The Principle of Least Action

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(2015) Buenos Aires
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In classical mechanics, this article obtains the principle of least action for a single particle in a didactic and simple way.

Introduction

Let us consider the following tautological equation for a single particle:

\[
\frac{d(\mathbf{v} \cdot \delta \mathbf{r})}{dt} = \delta \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \mathbf{a} \cdot \delta \mathbf{r}
\]

Now, integrating with respect to time from \(t_1\) to \(t_2\), yields:

\[
\int_{t_1}^{t_2} \left[ \frac{d(\mathbf{v} \cdot \delta \mathbf{r})}{dt} \right] dt = \int_{t_1}^{t_2} \left[ \delta \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \mathbf{a} \cdot \delta \mathbf{r} \right] dt
\]

The left side of the equation is zero, therefore, we obtain:

\[
0 = \int_{t_1}^{t_2} \left[ \delta \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \mathbf{a} \cdot \delta \mathbf{r} \right] dt
\]

In classical mechanics, this last tautological equation is the mathematical basis of the principle of least action for a single particle.
Now, multiplying by \( m \) (mass of the particle) the following tautological equation is obtained:

\[
0 = \int_{t_1}^{t_2} \left[ \delta \frac{1}{2} m (\mathbf{v} \cdot \mathbf{v}) + m \mathbf{a} \cdot \delta \mathbf{r} \right] \, dt
\]

Substituting \( \mathbf{a} = \mathbf{F} / m \) (Newton’s second law) the following empirical equation is obtained:

\[
0 = \int_{t_1}^{t_2} \left[ \delta \frac{1}{2} m (\mathbf{v} \cdot \mathbf{v}) + \mathbf{F} \cdot \delta \mathbf{r} \right] \, dt
\]

If the particle is only subject to conservative forces then \( \delta V = -\mathbf{F} \cdot \delta \mathbf{r} \) and since \( T = \frac{1}{2} m (\mathbf{v} \cdot \mathbf{v}) \) yields:

\[
0 = \int_{t_1}^{t_2} [\delta T - \delta V] \, dt
\]

That is:

\[
0 = \delta \int_{t_1}^{t_2} [T - V] \, dt
\]

Or else:

\[
\delta \int_{t_1}^{t_2} [T - V] \, dt = 0
\]

Finally we obtain:

\[
\delta \int_{t_1}^{t_2} L \, dt = 0
\]

Since \( L = T - V \).
Annex

\[ \frac{d}{dt} (v \cdot \delta r) = \ldots \]

\[ \frac{d}{dt} (m v \cdot \delta r) = \ldots \]

\[ \sum_i \frac{d}{dt} (m_i v_i \cdot \delta r_i) = \ldots \]

\[ \sum_{i,j} \frac{d}{dt} (m_i v_i \cdot \frac{\partial r_i}{\partial q_j} \delta q_j) = \ldots \]

\[ \sum_{i,j} \frac{d}{dt} (m_i v_i \cdot \frac{\partial r_i}{\partial q_j} \delta q_j) = \sum_{i,j} m_i v_i \cdot \frac{d}{dt} \left( \frac{\partial r_i}{\partial q_j} \right) \delta q_j + \sum_{i,j} m_i a_i \cdot \frac{\partial r_i}{\partial q_j} \delta q_j \]

\[ \sum_{i,j} \frac{d}{dt} (m_i v_i \cdot \frac{\partial r_i}{\partial q_j} \delta q_j) - \sum_{i,j} m_i v_i \cdot \frac{d}{dt} \left( \frac{\partial r_i}{\partial q_j} \right) \delta q_j = \sum_{i,j} m_i a_i \cdot \frac{\partial r_i}{\partial q_j} \delta q_j \]

\[ \sum_{i,j} \left[ \frac{d}{dt} (m_i v_i \cdot \frac{\partial r_i}{\partial q_j}) - m_i v_i \cdot \frac{d}{dt} \left( \frac{\partial r_i}{\partial q_j} \right) \right] \delta q_j = \sum_{i,j} m_i a_i \cdot \frac{\partial r_i}{\partial q_j} \delta q_j \]

\[ \sum_{i,j} \left[ \frac{d}{dt} (m_i v_i \cdot \frac{\partial r_i}{\partial q_j}) - m_i v_i \cdot \frac{d}{dt} \left( \frac{\partial r_i}{\partial q_j} \right) \right] \delta q_j = \sum_{i,j} F_i \cdot \frac{\partial r_i}{\partial q_j} \delta q_j \]

Bibliography


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