surah Ta Ha, 20:114.)

To my Wife Wahida

1 Introduction

In 1997, Andrew Beal [1] announced the following conjecture :

Conjecture 1.1. *Let A, B, C, m, n , and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{1.1}$$

then A, B , and C have a common factor.

In this paper, we give a complete proof of the Beal Conjecture. Our idea is to construct a polynomial $P(x)$ of three order having as roots A^m, B^n and $-C^l$ with the condition (1.1). In the next section, we do some preliminaries calculus to give the expressions of the three roots of $P(x) = 0$. The proof of the conjecture (1.1) is the subject of the section 3. At the end, a numerical example is presented.

We begin with the trivial case when $A^m = B^n$. The equation (1.1) becomes:

$$2A^m = C^l \tag{1.2}$$

then $2|C^l \implies 2|C \implies \exists c \in \mathbb{N}^* / C = 2c$, it follows $2A^m = 2^l c^l \implies A^m = 2^{l-1} c^l$.
As $l > 2$, then $2|A^m \implies 2|A \implies 2|B^n \implies 2|B$. The conjecture (1.1) is verified.

We suppose in the following that $A^m > B^n$.

2 Preliminaries Calculs

Let $m, n, l \in \mathbb{N}^* > 2$ and $A, B, C \in \mathbb{N}^*$ such:

$$A^m + B^n = C^l \quad (2.1)$$

We call:

$$\begin{aligned} P(x) &= (x - A^m)(x - B^n)(x + C^l) = x^3 - x^2(A^m + B^n - C^l) \\ &\quad + x[A^m B^n - C^l(A^m + B^n)] + C^l A^m B^n \end{aligned} \quad (2.2)$$

Using the equation (2.1), $P(x)$ can be written:

$$\boxed{P(x) = x^3 + x[A^m B^n - (A^m + B^n)^2] + A^m B^n(A^m + B^n)} \quad (2.3)$$

We introduce the notations:

$$p = (A^m + B^n)^2 - A^m B^n \quad (2.4)$$

$$q = A^m B^n(A^m + B^n) \quad (2.5)$$

As $A^m \neq B^n$, we have :

$$p > (A^m - B^n)^2 > 0 \quad (2.6)$$

Equation (2.3) becomes:

$$P(x) = x^3 - px + q \quad (2.7)$$

Using the equation (2.2), $P(x) = 0$ has three different real roots : A^m, B^n and $-C^l$.

Now, let us resolve the equation:

$$P(x) = x^3 - px + q = 0 \quad (2.8)$$

To resolve (2.8) let:

$$x = u + v \quad (2.9)$$

Then $P(x) = 0$ gives:

$$P(x) = P(u+v) = (u+v)^3 - p(u+v) + q = 0 \implies u^3 + v^3 + (u+v)(3uv - p) + q = 0 \quad (2.10)$$

To determine u and v , we obtain the conditions:

$$u^3 + v^3 = -q \quad (2.11)$$

$$uv = p/3 > 0 \quad (2.12)$$

Then u^3 and v^3 are solutions of the second ordre equation:

$$X^2 + qX + p^3/27 = 0 \quad (2.13)$$

Its discriminant Δ is written as :

$$\Delta = q^2 - 4p^3/27 = \frac{27q^2 - 4p^3}{27} = \frac{\bar{\Delta}}{27} \quad (2.14)$$

Let:

$$\begin{aligned} \bar{\Delta} &= 27q^2 - 4p^3 = 27(A^m B^n (A^m + B^n))^2 - 4[(A^m + B^n)^2 - A^m B^n]^3 \\ &= 27A^{2m} B^{2n} (A^m + B^n)^2 - 4[(A^m + B^n)^2 - A^m B^n]^3 \end{aligned} \quad (2.15)$$

Noting :

$$\alpha = A^m B^n > 0 \quad (2.16)$$

$$\beta = (A^m + B^n)^2 \quad (2.17)$$

we can write (2.15) as:

$$\bar{\Delta} = 27\alpha^2\beta - 4(\beta - \alpha)^3 \quad (2.18)$$

As $\alpha \neq 0$, we can also rewrite (2.18) as :

$$\bar{\Delta} = \alpha^3 \left(27\frac{\beta}{\alpha} - 4\left(\frac{\beta}{\alpha} - 1\right)^3 \right) \quad (2.19)$$

We call t the parameter :

$$t = \frac{\beta}{\alpha} \quad (2.20)$$

$\bar{\Delta}$ becomes :

$$\bar{\Delta} = \alpha^3 (27t - 4(t-1)^3) \quad (2.21)$$

Let us calling :

$$y = y(t) = 27t - 4(t-1)^3 \quad (2.22)$$

Since $\alpha > 0$, the sign of $\bar{\Delta}$ is also the signe of $y(t)$. Let us study the sign of y . We obtain $y'(t)$:

$$y'(t) = y' = 3(1+2t)(5-2t) \quad (2.23)$$

$y' = 0 \implies t_1 = -1/2$ and $t_2 = 5/2$, then the table of variations of y is given below:

t	$-\infty$	$-1/2$	$5/2$	4	$+\infty$
$1+2t$	-	0	+		+
$5-2t$	+		0	-	
$y'(t)$	-	0	+	0	-
$y(t)$	$+\infty$	0	54	0	$-\infty$

Fig. 1: The table of variation

The table of the variations of the function y shows that $y < 0$ for $t > 4$. In our case, we are interested for $t > 0$. For $t = 4$ we obtain $y(4) = 0$ and for $t \in]0, 4[\implies y > 0$. As we have $t = \frac{\beta}{\alpha} > 4$ because as $A^m \neq B^n$:

$$(A^m - B^n)^2 > 0 \implies \beta = (A^m + B^n)^2 > 4\alpha = 4A^m B^n \quad (2.24)$$

Then $y < 0 \implies \overline{\Delta} < 0 \implies \Delta < 0$. Then, the equation (2.13) does not have real solutions u^3 and v^3 . Let us find the solutions u and v with $x = u + v$ is a positive or a negative real and $u.v = p/3$.

2.1 Demonstration

Proof. The solutions of (2.13) are:

$$X_1 = \frac{-q + i\sqrt{-\Delta}}{2} \quad (2.25)$$

$$X_2 = \overline{X_1} = \frac{-q - i\sqrt{-\Delta}}{2} \quad (2.26)$$

We may resolve:

$$u^3 = \frac{-q + i\sqrt{-\Delta}}{2} \quad (2.27)$$

$$v^3 = \frac{-q - i\sqrt{-\Delta}}{2} \quad (2.28)$$

Writing X_1 in the form:

$$X_1 = \rho e^{i\theta} \quad (2.29)$$

with:

$$\rho = \frac{\sqrt{q^2 - \Delta}}{2} = \frac{p\sqrt{p}}{3\sqrt{3}} \quad (2.30)$$

$$\text{and } \sin\theta = \frac{\sqrt{-\Delta}}{2\rho} > 0 \quad (2.31)$$

$$\cos\theta = -\frac{q}{2\rho} < 0 \quad (2.32)$$

Then $\theta [2\pi] \in] + \frac{\pi}{2}, +\pi[$, let:

$$\boxed{\frac{\pi}{2} < \theta < +\pi \Rightarrow \frac{\pi}{6} < \frac{\theta}{3} < \frac{\pi}{3} \Rightarrow \frac{1}{2} < \cos\frac{\theta}{3} < \frac{\sqrt{3}}{2}} \quad (2.33)$$

and

$$\boxed{\frac{1}{4} < \cos^2\frac{\theta}{3} < \frac{3}{4}} \quad (2.34)$$

hence the expression of X_2 :

$$X_2 = \rho e^{-i\theta} \quad (2.35)$$

Let:

$$u = r e^{i\psi} \quad (2.36)$$

$$\text{and } j = \frac{-1 + i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \quad (2.37)$$

$$j^2 = e^{i\frac{4\pi}{3}} = -\frac{1 + i\sqrt{3}}{2} = \bar{j} \quad (2.38)$$

j is a complex cubic root of the unity $\iff j^3 = 1$. Then, the solutions u and v are:

$$u_1 = re^{i\psi_1} = \sqrt[3]{\rho}e^{i\frac{\theta}{3}} \quad (2.39)$$

$$u_2 = re^{i\psi_2} = \sqrt[3]{\rho}je^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+2\pi}{3}} \quad (2.40)$$

$$u_3 = re^{i\psi_3} = \sqrt[3]{\rho}j^2e^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+4\pi}{3}} \quad (2.41)$$

and similarly:

$$v_1 = re^{-i\psi_1} = \sqrt[3]{\rho}e^{-i\frac{\theta}{3}} \quad (2.42)$$

$$v_2 = re^{-i\psi_2} = \sqrt[3]{\rho}j^2e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi-\theta}{3}} \quad (2.43)$$

$$v_3 = re^{-i\psi_3} = \sqrt[3]{\rho}je^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{2\pi-\theta}{3}} \quad (2.44)$$

We may now choose u_k and v_h so that $u_k + v_h$ will be real. In this case, we have necessary :

$$v_1 = \overline{u_1} \quad (2.45)$$

$$v_2 = \overline{u_2} \quad (2.46)$$

$$v_3 = \overline{u_3} \quad (2.47)$$

We obtain as real solutions of the equation (2.10):

$$x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}\cos\frac{\theta}{3} > 0 \quad (2.48)$$

$$x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}\cos\frac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right) < 0 \quad (2.49)$$

$$x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}\cos\frac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right) > 0 \quad (2.50)$$

We compare the expressions of x_1 and x_3 , we obtain:

$$\begin{aligned} 2\sqrt[3]{\rho}\cos\frac{\theta}{3} &\stackrel{?}{>} \sqrt[3]{\rho}\left(-\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right) \\ 3\cos\frac{\theta}{3} &\stackrel{?}{>} \sqrt{3}\sin\frac{\theta}{3} \end{aligned} \quad (2.51)$$

As $\frac{\theta}{3} \in] + \frac{\pi}{6}, + \frac{\pi}{3}[$, then $\sin\frac{\theta}{3}$ and $\cos\frac{\theta}{3}$ are > 0 . Taking the square of the two members of the last equation, we get:

$$\frac{1}{4} < \cos^2\frac{\theta}{3} \quad (2.52)$$

which is true since $\frac{\theta}{3} \in] + \frac{\pi}{6}, + \frac{\pi}{3}[$ then $x_1 > x_3$. As A^m, B^n and $-C^l$ are the only real solutions of (2.8), we consider, as A^m is supposed great than B^n , the expressions:

$$\left\{ \begin{array}{l} A^m = x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}\cos\frac{\theta}{3} \\ B^n = x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}\cos\frac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right) \\ -C^l = x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}\cos\frac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right) \end{array} \right. \quad (2.53)$$

□

3 Proof of the Main Theorem

Main Theorem: Let A, B, C, m, n , and l be positive integers with $m, n, l > 2$. If:

$$A^m + B^n = C^l \quad (3.1)$$

then A, B , and C have a common factor.

Proof. $A^m = 2\sqrt[3]{\rho}\cos^2\frac{\theta}{3}$ is an integer $\Rightarrow A^{2m} = 4\sqrt[3]{\rho^2}\cos^2\frac{\theta}{3}$ is an integer. But:

$$\sqrt[3]{\rho^2} = \frac{p}{3} \quad (3.2)$$

Then:

$$A^{2m} = 4\sqrt[3]{\rho^2}\cos^2\frac{\theta}{3} = 4\frac{p}{3}\cos^2\frac{\theta}{3} = p\cdot\frac{4}{3}\cos^2\frac{\theta}{3} \quad (3.3)$$

As A^{2m} is an integer, and p is an integer then $\cos^2\frac{\theta}{3}$ must be written in the form:

$$\boxed{\cos^2\frac{\theta}{3} = \frac{1}{b} \quad \text{or} \quad \cos^2\frac{\theta}{3} = \frac{a}{b}} \quad (3.4)$$

with $b \in \mathbb{N}^*$, for the last condition $a \in \mathbb{N}^*$ and a, b co-primes.

3.1 Case $\cos^2\frac{\theta}{3} = \frac{1}{b}$

we obtain :

$$A^{2m} = p\cdot\frac{4}{3}\cos^2\frac{\theta}{3} = \frac{4\cdot p}{3\cdot b} \quad (3.5)$$

As $\frac{1}{4} < \cos^2\frac{\theta}{3} < \frac{3}{4} \Rightarrow \frac{1}{4} < \frac{1}{b} < \frac{3}{4} \Rightarrow b < 4 < 3b \Rightarrow b = 1, 2, 3$.

3.1.1 $b = 1$

$b = 1 \Rightarrow 4 < 3$ which is impossible.

3.1.2 $b = 2$

$b = 2 \Rightarrow A^{2m} = p\cdot\frac{4}{3}\cdot\frac{1}{2} = \frac{2\cdot p}{3} \Rightarrow 3|p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{2p}{3} = 2\cdot p' \quad (3.6)$$

But :

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 - 4\cos^2\frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4\frac{1}{2} \right) = \frac{p}{3} = \frac{3p'}{3} = p' \quad (3.7)$$

On the one hand:

$$\begin{aligned} A^{2m} &= (A^m)^2 = 2p' \Rightarrow 2|p' \Rightarrow p' = 2p'' \Rightarrow A^{2m} = 4p''^2 \\ &\Rightarrow A^m = 2p'' \Rightarrow 2|A^m \Rightarrow 2|A \end{aligned}$$

On the other hand:

$B^n C^l = p' = 2p'^2 \Rightarrow 2|B^n$ or $2|C^l$. If $2|B^n \Rightarrow 2|B$. As $C^l = A^m + B^n$ and $2|A$ and $2|B$, it follows $2|A^m$ and $2|B^n$ then $2|(A^m + B^n) \Rightarrow 2|C^l \Leftrightarrow 2|C$.

Then, we have : A, B and C solutions of (2.1) have a common factor. Also if $2|C^l$, we obtain the same result : A, B and C solutions of (2.1) have a common factor.

3.1.3 $b = 3$

$b = 3 \Rightarrow A^{2m} = p \cdot \frac{4}{3} \cdot \frac{1}{3} = \frac{4p}{9} \Rightarrow 9|p \Rightarrow p = 9p'$ with $p' \neq 1$ since $9 \ll p$ then $A^{2m} = 4p' \Rightarrow p'$ is not a prime. Let $\hat{A}\mu$ a prime with $\hat{A}\mu|p' \Rightarrow \hat{A}\mu|A^{2m} \Rightarrow \hat{A}\mu|A$.

On the other hand:

$$B^n C^l = \frac{p}{3} \left(3 - 4\cos^2 \frac{\theta}{3} \right) = 5p'$$

Then $\hat{A}\mu|B^n$ or $\hat{A}\mu|C^l$. If $\hat{A}\mu|B^n \Rightarrow \hat{A}\mu|B$. As $C^l = A^m + B^n$ and $\hat{A}\mu|A$ and $\hat{A}\mu|B$, it follows $\hat{A}\mu|A^m$ and $\hat{A}\mu|B^n$ then $\hat{A}\mu|(A^m + B^n) \Rightarrow \hat{A}\mu|C^l \Rightarrow \hat{A}\mu|C$.

Then, we have : A, B and C solutions of (2.1) have a common factor. Also if $\hat{A}\mu|C^l$, we obtain the same result : A, B and C solutions of (2.1) have a common factor.

3.2 Case $a > 1$, $\cos^2 \frac{\theta}{3} = \frac{a}{b}$

That is to say:

$$\cos^2 \frac{\theta}{3} = \frac{a}{b} \tag{3.8}$$

$$A^{2m} = p \cdot \frac{4}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{4 \cdot p \cdot a}{3 \cdot b} \tag{3.9}$$

and a, b verify one of the two conditions:

$$\boxed{\{3|p \text{ and } b|4p\}} \quad \text{or} \quad \boxed{\{3|a \text{ and } b|4p\}} \tag{3.10}$$

and using the equation (2.34), we obtain a third condition:

$$\boxed{b < 4a < 3b} \tag{3.11}$$

In these conditions, respectively, $A^{2m} = 4\sqrt[3]{\rho^2} \cos^2 \frac{\theta}{3} = 4\frac{p}{3} \cdot \cos^2 \frac{\theta}{3}$ is an integer.

Let us study the conditions given by the equation (3.10).

3.2.1 Hypothesis: $\{3|p \text{ and } b|4p\}$

3.2.1.1. Case $b = 2$ and $3|p$: $3|p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{4p \cdot a}{3b} = \frac{4 \cdot 3p' \cdot a}{3b} = \frac{4 \cdot p' \cdot a}{2} = 2 \cdot p' \cdot a \tag{3.12}$$

As:

$$\frac{1}{4} < \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{2} < \frac{3}{4} \Rightarrow a < 2 \Rightarrow a = 1 \quad (3.13)$$

But $a > 1$ then the case $b = 2$ and $3|p$ is impossible.

3.2.1.2. Case $b = 4$ and $3|p$: We have $3|p \Rightarrow p = 3p'$ with $p' \in \mathbb{N}^*$, it follows:

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3 \times 4} = p'.a \quad (3.14)$$

and:

$$\frac{1}{4} < \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{4} < \frac{3}{4} \Rightarrow 1 < a < 3 \Rightarrow a = 2 \quad (3.15)$$

But a, b are co-primes. Then the case $b = 4$ and $3|p$ is impossible.

3.2.1.3. Case: $b \neq 2, b \neq 4, b|p$ and $3|p$: As $3|p$ then $p = 3p'$ and :

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{a}{b} = \frac{4 \times 3p' a}{3 b} = \frac{4p'a}{b} \quad (3.16)$$

We consider the case: $b|p' \Rightarrow p' = bp''$ and $p'' \neq 1$ (if $p'' = 1$, then $p = 3b$, see sub-paragraph 2^{sd} sous-case equation (3.36)). Hence :

$$A^{2m} = \frac{4bp''a}{b} = 4ap'' \quad (3.17)$$

Let us calculate $B^n C^l$:

$$B^n C^l = \frac{p}{3} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = p' \left(3 - 4 \frac{a}{b} \right) = b.p'' \cdot \frac{3b - 4a}{b} = p'' \cdot (3b - 4a) \quad (3.18)$$

Finally, we have the two equations:

$$A^{2m} = \frac{4bp''a}{b} = 4ap'' \quad (3.19)$$

$$B^n C^l = p'' \cdot (3b - 4a) \quad (3.20)$$

Sous-case 1: p'' is prime. From (3.19), $p''|A^{2m} \Rightarrow p''|A^m \Rightarrow p''|A$. From (3.20), $p''|B^n$ or $p''|C^l$. If $p''|B^n \Rightarrow p''|B$, as $C^l = A^m + B^n \Rightarrow p''|C^l \Rightarrow p''|C$. If $p''|C^l \Rightarrow p''|C$, as $B^n = C^l - A^m \Rightarrow p''|B^n \Rightarrow p''|B$.

Then A, B and C solutions of (2.1) have a common factor.

Sous-case 2: p'' is not prime. Let λ one prime divisor of p'' . From (3.19), we have :

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as } \lambda \text{ is prime then } \lambda|A \quad (3.21)$$

From (3.20), as $\lambda|p''$ we have:

$$\lambda|B^n C^l \Rightarrow \lambda|B^n \quad \text{or } \lambda|C^l \quad (3.22)$$

If $\lambda|B^n$, λ is prime $\lambda|B$, and as $C^l = A^m + B^n$ then we have also :

$$\lambda|C^l \quad \text{as } \lambda \text{ is prime, then } \lambda|C \quad (3.23)$$

By the same way, if $\lambda|C^l$, we obtain $\lambda|B$.

Then: A, B and C solutions of (2.1) have a common factor.

Let us verify the condition (3.11) given by:

$$b < 4a < 3b$$

In our case, the last equation becomes:

$$p < 3A^{2m} < 3p \quad \text{with} \quad p = A^{2m} + B^{2n} + A^m B^n \quad (3.24)$$

The $3A^{2m} < 3p \implies A^{2m} < p$ is verified.

If :

$$p < 3A^{2m} \implies 2A^{2m} - A^m B^n - B^{2n} > 0$$

We put $Q(Y) = 2Y^2 - B^n Y - B^{2n}$, the roots of $Q(Y) = 0$ are $Y_1 = -\frac{B^n}{2}$ and $Y_2 = B^n$. $Q(Y) > 0$ for $Y < Y_1$ and $Y > Y_2 = B^n$. In our case, we take $Y = A^m$. As $A^m > B^n$ then $p < 3A^{2m}$ is verified. Then the condition $b < 4a < 3b$ is true.

In the following of the paper, we verify easily that the condition $b < 4a < 3b$ implies to verify $A^m > B^n$ which is true.

3.2.1.4. Case $b = 3$ and $3|p$: As $3|p \implies p = 3p'$ and we write :

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{a}{b} = \frac{4 \times 3p'}{3} \frac{a}{3} = \frac{4p'a}{3} \quad (3.25)$$

As A^{2m} is an integer and that a and b are co-primes and $\cos^2 \frac{\theta}{3}$ can not be one in reference to the equation (2.33), then we have necessary $3|p' \implies p' = 3p''$ with $p'' \neq 1$, if not $p = 3p' = 3 \times 3p'' = 9$ but $p = A^{2m} + B^{2n} + A^m B^n > 9$, the hypothesis $p'' = 1$ is impossible, then $p'' > 1$. hence:

$$A^{2m} = \frac{4p'a}{3} = \frac{4 \times 3p''a}{3} = 4p''a \quad (3.26)$$

$$B^n C^l = \frac{p}{3} \left(3 - 4\cos^2 \frac{\theta}{3} \right) = p' \left(3 - 4\frac{a}{b} \right) = \frac{3p''(9 - 4a)}{3} = p'' \cdot (9 - 4a) \quad (3.27)$$

As $\frac{1}{4} < \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{3} < \frac{3}{4} \implies 3 < 4a < 9 \implies a = 2$ as $a > 1$.
 $a = 2$, we obtain:

$$A^{2m} = \frac{4p'a}{3} = \frac{4 \times 3p''a}{3} = 4p''a = 8p'' \quad (3.28)$$

$$B^n C^l = \frac{p}{3} \left(3 - 4\cos^2 \frac{\theta}{3} \right) = p' \left(3 - 4\frac{a}{b} \right) = \frac{3p''(9 - 4a)}{3} = p'' \quad (3.29)$$

The two last equations give that p'' is not prime. Then we use the same methodology described above for the case 3.2.1.3., and we have : A, B and C solutions of (2.1) have a common factor.

3.2.1.5. Case $3|p$ and $b = p$: We have :

$$\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{a}{p}$$

and :

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{p} = \frac{4a}{3} \quad (3.30)$$

As A^{2m} is an integer, this implies that $3|a$, but $3|p \implies 3|b$. As a and b are co-primes, hence the contradiction. Then the case $3|p$ and $b = p$ is impossible.

3.2.1.6. Case $3|p$ and $b = 4p$: $3|p \implies p = 3p'$, $p' \neq 1$ because $3 \ll p$, hence $b = 4p = 12p'$.

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{a}{3} \implies 3|a \quad (3.31)$$

because A^{2m} is an integer. But $3|p \implies 3|(4p) = b$, that is in contradiction with the hypothesis a, b are co-primes. Then the case $b = 4p$ is impossible.

3.2.1.7. Case $3|p$ and $b = 2p$: $3|p \implies p = 3p'$, $p' \neq 1$ because $3 \ll p$, hence $b = 2p = 6p'$.

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{2a}{3} \implies 3|a \quad (3.32)$$

because A^{2m} is an integer. But $3|p \implies 3|(2p) \implies 3|b$, that is in contradiction with the hypothesis a, b are co-primes. Then the case $b = 2p$ is impossible.

3.2.1.8. Case $3|p$ and $b \neq 3$ is a divisor of p : We have $b = p' \neq 3$, and p is written as:

$$p = kp' \quad \text{with} \quad 3|k \implies k = 3k' \quad (3.33)$$

and :

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{4 \times 3k'p' a}{3 p'} = 4ak' \quad (3.34)$$

We calculate $B^n C^l$:

$$B^n C^l = \frac{p}{3} \cdot \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = k'(3p' - 4a) \quad (3.35)$$

1st Sous-case: $k' \neq 1$, we use the same methodology described for the case 3.1.2.3., and we obtain: A, B and C solutions of (2.1) have a common factor.

2nd sous-case:

$$k' = 1 \implies p = 3b \quad (3.36)$$

then we have:

$$A^{2m} = 4a \implies a \quad \text{is even} \quad (3.37)$$

and :

$$A^m B^n = 2 \sqrt[3]{\rho} \cos \frac{\theta}{3} \cdot \sqrt[3]{\rho} \left(\sqrt{3} \sin \frac{\theta}{3} - \cos \frac{\theta}{3} \right) = \frac{p\sqrt{3}}{3} \sin \frac{2\theta}{3} - 2a \quad (3.38)$$

let:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3} \sin \frac{2\theta}{3} = 2b\sqrt{3} \sin \frac{2\theta}{3} \quad (3.39)$$

The left member of (3.39) is an integer and b also, then $2\sqrt{3} \sin \frac{2\theta}{3}$ can be written in the form:

$$2\sqrt{3} \sin \frac{2\theta}{3} = \frac{k_1}{k_2} \quad (3.40)$$

where k_1, k_2 are two co-primes integers and $k_2|b \implies b = k_2.k_3$.

◇ - We suppose $k_3 \neq 1$. Hence:

$$A^{2m} + 2A^m B^n = k_3.k_1 \quad (3.41)$$

Let μ is a prime integer such that $\mu|k_3$. If $\mu = 2 \implies 2|b$ but $2|a$ that is contradiction with a, b co-primes. We suppose $\mu \neq 2$ and $\mu|k_3$, then $\mu|A^m(A^m + 2B^n) \implies \mu|A^m$ or $\mu|(A^m + 2B^n)$.

*A-1- If $\mu|A^m \implies \mu|A^{2m} \implies \mu|4a \implies \mu|a$. As $\mu|k_3 \implies \mu|b$ and that a, b are co-primes hence the contradiction.

*A-2- If $\mu|(A^m + 2B^n) \implies \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu|(A^m + 2B^n)$, we can write:

$$A^m + 2B^n = \mu.t' \quad t' \in \mathbb{N}^* \quad (3.42)$$

It follows:

$$A^m + B^n = \mu t' - B^n \implies A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}$$

Using the expression of p , we obtain:

$$p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^m) \quad (3.43)$$

As $p = 3b = 3k_2.k_3$ and $\mu|k_3$ hence $\mu|p \implies p = \mu\mu'$, so we have :

$$\mu' \mu = \mu(\mu t'^2 - 2t' B^n) + B^n (B^n - A^m) \quad (3.44)$$

and $\mu|B^n(B^n - A^m) \implies \mu|B^n$ or $\mu|(B^n - A^m)$.

*A-2-1- If $\mu|B^n \implies \mu|B$ which is in contradiction with *A-2.

*A-2-2- If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we obtain:

$$\mu|3B^n \implies \begin{cases} \mu|B^n \implies \mu|B \text{ which is impossible} \\ or \\ \mu = 3 \end{cases} \quad (3.45)$$

*A-2-2-1- If $\mu = 3 \implies 3|k_3 \implies k_3 = 3k'_3$, and we have $b = k_2 k_3 = 3k_2 k'_3$, it follows $p = 3b = 9k_2 k'_3$ then $9|p$, but $p = (A^m - B^n)^2 + 3A^m B^n$ then :

$$9k_2 k'_3 - 3A^m B^n = (A^m - B^n)^2$$

we write it as :

$$3(3k_2k'_3 - A^mB^n) = (A^m - B^n)^2 \quad (3.46)$$

hence $3|(3k_2k'_3 - A^mB^n) \implies 3|A^mB^n \implies 3|A^m$ or $3|B^n$.

*A-2-2-1-1- If $3|A^m \implies 3|A$ and we have also $3|A^{2m}$, but $A^{2m} = 4a \implies 3|4a \implies 3|a$. As $b = 3k_2k'_3$ then $3|b$, but a, b are co-primes hence the contradiction. Then $3 \nmid A$.

*A-2-2-1-2- If $3|B^n \implies 3|B$, but the (3.46) gives $3|(A^m - B^n)^2 \implies 3|(A^m - B^n) \implies 3|A^m \implies 3|A$. But using the result of the last paragraph *A-2-2-1-1, we obtain $3 \nmid A$. Then the hypothesis $k_3 \neq 1$ is impossible.

◇- Now we suppose that $k_3 = 1 \implies b = k_2$ and $p = 3b = 3k_2$. We have then:

$$2\sqrt{3}\sin\frac{2\theta}{3} = \frac{k_1}{b} \quad (3.47)$$

with k_1, b co-primes. We write (3.47) as :

$$4\sqrt{3}\sin\frac{\theta}{3}\cos\frac{\theta}{3} = \frac{k_1}{b}$$

Taking the square of the two members and replacing $\cos^2\frac{\theta}{3}$ by $\frac{a}{b}$, we obtain:

$$3 \times 4^2 \cdot a(b - a) = k_1^2 \quad (3.48)$$

which implies that :

$$3|a \quad \text{or} \quad 3|(b - a)$$

*B-1- If $3|a$, as $A^{2m} = 4a \implies 3|A^{2m} \implies 3|A$. But $p = (A^m - B^n)^2 + 3A^mB^n$ and that $3|p \implies 3|(A^m - B^n)^2 \implies 3|(A^m - B^n)$. But $3|A$ hence $3|B^n \implies 3|B$, it follows $3|C^l \implies 3|C$.

We obtain: A, B and C solutions of (2.1) have a common factor.

*B-2- Considering now that $3|(b - a)$. As $k_1 = A^m(A^m + 2B^n)$ by the equation (3.41) and that $3|k_1 \implies 3|A^m(A^m + 2B^n) \implies 3|A^m$ or $3|(A^m + 2B^n)$.

*B-2-1- If $3|A^m \implies 3|A \implies 3|A^{2m}$ then $3|4a \implies 3|a$. But $3|(b - a) \implies 3|b$ hence the contradiction with a, b are co-primes.

*B-2-2- If:

$$3|(A^m + 2B^n) \implies 3|(A^m - B^n) \quad (3.49)$$

But $p = A^{2m} + B^{2n} + A^mB^n = (A^m - B^n)^2 + 3A^mB^n$ then $p - 3A^mB^n = (A^m - B^n)^2 \implies 9|(p - 3A^mB^n)$ or $9|(3b - 3A^mB^n)$, then $3|(b - A^mB^n)$ but $3|(b - a) \implies 3|(a - A^mB^n)$. As $A^{2m} = 4a = (A^m)^2 \implies \exists a' \in \mathbb{N}^*$ and $a = a'^2 \implies A^m = 2a'$. We arrive to $3|(a'^2 - 2a'B^n) \implies 3|a'(a' - 2B^n)$.

*B-2-2-1- If $3|a' \implies 3|A^m \implies 3|A$, but $3|(A^m + 2B^n) \implies 3|2B^n \implies 3|B^n \implies 3|B$, it follows $3|C$.

Hence A, B and C solutions of (2.1) have a common factor.

*B-2-2-2- Now if $3|(a' - 2B^n) \implies 3|(2a' - 4B^n) \implies 3|(A^m - 4B^n) \implies 3|(A^m - B^n)$, we refine the hypothesis (3.49) above.

The study of the case 3.2.1.8. is finished.

3.2.1.9 Case $3|p$ and $b|4p$: As $3|p \implies p = 3p'$ and $b|4p \implies \exists k_1 \in \mathbb{N}^*$ and $4p = 12p' = k_1 b$.

** - $\boxed{k_1 = 1}$ then $b = 12p'$, ($p' \neq 1$ if not $p = 3 \ll A^{2m} + B^{2n} + A^m B^n$). But $A^{2m} = \frac{4p}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{12p' a}{3 b} = \frac{4p' a}{12p'} = \frac{a}{3} \implies 3|a$ because A^{2m} is an integer, then the contradiction with a, b co-primes.

** - $\boxed{k_1 = 3}$, then $b = 4p'$ and $A^{2m} = \frac{4p}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{k_1 a}{3} = a$.

Let us calculate $A^m B^n$:

$$A^m B^n = 2\sqrt[3]{\rho} \cos \frac{\theta}{3} \cdot \sqrt[3]{\rho} \left(\sqrt{3} \sin \frac{\theta}{3} - \cos \frac{\theta}{3} \right) = \frac{p\sqrt{3}}{3} \sin \frac{2\theta}{3} - \frac{a}{2} \quad (3.50)$$

let:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3} \sin \frac{2\theta}{3} = 2p' \sqrt{3} \sin \frac{2\theta}{3} \quad (3.51)$$

The left member of the equation (3.51) is an integer and also p' , then $2\sqrt{3} \sin \frac{2\theta}{3}$ can be written as :

$$2\sqrt{3} \sin \frac{2\theta}{3} = \frac{k_2}{k_3} \quad (3.52)$$

where k_2, k_3 are two co-primes integers and $k_3|p' \implies p' = k_3.k_4$.

◇ - We suppose that $\boxed{k_4 \neq 1}$, then:

$$A^{2m} + 2A^m B^n = k_2.k_4 \quad (3.53)$$

Let μ one prime integer with $\mu|k_4$. Then $\mu|A^m(A^m + 2B^n) \implies \mu|A^m$ or $\mu|(A^m + 2B^n)$.

*A-1- If $\mu|A^m \implies \mu|A^{2m} \implies \mu|a$. As $\mu|k_4 \implies \mu|p' \implies \mu|(4p' = b)$. But a, b are co-primes then the contradiction.

*A-2- If $\mu|(A^m + 2B^n) \implies \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu|(A^m + 2B^n)$, we can write:

$$A^m + 2B^n = \mu.t' \quad t' \in \mathbb{N}^* \quad (3.54)$$

It follows:

$$A^m + B^n = \mu t' - B^n \implies A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}$$

Using the expression of p , we obtain:

$$p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^m) \quad (3.55)$$

As $p = 3p'$ and $\mu|p' \Rightarrow \mu|(3p') \Rightarrow \mu|p$, we can write $\exists \mu' \in \mathbb{N}^*$ and $p = \mu\mu'$, then we obtain :

$$\mu'\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m) \quad (3.56)$$

and $\mu|B^n(B^n - A^m) \Rightarrow \mu|B^n$ or $\mu|(B^n - A^m)$.

*A-2-1- If $\mu|B^n \Rightarrow \mu|B$ which is in contradiction with *A-2.

*A-2-2- If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we obtain:

$$\mu|3B^n \Rightarrow \begin{cases} \mu|B^n \Rightarrow \mu|B \text{ which is impossible} \\ \text{ou} \\ \mu = 3 \end{cases} \quad (3.57)$$

*A-2-2-1- If $\mu = 3 \Rightarrow 3|k_4 \Rightarrow k_4 = 3k'_4$, and we obtain $p' = k_3k_4 = 3k_3k'_4$, it follows $p = 3p' = 9k_3k'_4$ then $9|p$, but $p = (A^m - B^n)^2 + 3A^mB^n$, then:

$$9k_4k'_5 - 3A^mB^n = (A^m - B^n)^2$$

that we write :

$$3(3k_4k'_5 - A^mB^n) = (A^m - B^n)^2 \quad (3.58)$$

then $3|(3k_4k'_5 - A^mB^n) \Rightarrow 3|A^mB^n \Rightarrow 3|A^m$ or $3|B^n$.

*A-2-2-1-1- If $3|A^m \Rightarrow 3|A^{2m} \Rightarrow 3|a$, but $3|p' \Rightarrow 3|(4p') \Rightarrow 3|b$, then the contradiction with a, b co-primes. Then $3 \nmid A$.

*A-2-2-1-2- If $3|B^n$ but $A^m = \mu t' - 2B^n = 3t' - 2B^n \Rightarrow 3|A^m$, which is in contradiction. Then the hypothesis $k_4 \neq 1$ is impossible.

◇- We suppose that $\boxed{k_4 = 1} \Rightarrow p' = k_3k_4 = k_3$. Then we obtain:

$$2\sqrt{3}\sin\frac{2\theta}{3} = \frac{k_2}{p'} \quad (3.59)$$

with k_2, p' co-primes, we write (3.59) as :

$$4\sqrt{3}\sin\frac{\theta}{3}\cos\frac{\theta}{3} = \frac{k_2}{p'}$$

Taking the square of the two members and replacing $\cos^2\frac{\theta}{3}$ by $\frac{a}{b}$ and $b = 4p'$, we obtain:

$$3.a(b - a) = k_2^2 \quad (3.60)$$

that implicate :

$$3|a \quad \text{or} \quad 3|(b - a)$$

*B-1- If $3|a \Rightarrow 3|A^{2m} \Rightarrow 3|A$, as $p = (A^m - B^n)^2 + 3A^mB^n$ and that $3|p \Rightarrow 3|(A^m - B^n)^2 \Rightarrow 3|(A^m - B^n)$. But $3|A$, then $3|B^n \Rightarrow 3|B$, it follows $3|C^l \Rightarrow 3|C$.

We obtain : A, B and C solutions of (2.1) have a common factor.

*B-2- We consider that $3|(b-a)$. As $k_2 = A^m(A^m + 2B^n)$ given by the equation (3.53) and that $3|k_2 \implies 3|A^m(A^m + 2B^n) \implies 3|A^m$ or $3|(A^m + 2B^n)$.

*B-2-1- If $3|A^m \implies 3|A^{2m} \implies 3|a$, but $3|(b-a) \implies 3|b$ then the contradiction with a, b co-primes.

*B-2-2- If:

$$3|(A^m + 2B^n) \implies 3|(A^m - B^n) \quad (3.61)$$

but $p = A^{2m} + B^{2n} + A^m B^n = (A^m - B^n)^2 + 3A^m B^n$ then $p - 3A^m B^n = (A^m - B^n)^2 \implies 9|(p - 3A^m B^n)$ or $9|(3p' - 3A^m B^n)$, then $3|(p' - A^m B^n) \implies 3|4(p' - 4A^m B^n) \implies 3|(b - 4A^m B^n)$ but $3|(b-a) \implies 3|(a - A^m B^n)$. As $3|(A^{2m} - 4A^m B^n) \implies 3|A^m(A^m - 4B^n)$.

*B-2-2-1- If $3|A^m \implies 3|A^{2m} \implies 3|a$, but $3|(b-a) \implies 3|b$ then the contradiction with a, b co-primes.

*B-2-2-2- Now if $3|(A^m - 4B^n) \implies 3|(A^m - B^n)$, we find the hypothesis of the beginning (3.61) above.

**- We suppose $k_1 \neq 3$ and $3|k_1 \Rightarrow \boxed{k_1 = 3k'_1}$ with $k'_1 \neq 1$. we have $4p = 12p' = k_1 b = 3k'_1 b \Rightarrow 4p' = k'_1 b$. A^{2m} can be written as :

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{3k'_1 b}{3} \frac{a}{b} = k'_1 a \quad (3.62)$$

and $B^n C^l$:

$$B^n C^l = \frac{p}{3} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{k'_1}{4} (3b - 4a) \quad (3.63)$$

As $B^n C^l$ is an integer, we must have $4|(3b - 4a)$ ou $4|k'_1$.

*** We suppose that $4|(3b - 4a) \Rightarrow \frac{3b - 4a}{4} = c \in \mathbb{N}^*$, and we obtain:

$$\begin{aligned} A^{2m} &= k'_1 a \\ B^n C^l &= k'_1 c \end{aligned}$$

C-1- If k'_1 is prime, then $k'_1|A^{2m} \Rightarrow k'_1|A$ and $k'_1|B^n C^l \Rightarrow k'_1|B^n$ or $k'_1|C^l$. If $k'_1|B^n \Rightarrow k'_1|B$, then $k'_1|C^l \Rightarrow k'_1|C$. With the same method if $k'_1|C^l$, we arrive to $k'_1|B$.

We obtain: A, B and C solutions of (2.1) have a common factor.

C-2- k'_1 not a prime. Let μ a prime divisor of k'_1 , as described in C-1- above, we obtain : A, B and C solutions of (2.1) have a common factor.

*** We suppose that $4|k'_1$.

C-3- $k'_1 = 4$ but this case is discussed in the second sous-case of the paragraph (3.2.1.8).

C-4- $k'_1 = 4k''_1$ with $k''_1 > 1$. Then, we have:

$$A^{2m} = 4k''_1 a \quad (3.64)$$

$$B^n C^l = k''_1 (3b - 4a) \quad (3.65)$$

C-4-1- If k''_1 is prime, then $k''_1 | A^{2m} \Rightarrow k''_1 | A$ and $k''_1 | B^n C^l \Rightarrow k''_1 | B^n$ or $k''_1 | C^l$. If $k''_1 | B^n \Rightarrow k''_1 | B$, then $k''_1 | C^l \Rightarrow k''_1 | C$. With the same method if $k''_1 | C^l$, we arrive to $k''_1 | B$.

We obtain: A, B and C solutions of (2.1) have a common factor.

C-4-2- k''_1 not a prime. Let μ a prime divisor of k''_1 , as described in C-4-1 above, we obtain : A, B and C solutions of (2.1) have a common factor.

3.2.2 Hypothesis : $\{3|a \text{ and } b|4p\}$

We have :

$$3|a \Rightarrow \exists a' \in \mathbb{N}^* / a = 3a' \quad (3.66)$$

3.2.2.1. Case $b = 2$ and $3|a$: A^{2m} is written as :

$$A^{2m} = \frac{4p}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{4p}{3} \cdot \frac{a}{2} = \frac{2 \cdot p \cdot a}{3} \quad (3.67)$$

Using the equation (3.66), A^{2m} becomes:

$$A^{2m} = \frac{2 \cdot p \cdot 3a'}{3} = 2 \cdot p \cdot a' \quad (3.68)$$

But $\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2} > 1$ which is impossible, then $b \neq 2$.

3.2.2.2. Case $b = 4$ and $3|a$: A^{2m} is written as :

$$A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{a}{b} = \frac{4p}{3} \cdot \frac{a}{4} = \frac{p \cdot a}{3} = \frac{p \cdot 3a'}{3} = p \cdot a' \quad (3.69)$$

$$\text{and } \cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4} < \left(\frac{\sqrt{3}}{2} \right)^2 = \frac{3}{4} \Rightarrow a' < 1 \quad (3.70)$$

which is impossible.

Then the case $b = 4$ is impossible.

3.2.2.3. Case $b = p$ and $3|a$: Then:

$$\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{p} \quad (3.71)$$

and:

$$A^{2m} = \frac{4p}{3} \cdot \cos^2 \frac{\theta}{3} = \frac{4p}{3} \cdot \frac{3a'}{p} = 4a' = (A^m)^2 \quad (3.72)$$

$$\exists a'' \in \mathbb{N}^* / a' = a''^2 \quad (3.73)$$

We calculate $A^m B^n$, hence:

$$\begin{aligned} A^m B^n &= p \cdot \frac{\sqrt{3}}{3} \sin \frac{2\theta}{3} - 2a' \\ \text{or } A^m B^n + 2a' &= p \cdot \frac{\sqrt{3}}{3} \sin \frac{2\theta}{3} \end{aligned} \quad (3.74)$$

The left member of (3.74) is an integer and p is also, then $2 \frac{\sqrt{3}}{3} \sin \frac{2\theta}{3}$ will be written as :

$$2 \frac{\sqrt{3}}{3} \sin \frac{2\theta}{3} = \frac{k_1}{k_2} \quad (3.75)$$

where k_1, k_2 are two co-primes integers and $k_2 | p \implies p = b = k_2 \cdot k_3, k_3 \in \mathbb{N}^*$.

◇ - We suppose that $k_3 \neq 1$. We obtain :

$$A^m(A^m + 2B^n) = k_1 \cdot k_3 \quad (3.76)$$

Let us μ a prime integer with $\mu | k_3$, then $\mu | b$ and $\mu | A^m(A^m + 2B^n) \implies \mu | A^m$ or $\mu | (A^m + 2B^n)$.

* If $\mu | A^m \implies \mu | A$ and $\mu | A^{2m}$, but $A^{2m} = 4a' \implies \mu | 4a' \implies (\mu = 2 \text{ but } 2 \nmid a')$ or $(\mu | a')$. Then $\mu | a$ hence the contradiction with a, b co-primes.

* If $\mu | (A^m + 2B^n) \implies \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. We write $\mu | (A^m + 2B^n)$ as:

$$A^m + 2B^n = \mu \cdot t' \quad t' \in \mathbb{N}^* \quad (3.77)$$

It follows:

$$A^m + B^n = \mu t' - B^n \implies A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n}$$

Using the expression of p :

$$p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^m) \quad (3.78)$$

Since $p = b = k_2 \cdot k_3$ and $\mu | k_3$ then $\mu | b \implies \exists \mu' \in \mathbb{N}^*$ and $b = \mu \mu'$, so we can write:

$$\mu' \mu = \mu (\mu t'^2 - 2t' B^n) + B^n (B^n - A^m) \quad (3.79)$$

From the last equation, we get $\mu | B^n (B^n - A^m) \implies \mu | B^n$ or $\mu | (B^n - A^m)$. If $\mu | B^n$ which is contradiction with $\mu \nmid B^n$. If $\mu | (B^n - A^m)$ and using $\mu | (A^m + 2B^n)$, on arrive to:

$$\mu | 3B^n \implies \begin{cases} \mu | B^n \implies \text{which is contradiction} \\ \text{or} \\ \mu = 3 \end{cases} \quad (3.80)$$

Si $\mu = 3$, then $3 | b$, but $3 \nmid a$ thus the contradiction with a, b co-primes.

◇ - We assume now $k_3 = 1$. Hence:

$$A^{2m} + 2A^m B^n = k_1 \quad (3.81)$$

$$b = k_2 \quad (3.82)$$

$$\frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3} = \frac{k_1}{b} \quad (3.83)$$

Taking the square of the last equation, we obtain:

$$\begin{aligned}\frac{4}{3}\sin^2\frac{2\theta}{3} &= \frac{k_1^2}{b^2} \\ \frac{16}{3}\sin^2\frac{\theta}{3}\cos^2\frac{\theta}{3} &= \frac{k_1^2}{b^2} \\ \frac{16}{3}\sin^2\frac{\theta}{3}\cdot\frac{3a'}{b} &= \frac{k_1^2}{b^2}\end{aligned}$$

Finally:

$$4^2a'(p-a) = k_1^2 \quad (3.84)$$

but $a' = a''^2$ then $p-a$ is a square. Let us:

$$\lambda^2 = p-a \quad (3.85)$$

The equation (3.116) becomes:

$$4^2a''^2\lambda^2 = k_1^2 \implies k_1 = 4a''\lambda \quad (3.86)$$

taking the positif square root. Using (3.113), we get :

$$k_1 = 4a''\lambda \quad (3.87)$$

But $k_1 = A^m(A^m + 2B^n) = 2a''(A^m + 2B^n)$, it follows:

$$A^m + 2B^n = 2\lambda \quad (3.88)$$

Let λ_1 prime $\neq 2$, a divisor of λ (if not $\lambda_1 = 2|\lambda \implies 2|\lambda^2 \implies 2|(p-a)$ but a is even, then $2|p \implies 2|b$ which is contradiction with a, b co-primes).

We consider $\lambda_1 \neq 2$ and :

$$\lambda_1|\lambda \implies \lambda_1|\lambda^2 \quad \text{and} \quad \lambda_1|(A^m + 2B^n) \quad (3.89)$$

$$\lambda_1|(A^m + 2B^n) \implies \lambda_1 \nmid A^m \quad \text{if not} \quad \lambda_1|2B^n \quad (3.90)$$

But $\lambda_1 \neq 2$ hence $\lambda_1|B^n \implies \lambda_1|B$, it follows:

$$\lambda_1|(p=b) \quad \text{and} \quad \lambda_1|A^m \implies \lambda_1|2a'' \implies \lambda_1|a \quad (3.91)$$

hence the contradiction with a, b co-primes.

We assume now $\lambda_1 \nmid A^m$. $\lambda_1|(A^m + 2B^n) \implies \lambda_1|(A^m + 2B^n)^2$ that is $\lambda_1|(A^{2m} + 4A^mB^n + 4B^{2n})$, we write it as $\lambda_1|(p + 3A^mB^n + 3B^{2n}) \implies \lambda_1|(p + 3B^n(A^m + 2B^n) - 3B^{2n})$. But $\lambda_1|(A^m + 2B^n) \implies \lambda_1|(p - 3B^{2n})$, as $\lambda_1|(p-a)$ hence by difference, we obtain $\lambda_1|(a - 3B^{2n})$ or $\lambda_1|(3a' - 3B^{2n}) \implies \lambda_1|3(a' - B^{2n}) \implies \lambda_1 = 3$ or $\lambda_1|(a' - B^{2n})$.

*A-1- If $\lambda_1 = 3$ but $3|a \implies 3|(p=b)$ hence the contradiction with a, b co-primes.

*A-2- If $\lambda_1|(a' - B^{2n}) \implies \lambda_1|(a''^2 - B^{2n}) \implies \lambda_1|(a'' - B^n)(a'' + B^n) \implies \lambda_1|(a'' + B^n)$ or $\lambda_1|(a'' - B^n)$, because $(a'' - B^n) \neq 1$ if not we obtain $a''^2 - B^{2n} =$

$a'' + B^n \Rightarrow a''^2 - a'' = B^n - B^{2n}$. The left member is positif and the right member is negatif, then the contradiction.

*A-2-1- If $\lambda_1|(a'' - B^n) \Rightarrow \lambda_1|2(a'' - B^n) \Rightarrow \lambda_1|(A^m - 2B^n)$ but $\lambda_1|(A^m + 2B^n)$ hence $\lambda_1|2A^m \Rightarrow \lambda_1|A^m$, $\lambda_1 \neq 2$, it follows $\lambda_1|A^m$ hence the contradiction with (3.122).

*A-2-2- If $\lambda_1|(a'' + B^n) \Rightarrow \lambda_1|2(a'' + B^n) \Leftrightarrow \lambda_1|(A^m + 2B^n)$. We refine the condition (3.121).

Then the case $k_3 = 1$ is impossible.

3.2.2.4. Case $b|p \Rightarrow p = b.p', p' > 1, b \neq 2, b \neq 4$ and $3|a$:

$$A^{2m} = \frac{4.p}{3} \cdot \frac{a}{b} = \frac{4.b.p'.3.a'}{3.b} = 4.p'.a' \quad (3.92)$$

We calculate $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 \sin^2 \frac{\theta}{3} - \cos^2 \frac{\theta}{3} \right) = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) \quad (3.93)$$

But $\sqrt[3]{\rho^2} = \frac{p}{3}$ hence using $\cos^2 \frac{\theta}{3} = \frac{3.a'}{b}$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \frac{3.a'}{b} \right) = p \cdot \left(1 - \frac{4.a'}{b} \right) = p'(b - 4a') \quad (3.94)$$

As $p = b.p'$, and $p' > 1$, we have then:

$$B^n C^l = p'(b - 4a') \quad (3.95)$$

$$\text{and } A^{2m} = 4.p'.a' \quad (3.96)$$

A - Let λ a prime divisor of p' (we suppose p' not prime). From (3.96), we have:

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as } \lambda \text{ is a prime, then } \lambda|A \quad (3.97)$$

From (3.95), as $\lambda|p'$ we have:

$$\lambda|B^n C^l \Rightarrow \lambda|B^n \quad \text{or } \lambda|C^l \quad (3.98)$$

If $\lambda|B^n$, λ is a prime $\lambda|B$, but $C^l = A^m + B^n$, then we have also :

$$\lambda|C^l \quad \text{as } \lambda \text{ is a prime, then } \lambda|C \quad (3.99)$$

By the same way, if $\lambda|C^l$, we obtain $\lambda|B$. then : A, B and C solutions of (2.1) have a common factor.

B - We suppose now that p' is prime, from the equations (3.95) and (3.96), we obtain then:

$$p'|A^{2m} \Rightarrow p'|A^m \Rightarrow p'|A \quad (3.100)$$

and:

$$p'|B^n C^l \Rightarrow p'|B^n \text{ or } p'|C^l \quad (3.101)$$

$$\text{If } p'|B^n \Rightarrow p'|B \quad (3.102)$$

$$\begin{aligned} \text{As } C^l = A^m + B^n \text{ and that } p'|A, p'|B \Rightarrow p'|A^m, p'|B^n \Rightarrow p'|C^l \\ \Rightarrow p'|C \end{aligned} \quad (3.103)$$

By the same way, if $p'|C^l$, we arrive to $p'|B$.

Hence: A, B and C solutions of (2.1) have a common factor.

3.2.2.5. Case $b = 2p$ and $3|a$: We have:

$$\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2p} \Rightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3} \cdot \frac{3a'}{2p} = 2a' \Rightarrow 2|A^m \Rightarrow 2|a \Rightarrow 2|a'$$

Then $2|a$ and $2|b$ which is contradiction with a, b co-primes.

3.2.2.6. Case $b = 4p$ and $3|a$: We have :

$$\cos^2 \frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4p} \Rightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3} \cdot \frac{3a'}{4p} = a'$$

Calculate $A^m B^n$, we obtain:

$$\begin{aligned} A^m B^n &= \frac{p\sqrt{3}}{3} \cdot \sin \frac{2\theta}{3} - \frac{2p}{3} \cos^2 \frac{\theta}{3} = \frac{p\sqrt{3}}{3} \cdot \sin \frac{2\theta}{3} - \frac{a'}{2} \Rightarrow \\ A^m B^n + \frac{A^{2m}}{2} &= \frac{p\sqrt{3}}{3} \cdot \sin \frac{2\theta}{3} \end{aligned} \quad (3.104)$$

let:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3} \sin \frac{2\theta}{3} \quad (3.105)$$

The left member of (3.105) is an integer and p is an integer, then $\frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3}$ will be written:

$$\frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3} = \frac{k_1}{k_2} \quad (3.106)$$

where k_1, k_2 are two co-primes integers and $k_2|p \Rightarrow p = k_2.k_3$.

◇ - Firstly, we suppose that $k_3 \neq 1$. Hence:

$$A^{2m} + 2A^m B^n = k_3.k_1 \quad (3.107)$$

Let μ a prime integer and $\mu|k_3$, then $\mu|A^m(A^m + 2B^n) \Rightarrow \mu|A^m$ or $\mu|(A^m + 2B^n)$.

* If $\mu|A^m \Rightarrow \mu|A$. As $\mu|k_3 \Rightarrow \mu|p$ and that $p = A^{2m} + B^{2n} + A^m B^n \Rightarrow \mu|B^{2n}$ then $\mu|B$, it follows $\mu|C^l$, hence A, B and C solutions of (2.1) have a common factor.

* If $\mu|(A^m + 2B^n) \Rightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then:

$$\mu \neq 2 \text{ and } \mu \nmid B^n \quad (3.108)$$

$\mu|(A^m + 2B^n)$, we write:

$$A^m + 2B^n = \mu \cdot t' \quad t' \in \mathbb{N}^* \quad (3.109)$$

Then :

$$\begin{aligned} A^m + B^n = \mu t' - B^n &\implies A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t' \mu B^n + B^{2n} \\ &\implies p = t'^2 \mu^2 - 2t' B^n \mu + B^n (B^n - A^m) \end{aligned} \quad (3.110)$$

As $b = 4p = 4k_2 \cdot k_3$ and $\mu|k_3$ then $\mu|b \implies \exists \mu' \in \mathbb{N}^*$ that $b = \mu \mu'$, we obtain:

$$\mu' \mu = \mu(4\mu t'^2 - 8t' B^n) + 4B^n (B^n - A^m) \quad (3.111)$$

The last equation implies $\mu|4B^n(B^n - A^m)$, but $\mu \neq 2$ then $\mu|B^n$ or $\mu|(B^n - A^m)$. If $\mu|B^n \implies$ it is contradiction with (3.108). If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we have:

$$\mu|3B^n \implies \begin{cases} \mu|B^n & \text{it is contradiction with 3.108} \\ \text{or} \\ \mu = 3 \end{cases} \quad (3.112)$$

If $\mu = 3$, then $3|b$, but $3|a$ which is contradiction with a, b co-primes.

\diamond - We assume now $k_3 = 1$. Hence:

$$A^{2m} + 2A^m B^n = k_1 \quad (3.113)$$

$$p = k_2 \quad (3.114)$$

$$\frac{2\sqrt{3}}{3} \sin \frac{2\theta}{3} = \frac{k_1}{p} \quad (3.115)$$

Taking the square of the last equation, we obtain:

$$\frac{4}{3} \sin^2 \frac{2\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3} \sin^2 \frac{\theta}{3} \cos^2 \frac{\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3} \sin^2 \frac{\theta}{3} \cdot \frac{3a'}{b} = \frac{k_1^2}{p^2}$$

Finally:

$$a'(4p - 3a') = k_1^2 \quad (3.116)$$

but $a' = a''^2$ then $4p - 3a'$ is a square. Let us:

$$\lambda^2 = 4p - 3a' = 4p - a = b - a \quad (3.117)$$

The equation (3.116) becomes:

$$a''^2 \lambda^2 = k_1^2 \implies k_1 = a'' \lambda \quad (3.118)$$

taking the positif square root. Using (3.113), we get :

$$k_1 = a'' \lambda \quad (3.119)$$

But $k_1 = A^m(A^m + 2B^n) = a''(A^m + 2B^n)$, it follows:

$$(A^m + 2B^n) = \lambda \quad (3.120)$$

Let λ_1 prime $\neq 2$, a divisor of λ (if not $\lambda_1 = 2|\lambda \Rightarrow 2|\lambda^2$. As $2|(b = 4p) \Rightarrow 2|(a = 3a')$ which is contradiction with a, b co-primes).

We consider $\lambda_1 \neq 2$ and :

$$\lambda_1|\lambda \Rightarrow \lambda_1|(A^m + 2B^n) \quad (3.121)$$

$$\Rightarrow \lambda_1 \nmid A^m \text{ if not } \lambda_1|2B^n \quad (3.122)$$

But $\lambda_1 \neq 2$ hence $\lambda_1|B^n \Rightarrow \lambda_1|B$, it follows:

$$\lambda_1|(b = 4p) \text{ and } \lambda_1|A^m \Rightarrow \lambda_1|2a'' \Rightarrow \lambda_1|a \quad (3.123)$$

hence the contradiction with a, b co-primes.

We assume now $\lambda_1 \nmid A^m$. $\lambda_1|(A^m + 2B^n) \Rightarrow \lambda_1|(A^m + 2B^n)^2$ that is $\lambda_1|(A^{2m} + 4A^mB^n + 4B^{2n})$, we write it as $\lambda_1|(p + 3A^mB^n + 3B^{2n}) \Rightarrow \lambda_1|(p + 3B^n(A^m + 2B^n) - 3B^{2n})$. But $\lambda_1|(A^m + 2B^n) \Rightarrow \lambda_1|(p - 3B^{2n})$, as $\lambda_1|(4p - a)$ hence by difference, we obtain $\lambda_1|(a - 3(B^{2n} + p))$ or $\lambda_1|(3a' - 3(B^{2n} + p)) \Rightarrow \lambda_1|3(a' - B^{2n} - p) \Rightarrow \lambda_1 = 3$ or $\lambda_1|(a' - (B^{2n} + p))$.

*A-1- If $\lambda_1 = 3|\lambda \Rightarrow 3|\lambda^2 \Rightarrow 3|b - a$ but $3|a \Rightarrow 3|(p = b)$ hence the contradiction with a, b co-primes.

*A-2- If $\lambda_1 \neq 3$ and $\lambda_1|(a' - B^{2n} - p) \Rightarrow \lambda_1|(A^mB^n + B^{2n}) \Rightarrow \lambda_1|B^n(A^m + 2B^n) \Rightarrow \lambda_1|B^n$ or $\lambda_1|(A^m + 2B^n)$. The case $\lambda_1|B^n$ was studied above.

*A-2-1- If $\lambda_1|(A^n + 2B^n)$. We refine the condition (3.121).

Then the case $k_3 = 1$ is impossible.

3.2.2.7. Case $3|a$ and $b = 2p'$ $b \neq 2$ with $p'|p$: $3|a \Rightarrow a = 3a'$, $b = 2p'$ with $p = k.p'$, hence:

$$A^{2m} = \frac{4.p}{3} \cdot \frac{a}{b} = \frac{4.k.p'.3.a'}{6p'} = 2.k.a' \quad (3.124)$$

Calculate $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 \sin^2 \frac{\theta}{3} - \cos^2 \frac{\theta}{3} \right) = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) \quad (3.125)$$

But $\sqrt[3]{\rho^2} = \frac{p}{3}$ hence en using $\cos^2 \frac{\theta}{3} = \frac{3.a'}{b}$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \frac{3.a'}{b} \right) = p \cdot \left(1 - \frac{4.a'}{b} \right) = k(p' - 2a') \quad (3.126)$$

As $p = b.p'$, and $p' > 1$, we have then:

$$B^n C^l = k(p' - 2a') \quad (3.127)$$

$$\text{and } A^{2m} = 2k.a' \quad (3.128)$$

A - Soit λ a prime divisor of k (we suppose k not a prime). From (3.128), we have:

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as } \lambda \text{ is prime then } \lambda|A \quad (3.129)$$

From (3.127), as $\lambda|k$, we have:

$$\lambda|B^n C^l \Rightarrow \lambda|B^n \quad \text{or} \quad \lambda|C^l \quad (3.130)$$

If $\lambda|B^n$, λ is prime $\lambda|B$, and as $C^l = A^m + B^n$ then we have also:

$$\lambda|C^l \quad \text{as } \lambda \text{ is prime then } \lambda|C \quad (3.131)$$

By the same way, if $\lambda|C^l$, we obtain $\lambda|B$. Then : A, B and C solutions of (2.1) have a common factor.

B - We suppose now that k is prime, from the equations (3.127) and (3.128), we obtain:

$$k|A^{2m} \Rightarrow k|A^m \Rightarrow k|A \quad (3.132)$$

and:

$$k|B^n C^l \Rightarrow k|B^n \quad \text{or} \quad k|C^l \quad (3.133)$$

$$\text{if } k|B^n \Rightarrow k|B \quad (3.134)$$

$$\begin{aligned} \text{as } C^l = A^m + B^n \quad \text{and that } k|A, k|B \Rightarrow k|A^m, k|B^n \Rightarrow k|C^l \\ \Rightarrow k|C \end{aligned} \quad (3.135)$$

By the same way, if $k|C^l$, we arrive to $k|B$.

Hence: A, B and C solutions of (2.1) have a common factor.

3.2.2.8. Case $3|a$ and $b = 4p'$ $b \neq 2$ with $p'|p$: $3|a \Rightarrow a = 3a'$, $b = 4p'$ with $p = k.p'$, $k \neq 1$ if not $b = 4p$ a case has been studied (paragraph **3.2.2.6**), then we have :

$$A^{2m} = \frac{4.p}{3} \cdot \frac{a}{b} = \frac{4.k.p'.3.a'}{12p'} = k.a' \quad (3.136)$$

Writing $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 \sin^2 \frac{\theta}{3} - \cos^2 \frac{\theta}{3} \right) = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) \quad (3.137)$$

But $\sqrt[3]{\rho^2} = \frac{p}{3}$, hence en using $\cos^2 \frac{\theta}{3} = \frac{3.a'}{b}$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \frac{3.a'}{b} \right) = p \cdot \left(1 - \frac{4.a'}{b} \right) = k(p' - a') \quad (3.138)$$

As $p = b.p'$, and $p' > 1$, we have:

$$B^n C^l = k(p' - 2a') \quad (3.139)$$

$$\text{and } A^{2m} = 2k.a' \quad (3.140)$$

A - Let λ a prime divisor of k (we suppose k not a prime). From (3.140), we have:

$$\lambda|A^{2m} \Rightarrow \lambda|A^m \quad \text{as } \lambda \text{ is prime then } \lambda|A \quad (3.141)$$

From (3.139), as $\lambda|k$ we obtain:

$$\lambda|B^n C^l \Rightarrow \lambda|B^n \quad \text{or } \lambda|C^l \quad (3.142)$$

If $\lambda|B^n$, λ is a prime $\lambda|B$, and as $C^l = A^m + B^n$, then we have:

$$\lambda|C^l \quad \text{as } \lambda \text{ is prime, then } \lambda|C \quad (3.143)$$

By the same way if $\lambda|C^l$, we obtain $\lambda|B$. Then : A, B and C solutions of (2.1) have a common factor.

B - We suppose now that k is prime, from the equations (3.139) and (3.140), we have:

$$k|A^{2m} \Rightarrow k|A^m \Rightarrow k|A \quad (3.144)$$

and:

$$k|B^n C^l \Rightarrow k|B^n \quad \text{or } k|C^l \quad (3.145)$$

$$\text{if } k|B^n \Rightarrow k|B \quad (3.146)$$

$$\begin{aligned} \text{as } C^l = A^m + B^n \quad \text{and that } k|A, k|B \Rightarrow k|A^m, k|B^n \Rightarrow k|C^l \\ \Rightarrow k|C \end{aligned} \quad (3.147)$$

By the same way if $k|C^l$, we arrive to $k|B$.

Hence: A, B and C solutions of (2.1) have a common factor.

3.2.2.9. Case $3|a$ and $b|4p$: $a = 3a'$ and $4p = k_1 b$ with $k_1 \in \mathbb{N}^*$. As $A^{2m} = \frac{4p}{3} \cos^2 \frac{\theta}{3} = \frac{4p}{3} \frac{3a'}{b} = k_1 a'$ and $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2} \left(3 \sin^2 \frac{\theta}{3} - \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \cos^2 \frac{\theta}{3} \right) = \frac{p}{3} \left(3 - 4 \frac{3a'}{b} \right) = \frac{k_1}{4} (b - 4a') \quad (3.148)$$

As $B^n C^l$ is an integer, we must have $4|k_1$ or $4|(b - 4a')$.

1- If $k_1 = 1 \Rightarrow b = 4p$: it is the case **(3.2.2.6) above.

2- If $k_1 = 4 \Rightarrow p = b$: it is the case **(3.2.2.3) above.

**3- We suppose that $4|k_1$ with $k_1 > 4 \Rightarrow k_1 = 4k'_1$, then we have:

$$\begin{aligned} A^{2m} &= 4k'_1 a' \\ B^n C^l &= k'_1 (b - 4a') \end{aligned}$$

By discussing k'_1 is a prime integer or not, we arrive easily to: A, B and C solutions of (2.1) have a common factor.

**4- If $4 \nmid (b - 4a')$ and $4 \nmid k'_1$ it is impossible. If $4|(b - 4a') \Rightarrow (b - 4a') = 4c$, with $c \in \mathbb{N}^*$, then we obtain:

$$\begin{aligned} A^{2m} &= k_1 a' \\ B^n C^l &= k_1 c \end{aligned}$$

By discussing k_1 is a prime integer or not, we arrive easily to: A, B and C solutions of (2.1) have a common factor.

□

The main theorem is proved.

4 A Numerical Example

We consider the example:

$$6^3 + 3^3 = 3^5 \quad (4.1)$$

with $A^m = 6^3$, $B^n = 3^3$ and $C^l = 3^5$. With the notations used in the paper, we obtain:

$$p = 3^6 \times 73, \quad (4.2)$$

$$q = 8 \times 3^{11}, \quad (4.3)$$

$$\bar{\Delta} = 4 \times 3^{11} (3^6 \times 4^2 - 73^3) < 0, \quad (4.4)$$

$$\rho = \frac{p\sqrt{p}}{3\sqrt{3}} = \frac{3^8 \times 73\sqrt{73}}{3}, \quad (4.5)$$

$$\cos\theta = -\frac{4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \quad (4.6)$$

As $A^{2m} = \frac{4p}{3} \cdot \cos^2 \frac{\theta}{3} \Rightarrow \cos^2 \frac{\theta}{3} = \frac{3A^{2m}}{4p} = \frac{3 \times 2^4}{73} = \frac{a}{b} \Rightarrow a = 3 \times 2^4, b = 73$;
then:

$$\cos \frac{\theta}{3} = \frac{4\sqrt{3}}{\sqrt{73}} \quad (4.7)$$

$$p = 3^6 b \quad (4.8)$$

Let us verify the equation (4.6) using the equation (4.7):

$$\cos\theta = \cos 3(\theta/3) = 4\cos^3 \frac{\theta}{3} - 3\cos \frac{\theta}{3} = 4 \left(\frac{4\sqrt{3}}{\sqrt{73}} \right)^3 - 3 \frac{4\sqrt{3}}{\sqrt{73}} = -\frac{4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \quad (4.9)$$

That's OK. For this example, we can use the two conditions of (3.10) as $3|p, b|4p$ and $3|a$. The cases **3.2.1.3** and **3.2.2.4** are respectively used. We find for both cases that A^m, B^n and C^l of the equation (4.1) have a common prime factor which is true.

References

- [1] R. DANIEL MAULDIN. *A Generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem*. Notice of AMS, Vol 44, n°11, 1997, pp 1436-1437.