The Collapse of the Schwarzschild Radius: The End of Black Holes A Revised Escape Velocity Is Valid Under Strong Gravitational Fields with Potential Significant Implications for Cosmology

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Abstract

In this paper we introduce an exact escape velocity that also holds under very strong gravitational fields, even below the Schwarzschild radius. The standard escape velocity known from modern physics is only valid under weak gravitational fields. This paper strongly indicates that an extensive series of interpretations around the Schwarzschild radius are wrong and were developed as a result of using an approximate escape velocity that not is accurate when we approach strong gravitational fields. Einstein's general relativity escape velocity as well as the gravitational time dilation and gravitational redshift that are derived from the Schwarzschild metric need to be modified; in reality, they are simply approximations that only give good predictions in low gravitational fields. This paper could have major implications for gravitational physics as well as a long series of interpretations in cosmology.

Key words: Escape velocity, strong gravitational field, Schwarzschild radius, gravitational time dilation, gravitational redshift, special relativity, general relativity, Planck quantization, Newton, Einstein, Schwarzschild.

1 Short Background on the Derivation on the Standard Escape Velocity

Derivation of the standard classical escape velocity is accomplished by solving the following equation with respect to v

$$\begin{aligned}
E_k - \frac{GmM}{r} &= 0 \\
\frac{1}{2}mv_e^2 - \frac{GmM}{r} &= 0 \\
v_e^2 &= \frac{\frac{GmM}{r}}{\frac{1}{2}m} \\
v_e^2 &= \frac{2GM}{r} \\
v_e &= \sqrt{\frac{2GM}{r}}
\end{aligned} \tag{1}$$

which is the well known escape velocity, with important applications in rocket science and cosmology. Exactly the same escape velocity formula can be derived directly from Einstein's general relativity using the Schwarzschild metric. However, as pointed out by Augousti and Radosz (2006), for example, the formula derived from the Schwarzschild metric under general relativity theory only holds for a weak gravitational field. That is when we are considerably far away from the Schwarzschild radius of the mass in question. Both the standard way of deriving the classical escape velocity and the same escape velocity

*e-mail espenhaug@mac.com. Thanks to Victoria Terces for helping me edit this manuscript. Thanks to Daniel J. Duffy and Tranden4Alpha for helping me simplify the gravitational redshift formula and (indirectly) also the gravitational time dilation formula dramatically. Also thanks to Jeremy Dunning-Davies for some very useful comments. When the end results are extremely simple, beautiful, and well-behaved mathematical expressions, then one has reason to suspect one really is on to something?

formula derived from general relativity theory using the Schwarzschild metric are only approximations that are very inaccurate in strong gravitational fields. At and below the Schwarzschild radius, the standard escape velocity formula has no logic and has led to a series of likely incorrect speculative conclusions that have had a significant affect on our view of gravitational physics and cosmology. See also Crothers (2014, 2015) for criticism of using this standard escape velocity in general relativity theory. In this paper we not only criticize the use of the classical "Newton" escape velocity that is only valid for velocities $v << c$ (that is it is only valid as a good approximation for weak gravitational fields), we also come up with a solution that collapses the Schwarzschild radius interpretations. The Black Hole interpretation in General relativity theory is nothing more than an incorrect interpretation of a gravitational theory that only holds for weak gravitational fields and breaks down at the Schwarzschild radius.

2 An Exact Escape Velocity That Also Holds Under Strong Gravitational Fields

In this section I will derive the exact escape velocity based on the exact kinetic energy formula. The kinetic energy formula typically used to demonstrate the derivation of the escape velocity is $E_k \approx \frac{1}{2}Mv^2$, see above. It is (or should be) understood that this is only an approximation formula for the exact kinetic energy formula that is given by:

$$E_k = \frac{Mc^2}{\sqrt{1-\frac{v^2}{c^2}}} - Mc^2 \tag{2}$$

where M is the rest mass. By performing a Taylor series expansion of Einstein's "moving mass" formula we get:

$$\begin{aligned} \frac{Mc^2}{\sqrt{1-\frac{v^2}{c^2}}} &= Mc^2 + \frac{1}{2}Mv^2 + \frac{3}{8}M\frac{v^4}{c^2} + \frac{5}{16}M\frac{v^6}{c^4} + \frac{35}{128}M\frac{v^8}{c^6} + \frac{63}{256}M\frac{v^{10}}{c^8} \\ &\quad + \frac{231}{1024}M\frac{v^{12}}{c^{10}} + \frac{429}{2048}M\frac{v^{14}}{c^{12}} + \frac{6435}{32768}M\frac{v^{16}}{c^{14}} + \frac{12155}{65536}M\frac{v^{18}}{c^{16}} \\ &\quad + \frac{46189}{262144}M\frac{v^{20}}{c^{18}} + \frac{88179}{524288}M\frac{v^{22}}{c^{20}} + \frac{676039}{4194304}M\frac{v^{24}}{c^{22}} + \cdots . \end{aligned}$$

The Taylor series consists of an infinite array of terms, but when $v << c$ then only the first two terms in the Taylor series are needed for a good approximation:

$$\frac{Mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \approx Mc^2 + \frac{1}{2}Mv^2.$$

By subtracting the rest mass energy Mc^2 from the formula above, we get the classical kinetic energy formula $E_k \approx \frac{1}{2}Mv^2$. But again this approximate kinetic energy formula only holds when the velocity is much smaller than c. It is not an exact kinetic energy formula and it is very inaccurate for velocities approaching the speed of light. If we are using this approximate kinetic energy formula in deriving the escape velocity, then the escape velocity will also be approximation that only is valid for $v << c$.

Here we will derive the escape formula from the exact kinetic energy formula; the escape velocity should then be exact. We are basically combining Newton's gravitational potential with Einstein's special relativity theory to solve for the escape velocity:

$$\begin{aligned} 0 &= E_k - \frac{GmM}{r} \\ 0 &= \frac{mc^2}{\sqrt{1-\frac{\bar{v}_e^2}{c^2}}} - mc^2 - \frac{GmM}{r} \end{aligned} \tag{3}$$

we note the escape velocity here as $\bar{v}_e$ rather than v or v_e to distinguish the notation for the exact escape velocity from the standard (approximate) escape velocity v_e. We get

$$
\begin{aligned}
\bar{v}_e &= \frac{c\sqrt{GM}\sqrt{GM+2c^2r}}{GM+c^2r} \\
\bar{v}_e &= \frac{c\sqrt{G^2M^2+GM2c^2r}}{GM\left(1+\frac{c^2r}{GM}\right)} \\
\bar{v}_e &= c\frac{\sqrt{1+\frac{2c^2r}{GM}}}{1+\frac{c^2r}{GM}}
\end{aligned} \tag{4}
$$

This is the exact escape velocity formula. We can go further and obtain a quantized version of formula 4 based on the principle of Haug (2016a,b). We will set the Newton gravitational constant to

$$
G_p = \frac{\aleph^2 c^3}{\hbar} \tag{5}
$$

where $\hbar$ is the reduced Planck's constant and c is the well tested round-trip speed of light. We could call this Planck's form of the gravitational constant. The parameter $\aleph$ is unknown constant that can be set equal to the Planck length if this is know, or alternatively it can be calibrated to the measured gravitational constant and then we have found the Planck length indirectly. From this the Planck length is given by

$$
l_p = \sqrt{\frac{\hbar G_p}{c^3}} = \sqrt{\frac{\hbar\frac{\aleph^2c^3}{\hbar}}{c^3}} = \aleph \tag{6}
$$

and the Planck mass is given by

$$
m_p = \sqrt{\frac{\hbar c}{G_p}} = \sqrt{\frac{\hbar c}{\frac{\aleph^2c^3}{\hbar}}} = \frac{\hbar}{\aleph}\frac{1}{c} \tag{7}
$$

Using the gravitational constant in the Planck form, as well as the rewritten Planck units, we are easily able to rewrite the exact escape velocity in a quantized form as well

$$
\begin{aligned}
\bar{v}_e &= c\frac{\sqrt{1+\frac{2c^2r}{G_pM}}}{1+\frac{c^2r}{G_pM}} \\
\bar{v}_e &= c\frac{\sqrt{1+\frac{2c^2r}{G_pNm_p}}}{1+\frac{c^2r}{G_pNm_p}} \\
\bar{v}_e &= \frac{c\sqrt{1+\frac{2c^2r}{\frac{\aleph^2c^3}{\hbar}N\frac{\hbar}{\aleph}\frac{1}{c}}}}{\left(1+\frac{c^2r}{\frac{\aleph^2c^3}{\hbar}N\frac{\hbar}{\aleph}\frac{1}{c}}\right)} \\
\bar{v}_e &= c\frac{\sqrt{1+\frac{2r}{N\aleph}}}{1+\frac{r}{N\aleph}}
\end{aligned} \tag{8}
$$

where N is the total number of Planck masses, m_p, in the mass, M, we are trying to escape from. Formula 4 and 8 will give the exact same output values, they differ in that the formula 4 requires the gravitational constant as input and the mass in kg, while the formula 8 requires the number of Planck masses the mass makes up, the Planck length, and the reduced Planck constant. From the exact escape velocity formula we can see there is no radius where the escape velocity is larger than c. In other words, the formula predicts that light can always escape an object no matter how massive it is or how strong the gravitational field is. In other words, the notion of a black hole is a mathematical illusion from an approximate escape velocity formula that is not valid in the presence of strong gravitational fields.

An interesting case is what the exact escape velocity is at the Schwarzschild radius $r_s = \frac{GM}{c^2r} = 2N\aleph$. This gives

$$
\begin{aligned}
\bar{v}_e &= c\frac{\sqrt{1+\frac{2r_s}{N\aleph}}}{1+\frac{r_s}{N\aleph}} \\
\bar{v}_e &= c\frac{\sqrt{1+\frac{2\times 2N\aleph}{N\aleph}}}{1+\frac{2N\aleph}{N\aleph}} \\
\bar{v}_e &= c\frac{\sqrt{5}}{3} \approx 0.745355992c
\end{aligned} \tag{9}
$$

First, at a radius considerably below the Schwarzschild radius, the escape velocity is approaching c. This is in sharp contrast to the standard approximate escape velocity of modern physics that predicts that the escape velocity at the Schwarzschild radius is c and that the escape velocity inside the Schwarzschild radius is $> c$. In this scenario, the standard escape velocity predicts that not even a photon can escape if it passes inside the Schwarzschild radius. Such interpretations are likely the result of the misuse of approximations and artifact coordinates in the Schwarzschild metric. Table 1 shows the exact escape velocity and the approximate standard escape velocity. At the surface of earth we must go out to the 6 decimals to see differences between the exact and the approximate escape velocities. This is probably the reason that the standard escape velocity has been used so successfully and that no one has focused on the fact that it is only an approximation. Nevertheless, reliance on the standard escape velocity, an approximation that is not valid in strong gravitational fields, may very well have produced a series of deep misinterpretations in cosmology.

Table 1: The table shows the exact escape velocity from an Earth-sized mass at different radiuses compared to the standard escape velocity. Assumed mass: $N = 2.74388 \times 10^{32}$ Planck masses or 5.97197×10^{24} kg.

Multiples of the Schwarzschild radius :	**Radius meter :**	**Exact escape velocity meters per second :**	**Standard escape velocity meters per second :**
Surface earth: $718,306,435r_s$	6,371,000	11,185.768431	11,185.768436
$100r_s$	0.886947366	29,867,359.67	29,979,245.80
$10r_s$	0.088694737	91,409,921.62	94,802,699.26
$5r_s$	0.044347368	124,892,875.60	134,071,263.05
r_s :	**0.008869474**	$c\frac{\sqrt{5}}{3} \approx$ **223,452,105**	**299,792,458.00=c**
$0.5r_s$	0.004434737	259,627,884.49	$> c$ Impossible to escape=Black hole
$0.1r_s$	0.000886947	295,599,349.98	$> c$ Impossible to escape=Black hole
$0.001r_s$	8.86947E-06	299,791,860.81	$> c$ Impossible to escape=Black hole
$0.0001r_s$	8.86947E-07	299,792,452.01	$> c$ Impossible to escape=Black hole
$0.00001r_s$	8.86947E-08	299,792,457.94	$> c$ Impossible to escape=Black hole
$0r_s$	**0**	**299,792,458.00=c**	**Equation breaks= BH singularity**

I will claim that the Schwarzschild radius is nothing special in the physical world. The Schwarzschild radius and its misinterpretations arise from the use of an approximate escape velocity and likely are also from coordinate artifacts in the Schwarzschild solution of the Einstein field equation when $r = r_s$, see **?** and Crothers (2009). It is well known that some of the Schwarzschild metric components blow up at $r = r_s$ and $r = 0$. The misinterpretation of the Schwarzschild radius should become even clearer when we move on to gravitational time dilation and gravitational redshift.

3 Gravitational Time Dilation

Einstein's gravitational time dilation is given by

$$
t_o = t_f\sqrt{1-\frac{\frac{2GM}{r}}{c^2}} = \sqrt{1-\frac{v_e^2}{c^2}} \tag{10}
$$

where $v_e = \sqrt{\frac{2GM}{r}}$ the standard escape velocity and r is the radius out from the center of the mass, and t_f is the time gone by for a clock so far from the gravitational center that it is basically unaffected by the gravitational field. To calculate the Einstein gravitational time dilation, we need to know the escape velocity. Einstein's standard gravitational time dilation formula uses the approximate escape velocity formula that is only valid under weak gravitational fields. The approximate escape velocity formula gives extremely accurate values when we are at the surface of Earth or at radiuses similar to that of the GPS satellites, for example.

However, when we approach strong gravitational fields in the range of the Schwarzschild radius, then the standard Einstein approximate gravitational time dilation formula is likely to give incorrect values. At the Schwarzschild radius, the formula above gives highly inaccurate predictions and below the Schwarzschild radius the formula simply breaks down. It is also worth mentioning that Haug (2016b) has recently quantized the standard Einstein gravitational time dilation, see Appendix B. The quantization is not important for the conclusions in this paper, but we mention it here, as we will quantize the exact gravitational time dilation that holds at extremely strong gravitational fields.

Exact gravitational time dilation

The exact gravitational time dilation is obtained simply by replacing the approximate escape velocity used in modern physics with the exact escape velocity derived above

$$\begin{aligned} t_o &= t_f\sqrt{1-\frac{\bar{v}_e^2}{c^2}} \\ t_o &= t_f\sqrt{1-\frac{\left(c\frac{\sqrt{1+\frac{2c^2r}{GM}}}{1+\frac{c^2r}{GM}}\right)^2}{c^2}} \\ t_o &= t_f\sqrt{1-\frac{\left(1+\frac{2c^2r}{GM}\right)}{\left(1+\frac{c^2r}{GM}\right)^2}} \end{aligned} \tag{11}$$

This can be rewritten as simply

$$t_o = \frac{t_f}{1+\frac{GM}{rc^2}} \tag{12}$$

Alternatively we could write this in a more informative and elegant quantized form

$$\begin{aligned} t_o &= t_f\sqrt{1-\frac{\bar{v}_e^2}{c^2}} \\ t_o &= t_f\sqrt{1-\frac{\left(c\frac{\sqrt{1+\frac{2r}{N\aleph}}}{1+\frac{r}{N\aleph}}\right)^2}{c^2}} \\ t_o &= t_f\sqrt{1-\frac{1+\frac{2r}{N\aleph}}{\left(1+\frac{r}{N\aleph}\right)^2}} \end{aligned} \tag{13}$$

This can be rewritten as simply

$$t_o = \frac{t_f}{1+\frac{N\aleph}{r}} \tag{14}$$

The quantized and non-quantized forms give exactly the same output values, except that the quantized form comes in quantized steps. An interesting case is when we set the radius to the Schwarzschild radius, $r_s = \frac{2GM}{c^2} = 2N\aleph$. Then we get

$$\begin{aligned} t_o &= t_f\sqrt{1-\frac{1+\frac{2r}{N\aleph}}{\left(1+\frac{r}{N\aleph}\right)^2}} \\ t_o &= t_f\sqrt{1-\frac{1+\frac{2\times 2N\aleph}{N\aleph}}{\left(1+\frac{2N\aleph}{N\aleph}\right)^2}} \\ t_o &= t_f\sqrt{1-\frac{5}{9}} \approx t_f 0.666666667 \end{aligned} \tag{15}$$

or we could derive it as

$$\begin{aligned} t_o &= t_f \frac{1}{1+\frac{N\aleph}{r}} \\ t_o &= t_f \frac{1}{1+\frac{N\aleph}{2N\aleph}} \\ t_o &= t_f \frac{1}{1+\frac{1}{2}} \approx t_f 0.666666667 \end{aligned} \tag{16}$$

In other words, time does not stand still at the Schwarzschild radius. For the exact time dilation, the Schwarzschild radius is not unique and nothing special happens at this radius. The standard interpretation that time stands still at the Schwarzschild radius is very likely to be an incorrect interpretation rooted in the use of the approximate escape velocity.

In Table 2 one can study the differences in predictions between Einstein's gravitational time dilation rooted in the Schwarzschild metric and the modified exact solution presented here. The gravitational time dilation works all the way down to radius zero, which is far below the Schwarzschild radius.

Table 2: The table shows the exact time dilation for a Earth-sized mass at different radiuses compared to the standard gravitational time dilation. Assumed mass: $N = 2.74388 \times 10^{32}$ Planck masses or 5.97197×10^{24} kg.

Multiples of the Schwarzschild radius :	**Radius meters :**	**Time dilation from exact escape velocity :**	**Einstein time dilation factor :**
Mountain top radius $718,419,181 r_s$	6,372,000	0.999999999304027	0.999999999304027
Earth surface $718,306,435 r_s$	6,371,000	0.999999999303918	0.999999999303918
$100 r_s$	0.886947	0.995024875621891	0.994987437106620
$10 r_s$	0.088695	0.952380952380952	0.948683298050514
$5 r_s$	0.044347	0.909090909090909	0.894427190999916
r_s	**0.008869**	$\frac{1}{1+\frac{1}{2}}$ **≈0.666667**	**0 (time stands still)**
$0.5 r_s$	0.004435	$\frac{1}{2} = 0.5$	equation collapse=Black hole
$0.1 r_s$	0.000887	0.166666666666667	equation collapse=Black hole
$0.001 r_s$	8.8695E-06	0.001996007983993	equation collapse=Black hole
$0.0001 r_s$	8.8695E-07	0.000199960007936	equation collapse=Black hole
$0.00001 r_s$	8.8695E-08	0.000019999595592	equation collapse=Black hole
$0 r_s$	0	0.000000000000000	equation collapse=Black hole

Relative gravitational time dilation between two masses

The formula derived above gives the dime dilation in one frame relative to how much time has gone by in outer space (in an area with close to no gravitation, an extremely weak gravitational field):

$$t_o = \frac{t_f}{1+\frac{GM}{rc^2}} \tag{17}$$

If we want to compare how much time has gone by on Earth with how much time gone by on Mars, for example *or we can substitute any two masses) we get the following formula

$$t_1 = t_2 \frac{\left(1+\frac{GM_2}{r_2c^2}\right)}{\left(1+\frac{GM_1}{r_1c^2}\right)} \tag{18}$$

and naturally the other way around

$$t_2 = t_1 \frac{\left(1+\frac{GM_1}{r_1c^2}\right)}{\left(1+\frac{GM_2}{r_2c^2}\right)} \tag{19}$$

Alternatively we can use the Planck length and the number of Planck masses as input and we then get

$$t_1 = t_2 \frac{\left(1 + \frac{N_2 \aleph}{r_2}\right)}{\left(1 + \frac{N_1 \aleph}{r_1}\right)} \tag{20}$$

where N_1 is the number of Planck masses in the mass 1 (for example Earth) and N_2 is the number of Planck masses in mass two (for example Mars). And naturally we have

$$t_2 = t_1 \frac{\left(1 + \frac{N_1 \aleph}{r_1}\right)}{\left(1 + \frac{N_2 \aleph}{r_2}\right)} \tag{21}$$

Same object different radius time dilation formula

Assuming one wants to compare the time difference on the same planet, but at two different altitudes, then we get the following formula

$$t_1 = t_2 \frac{\left(1 + \frac{GM}{(r+A)c^2}\right)}{\left(1 + \frac{GM}{rc^2}\right)} \tag{22}$$

where A is the altitude above the other radius.

$$t_2 = t_1 \frac{\left(1 + \frac{GM}{rc^2}\right)}{\left(1 + \frac{GM}{(r+A)c^2}\right)} \tag{23}$$

Or in the quantized Plank length form

$$t_1 = t_2 \frac{\left(1 + \frac{N \aleph}{r+A}\right)}{\left(1 + \frac{N \aleph}{r}\right)} \tag{24}$$

and we must naturally have

$$t_2 = t_1 \frac{\left(1 + \frac{N \aleph}{r}\right)}{\left(1 + \frac{N \aleph}{r+A}\right)} \tag{25}$$

4 Gravitational Redshift

Redshift is often described with the dimensionless variable z, that is defined as the fractional change of the wavelength:

$$z = \frac{\lambda_R - \lambda_e}{\lambda_e} \tag{26}$$

where λ_o is the wavelength of the photon as measured by the receiver and λ_e is the wavelength of the so-called photon as measured from the source where it is emitted. The Einstein gravitational redshift derived from the Schwarzschild metric is given by

$$\begin{aligned} \lim_{r \to +\infty} z(r) &= \frac{1}{\sqrt{1 - \frac{\frac{2GM}{c^2}}{R_e}}} - 1 \\ \lim_{r \to +\infty} z(r) &= = \frac{1}{\sqrt{1 - \frac{v_e^2}{c^2}}} - 1 \end{aligned} \tag{27}$$

where R_e is the distance between the center of the mass of the gravitating body and the point at which the photon is emitted and v_e is the well known standard escape velocity and $\lim_{r \to +\infty} z(r)$ indicates this is how it is observed very far away from the mass .[1] The escape velocity used in Einstein's gravitational redshift formula is an approximate escape velocity that does not work well when we approach the Schwarzschild radius. In other words, Einstein's gravitational redshift very likely gives the wrong redshift predictions for photons emitted from strong gravitational fields.

The gravitational redshift based on the exact escape velocity formula that holds under strong gravitational fields must be

[1] See Haug (2016b) for quantization of this formula, even if that not is important in this context.

$$\lim_{r\to+\infty} z(r) = \frac{1}{\sqrt{1-\frac{\bar{v}_e^2}{c^2}}} - 1 \tag{28}$$

We can rewrite this as

$$\begin{aligned}
\lim_{r\to+\infty} z(r) &= \frac{1}{\sqrt{1-\frac{\left(c\frac{\sqrt{1+\frac{2c^2R_e}{GM}}}{1+\frac{c^2r}{GM}}\right)^2}{c^2}}} - 1 \\
\lim_{r\to+\infty} z(r) &= \frac{1}{\sqrt{1-\frac{1+\frac{2c^2R_e}{GM}}{\left(1+\frac{c^2R_e}{GM}\right)^2}}} - 1
\end{aligned} \tag{29}$$

This we can rewrite as simply

$$\lim_{r\to+\infty} z(r) = \frac{GM}{R_e c^2} \tag{30}$$

Based on a different method than shown here, Adler, Bazin, and Schiffer (1965) as well as Evans and Dunning-Davies (2004) has derived an identical redshift formula with no recourse to the general relativity theory, nor to the principle of equivalence. We will remark also that the gravitational redshift that was derived based on the exact escape velocity, or the method described by Adler, Bazin, and Schiffer (1965) and Evans and Dunning-Davies (2004), is equal to what is considered a approximation redshift formula in general relativity. With great interest we notice that mainstream gravitational researchers consider formula 27 from general relativity to be the exact formula despite its connection to the approximate escape velocity, and they consider formula 30 an approximate gravitational redshift formula despite the fact that the latter one can be derived from the exact escape velocity. Based on our analysis and derivations we actually suspect that the gravitational redshift formula derived from the exact escape velocity formula, or alternatively in the Evans and Dunning-Davies way, must be the correct gravitational redshift formula that also holds under strong gravitational fields, and that the standard gravitational redshift formula must be the approximation that only holds for weak gravitational fields.

We can alternatively write the redshift formula above on the quantized form

$$\begin{aligned}
\lim_{r\to+\infty} z(r) &= \frac{1}{\sqrt{1-\frac{\left(c\frac{\sqrt{1+\frac{2R_e}{N\aleph}}}{1+\frac{R_e}{N\aleph}}\right)^2}{c^2}}} - 1 \\
\lim_{r\to+\infty} z(r) &= \frac{1}{\sqrt{1-\frac{1+\frac{2R_e}{N\aleph}}{\left(1+\frac{R_e}{N\aleph}\right)^2}}} - 1
\end{aligned} \tag{31}$$

This we can rewrite as simply

$$\lim_{r\to+\infty} z(r) = \frac{N\aleph}{R_e} \tag{32}$$

And interesting special case is the gravitational redshift at the Schwarzschild radius. The Planck-quantized Schwarzschild radius is given by Haug (2016a) and is

$$\begin{aligned}
r_s &= \frac{2G_pM}{c^2} \\
r_s &= \frac{\frac{2\aleph^2c^3}{\hbar}N\frac{\hbar}{\aleph}\frac{1}{c}}{c^2r} \\
r_s &= 2N\aleph
\end{aligned} \tag{33}$$

This gives the redshift for photons emitted at the Schwarzschild radius

$$\begin{aligned}
\lim_{r\to+\infty} z(r) &= \frac{N\aleph}{R_e} \\
\lim_{r\to+\infty} z(r) &= \frac{N\aleph}{N2\aleph} = \frac{1}{2}
\end{aligned} \tag{34}$$

That is for photons emitted at the Schwarzschild radius, the gravitational redshift factor z is 0.5. We can also rewrite the gravitational redshift as a function of how many Schwarzschild radiuses the photons are emitted from, rather than the radius itself. Let's use the symbol y for how many Schwarzschild radiuses we are emitting the photons from; this gives the following neat formula

$$\begin{aligned}\lim_{r\to+\infty} z(y) &= \frac{N\aleph}{R_e} \\ \lim_{r\to+\infty} z(y) &= \frac{N\aleph}{yN2\aleph} = \frac{1}{2y} \end{aligned} \tag{35}$$

Formulas 30, 32, and 35 will all give exactly the same output, but require different inputs. In formula 30 one must input the mass, the gravitational constant, the speed of light, and the radius the photons are emitted from. In formula 32 one must input the number of Planck masses in the mass and the Planck length. In formula 35 one must input only the multiples of Schwarzschild radiuses the photons are emitted from.

In Table 3 we have calculated predicted gravitational redshifts for a mass containing 10 solar masses with the standard Einstein gravitational redshift formula and our modified gravitational redshift formula that also holds down to and even below the Schwarzschild radius. It is clear from the table that massive dense objects can have a very high gravitational redshift. The standard model is not able to give predictions for photons emitted from an area below the Schwarzschild radius. As we have discussed, the predictions from the standard theory break down at the Schwarzschild radius in this regard and are thus interpreted as black holes.

Table 3: The table shows the exact redshift for a 10 Solar-sized mass at different radiuses compared to the standard gravitational redshift. Assumed mass of object, 10 solar masses : $N = 9.134 \times 10^{38}$ Planck masses or 1.98855×10^{31} kg.

Multiples of Schwarzschild radius :	**Radius photons emitted from (meters) :**	**Robust exact redshift z(r) :**	**Einstein/Schwarzschild redshift z(r) :**
$23,584.62r_s$	Radius Sun : 696,342,000.00	0.00002120025844	0.00002120093264
$216r_s$	Radius Earth : 6,371,000	0.00231716062796	0.00232524570812
$100r_s$	2,952,526.072	0.005	0.00503781525921
$10r_s$	295,252.607	0.05	0.05409255338946
$5r_s$	147,626.304	0.1	0.11803398874989
$4r_s$	118,101.043	0.125	0.15470053837925
$3r_s$	88,575.782	0.16666666667	0.22474487139159
$2r_s$	59,050.521	0.25	0.41421356237310
$1.5r_s$	44,287.891	0.33333333333	0.73205080756888
$1.25r_s$	36,906.576	0.4	1.23606797749979
$1.01r_s$	29,820.513	0.49504950495	9.04987562112077
$1.001r_s$	29,554.786	0.49950049950	30.63858403911150
r_s	**29,525.261**	**0.5**	**Equation break down**
$0.5r_s$	14,762.63	1	Equation break down
$0.2r_s$	5,905.052	2.5	Equation break down
$0.1r_s$	2,952.526	5	Equation break down
$0.01r_s$	295.253	50	Equation break down
$0.001r_s$	29.525	500	Equation break down
$0.0001r_s$	2.953	5000	Equation break down
$0.00001r_s$	0.295	50000	Equation break down
$0r_s$	0	Equation break down	Equation break down

The next table shows the redshift for an Earth-mass sized object. At the surface and 100 meters above the surface of earth we see that our modified approach and the standard Einstein Schwarzschild approach gives indistinguishable values with 15-digit precision. This is no surprise, since the Einstein Schwarzschild framework is, in our view, an excellent approximation in weak gravitational fields. However, as we are approaching the Schwarzschild radius (approaching stronger gravitational fields) the differences in the two methods vary dramatically. One should ask how physicists can depend on a gravitational redshift formula that indirectly relys on a escape velocity that we know must be inaccurate at high escape velocities (high gravitational fields)?

Table 4: The table shows the exact redshift for a Earth-sized mass at different radiuses compared to the standard gravitational redshift. Assumed mass of object : $N = 2.74388 \times 10^{32}$ Planck masses or 5.97197×10^{24} kg.

Multiples of Schwarzschild radius :	Radius photons emitted from (meters) :	Robust exact redshift z(r) :	Einstein/Schwarzschild redshift z(r) :
$718,317,709.25r_s$	100m above surface 6,371,100.00	0.000000000696071	0.000000000696071
$718,306,434.63r_s$	Earth surface 6,371,000.00	0.000000000696082	0.000000000696082
$100r_s$	0.886947366	0.005	0.005037815259212
$10r_s$	0.088694737	0.05	0.054092553389460
$5r_s$	0.044347368	0.1	0.118033988749895
$4r_s$	0.035477895	0.125	0.154700538379252
$3r_s$	0.026608421	0.16666666667	0.224744871391589
$2r_s$	0.017738947	0.25	0.414213562373095
$1.5r_s$	0.013304210	0.33333333333	0.732050807568878
$1.25r_s$	0.011086842	0.4	1.236067977499790
r_s	**0.008869474**	**0.5**	**Equation break down**
$0.5r_s$	0.004434737	1	Equation break down
$0.2r_s$	0.001773895	2.5	Equation break down
$0.1r_s$	0.000886947	5	Equation break down
$0.01r_s$	0.000088695	50	Equation break down
$0.001r_s$	0.000008869	500	Equation break down
$0.0001r_s$	0.000000887	5000	Equation break down
$0.00001r_s$	0.000000089	50000	Equation break down
$0r_s$	0	Equation break down	Equation break down

Gravitational Red Shift From Gravitational Time Dilation

Above we simply replaced the escape velocity embedded in Einstein's gravitational redshift formula with our modified escape velocity. We can also derive the redshift from the Einstein time dilation formula (which is the same thing). We have

$$\lambda = c\Delta\tau \tag{36}$$

where λ is the observed wavelength and $\Delta\tau$ is the time interval as measured by an observer required for a single "wavelength" to be emitted or received. Based on this we must have

$$\frac{\lambda_R}{\lambda_e} = \frac{\tau_r}{\tau_e} \tag{37}$$

where λ_R is the wavelength as observed from the receiver and λ_e is the wavelength as observed from the emitter. This gives us the following relativistic redshift

$$\frac{\lambda_R}{\lambda_e} = \frac{1 + \frac{GM_e}{r_e c^2}}{1 + \frac{GM_r}{r_r c^2}} \tag{38}$$

Further the so-called fractional redshift is given by

$$\frac{\lambda_R - \lambda_e}{\lambda_e} = \frac{\tau_r}{\tau_e} - 1 = \frac{1 + \frac{GM_e}{R_e c^2}}{1 + \frac{GM_r}{R_r c^2}} - 1 \tag{39}$$

In the special case where the receiver is in outer space far from any gravitational field we have that $M_r = 0$ (only the measuring aparatus must contain some mass so we could argue $M_r \approx 0$, but this would hardly alter the result when reciving llight emitted from a large mass.) we get

$$
\begin{aligned}
\frac{\lambda_R-\lambda_e}{\lambda_e} &= \frac{1+\frac{GM_e}{r_e c^2}}{1+\frac{GM_r}{r_r c^2}}-1 \\
\lim_{r\to+\infty} z(r)=\frac{\lambda_R-\lambda_e}{\lambda_e} &= \frac{1+\frac{GM_e}{r_e c^2}}{1+\frac{G\times 0}{r_r c^2}}-1 \\
\lim_{r\to+\infty} z(r)=\frac{\lambda_R-\lambda_e}{\lambda_e} &= 1+\frac{GM_e}{r_e c^2}-1 \\
\lim_{r\to+\infty} z(r)=\frac{\lambda_R-\lambda_e}{\lambda_e} &= \frac{GM_e}{r_e c^2}
\end{aligned}
\tag{40}
$$

This is naturally the same formula as we got from simply replacing the escape velocity in Einstein's gravitational redshift formula earlier. We can also quantify the formulas above. That has been done in the table summary in the end of the chapter.

One of the famous experiments that is claimed to have confirmed general relativity with very high precision is the Pound and Rebka Jr. (1959) experiment. They measured the gravitational redshift in a tower over a distance of approximately 22.5 meters. This was an excellent experiment that got the same result as predicted by Einstein's general relativity theory. However, this experiment is provide evidence that the general relativity theory is a complete theory. The experiment was done in a very weak gravitational field where we know the approximation formulas should work very well.

5 Possibly Cosmological Implications

Hawkins (2010) has done an impressive empirical job in observing and studying redshift in quasars. Surprisingly he did not find excess time dilation in the High-z quasars as expected by predictions from standard cosmology. However, instead of claiming that the data were right and the current cosmology theories were incomplete, he introduced new ideas such as the existence of growing black holes that would offset the lacking excess gravitational time dilation. We suggest that the correct explanation of the High-z quasar studies may be based in the fact that the standard gravitational theory does not have a gravitational redshift theory that works well close to and below the Schwarzschild radius. As a result, many of the High-z redshift interpretations in cosmology are possibly wrong. Objects with for example $z > 0.5$ is in our theory interpreted simply as objects that sends out photons from inside the so-called Schwarzschild radius. Or should the so-called Schwarzschild radius even be considering a radius? Even this is questionable. Based on our theory, it is also not surprising if one should find some High-z objects in front of lower z objects, as those claimed to be observed by Arp (1987, 1998).

Further, in light of this theory many other predictions in cosmology, including the theory of the Big Bang interpretation of the universe could also be viewed as misguided. The Big Bang interpretation is largely built on a given interpretation of cosmological redshift. It is a misconception that the Big Bang theory and expanding universe is well tested and stands empirically out against alternative hypothesis, see Lòpez-Corredoira (2014) for a interesting summary. What if the predictions of mainstream models are misinterpreting redshifts from strong gravitational fields?

The entire logic around black holes is questionable. First, black holes were considered to be totally black due to the fact that nothing could escape from inside the Schwarzschild radius. Then, suddenly, Quasars where interpreted as black holes. Quasars are also considered the brightest objects on the sky that shine light on us from the other side of the universe. Therefore, black holes are not only shining, but also have to be growing rapidly to offset the lack of expected excess time dilation as predicted by the standard model?

6 Table Summary

The table below summarizes a series of the gravitational formulas described in this chapter.

7 Self Criticism

This paper can be criticized for simply combining classical mechanics with relativity theory and then mixing in gravity without making sure it is consistent with General relativity, or without being derived from scratch from a solid framework. Such criticism is valid and interesting, but does not necessarily mean that the approach presented in this paper not is on the right track. It could be important to see this paper in light of the recent development in relativity theory derived from scratch by Haug (2014)

Table 5: The table summarize many of the formulas given in this paper.

	Robust form traditional input :	**Robust form Planck input :**	**Einstein form traditional input weak field approx :**	**Einstein form Planck input weak field approx :**
Field strength	Any	Any	Weak field only	Weak field only
Limitations	"No" limitations	"No" limitations	$r >> r_s$	$r >> r_s$
Escape velocity	$\bar{v}_e = c\frac{\sqrt{1+\frac{2c^2 r}{GM}}}{1+\frac{c^2 r}{GM}}$	$\bar{v}_e = c\frac{\sqrt{1+\frac{2r}{N\aleph}}}{1+\frac{r}{N\aleph}}$	$v_e = \sqrt{\frac{2GM}{r}}$	$v_e = \sqrt{N\frac{2\aleph}{r}}$
Time dilation	$t_o = \frac{t_f}{1+\frac{GM}{rc^2}}$	$t_o = \frac{t_f}{1+\frac{N\aleph}{r}}$	$t_o = t_f\sqrt{1-\frac{\frac{2GM}{r}}{c^2}}$	$t_o = t_f\sqrt{1-\frac{2N\aleph}{r}}$
Time dilation	$t_1 = t_2\frac{\left(1+\frac{GM_2}{r_2 c^2}\right)}{\left(1+\frac{GM_1}{r_1 c^2}\right)}$	$t_1 = t_2\frac{\left(1+\frac{N_2\aleph}{r_2}\right)}{\left(1+\frac{N_1\aleph}{r_1}\right)}$	$t_1 = t_2\frac{\sqrt{1-\frac{\frac{2GM_1}{r_1}}{c^2}}}{\sqrt{1-\frac{\frac{2GM_2}{r_2}}{c^2}}}$	$t_1 = t_2\frac{\sqrt{1-\frac{2N_1\aleph}{r_1}}}{\sqrt{1-\frac{2N_2\aleph}{r_2}}}$
Time dilation different altitude	$t_1 = t_2\frac{\left(1+\frac{GM}{(r+A)c^2}\right)}{\left(1+\frac{GM}{rc^2}\right)}$	$t_1 = t_2\frac{\left(1+\frac{N\aleph}{r+A}\right)}{\left(1+\frac{N\aleph}{r}\right)}$	$t_1 = t_2\frac{\sqrt{1-\frac{\frac{2GM}{r}}{c^2}}}{\sqrt{1-\frac{\frac{2GM}{r+A}}{c^2}}}$	$t_1 = t_2\frac{\sqrt{1-\frac{2N\aleph}{r}}}{\sqrt{1-\frac{2N\aleph}{r+A}}}$
Redshift $\lim\limits_{r\to+\infty}$	$z(r) = \frac{GM_e}{R_e c^2}$	$z(r) = \frac{N\aleph}{R_e}$	$z(r) = \frac{1}{\sqrt{1-\frac{\frac{2GM_e}{c^2}}{R_e}}} - 1$	$z(r) = \frac{1}{\sqrt{1-\frac{2N\aleph}{R_e}}} - 1$
Redshift	$\frac{\lambda_R-\lambda_e}{\lambda_e} = \frac{1+\frac{GM_e}{R_e c^2}}{1+\frac{GM_r}{R_r c^2}} - 1$	$\frac{\lambda_R-\lambda_e}{\lambda_e} = \frac{1+\frac{N_e\aleph}{R_e}}{1+\frac{N_r\aleph}{R_r}} - 1$	$\frac{\lambda_R-\lambda_e}{\lambda_e} = \frac{\sqrt{1-\frac{\frac{2GM_r}{R_r}}{c^2}}}{\sqrt{1-\frac{\frac{2GM_e}{R_e}}{c^2}}} - 1$	$\frac{\lambda_R-\lambda_e}{\lambda_e} = \frac{\sqrt{1-\frac{\frac{2N_r\aleph}{R_r}}{c^2}}}{\sqrt{1-\frac{\frac{2N_e\aleph}{R_e}}{c^2}}} - 1$
Redshift different altitude	$\frac{\lambda_R-\lambda_e}{\lambda_e} = \frac{1+\frac{GM}{(r+A)c^2}}{1+\frac{GM}{rc^2}} - 1$	$\frac{\lambda_R-\lambda_e}{\lambda_e} = \frac{1+\frac{N\aleph}{r+A}}{1+\frac{N\aleph}{r}} - 1$	$\frac{\lambda_R-\lambda_e}{\lambda_e} = \frac{\sqrt{1-\frac{\frac{2GM}{r}}{c^2}}}{\sqrt{1-\frac{\frac{2GM}{r+A}}{c^2}}} - 1$	$\frac{\lambda_R-\lambda_e}{\lambda_e} = \frac{\sqrt{1-\frac{\frac{2N\aleph}{r}}{c^2}}}{\sqrt{1-\frac{\frac{2N\aleph}{r+A}}{c^2}}} - 1$
	Function of y :		**Function of y :**	
Redshift $\lim\limits_{r\to+\infty}$	$z(r) = \frac{1}{2y}$		$z(r) = \frac{1}{\sqrt{1-y}} - 1$	

from classical mechanical particles. That is from atomism, where the fundament is very similar to that of the Newton corpuscular. The new indivisible relativity theory derived by Haug gives all the same mathematical end results as Einstein's special relativity theory, but at the same time shows that special relativity is incomplete, and the work provides a series of additional results. Most of the new results given in this paper can potentially be derived from scratch from atomism. This is unclear, but is something we will look into over the next few years.

Based on the extreme simplicity and common sense logic in indivisible relativity theory, it would not surprise me if this is also the path to a better gravity theory that holds for strong gravitational fields and does not have the current strange set of interpretations, including the Schwarzschild (Hilbert) metric of the Einstein field equation.

What is more important is if the formulas given in this paper stand up against tests on gravitational redshifts, gravitational time dilations, and escape velocities, for example. From what we can see, there has not yet been a single experiment that is inconsistent with the formulas given in this paper. Most experiments that we truly have control over have been done in very weak gravitational fields, taking the famous Pound and Rebka Jr. (1959) experiment, for example, where the theory presented here and the results given by Einstein are almost indistinguishable.

Possibly some high precision redshift studies have or can be done that would distinguish the results as predicted by this paper and the results as predicted by the standard formulas. We think this should be investigated further, and this it is one of my first papers on gravity; it will not be the last.

8 Conclusion

We have derived an exact escape velocity based on Newton and special relativity theory that also holds for very strong gravitational fields. The standard escape velocity used in modern physics is only an approximate escape velocity that not is valid in strong gravitational fields. This paper suggests that the interpretations of the Schwarzschild radius in modern physics are incorrect. There are likely no black holes, the escape velocity at the Schwarzschild radius is not c, and time does not stand still at the Schwarzschild radius. There is likely "nothing" special about the Schwarzschild radius, except perhaps a set of mathematical artifacts that are the result of mathematical approximations, indeed approximations that are not valid in strong gravitational fields. In addition to the central discussion, we have also quantified the escape velocity, the gravitational time dilation and the gravitational redshift. Some people may claim that the solutions given in this paper must be incomplete since they do not use GR to be derived. We look forward to a debate on these topics. The new escape velocity, gravitational time dilation, and gravitational redshift introduced this paper can hopefully be a small piece in helping to bring physics and cosmology back on the right track?

Appendix A

Derivation of the standard escape velocity from Planck scale as first shown by Haug (2016b)

$$\begin{aligned}
E &\approx \frac{1}{2}mv^2 - \frac{GmM}{r} \\
E &\approx \frac{1}{2}N_1 m_p v^2 - \frac{GN_1 m_p N_2 m_p}{r} \\
E &\approx \frac{1}{2}N_1 \frac{\hbar}{\aleph}\frac{1}{c}v^2 - \frac{N_1 \frac{\aleph^2 c^3}{\hbar}\frac{\hbar}{\aleph}\frac{1}{c}N_2\frac{\hbar}{\aleph}\frac{1}{c}}{r} \\
E &\approx \frac{1}{2}N_1 \frac{\hbar}{\aleph}\frac{1}{c}v^2 - N_1 N_2 \frac{\hbar}{r}c
\end{aligned} \tag{41}$$

where N_1 is the number of Planck masses in the smaller mass m (for example a rocket) and N_2 is the number of Planck masses in the other mass. This we have to set to 0 and solve with respect to v to find the escape velocity:

$$\begin{aligned}
\frac{1}{2}N_1 \frac{\hbar}{\aleph}\frac{1}{c}v_e^2 - N_1 N_2 \frac{\hbar}{r}c &= 0 \\
v_e^2 &= 2\frac{N_1 N_2 \frac{\hbar}{r}c}{N_1 \frac{\hbar}{\aleph}\frac{1}{c}} \\
v_e^2 &= 2N_2 \frac{\aleph c^2}{r} \\
v_e &= c\sqrt{N_2 \frac{2\aleph}{r}}
\end{aligned} \tag{42}$$

This is a quantized escape velocity. Bear in mind that the kinetic energy of $\frac{1}{2}mv^2$ is only a good approximation for $v << c$. Still, for all planets in our solar system and even for the massive Sun itself, the escape velocity from the surface of these "objects" will be so small that $v << c$. Only when we approach the escape velocity at the Schwarzschild radius are the approximations in this Appendix inaccurate. Since N_1 cancels out, we can simply call N_2 for N and write the escape velocity as

$$v_e = c\sqrt{N\frac{2\aleph}{r}} \tag{43}$$

where N is the number of Planck masses in the mass we are trying to escape from.

Appendix B: Gravitational Time Dilation at Planck Scale

We can rewrite the standard Einstein gravitational time dilation in the form of quantized escape velocity (derived above).

$$\begin{aligned}
t_o &= t_f\sqrt{1 - \frac{v_e^2}{c^2}} \\
t_o &= t_f\sqrt{1 - \frac{\left(c\sqrt{2N\frac{\aleph}{r}}\right)^2}{c^2}} \\
t_o &= t_f\sqrt{1 - \frac{2N\aleph}{r}}
\end{aligned} \tag{44}$$

Let's see if we can calculate the time dilation at, for example, the surface of the Earth from Planck scale gravitational time dilation. The Earth's mass is 5.972×10^{24} kg. And again, the Earth's mass in terms of the Planck mass must be $\frac{5.972\times10^{24}}{2.17651\times10^{-8}} \approx 2.74388 \times 10^{32}$. Further, the radius of the Earth is $r \approx 6\ 371\ 000$ meters. We can now just plug this into the quantized gravitational time dilation

$$
\begin{aligned}
t_o &= t_f\sqrt{1-\frac{2N\aleph}{r}} \\
t_o &= t_f\sqrt{1-\frac{2\times 2.74388\times 10^{32}\times 1.61622837\times 10^{-35}}{6\ 371\ 000}} \approx t_f \times 0.999999999303915
\end{aligned}
$$

That is for every second that goes by in outer space (a clock far away from the massive object), 0.99999999930391500 seconds goes by on the surface of the Earth. That is, for every year in in outer space (very far from the Earth), there are about 22 milliseconds left to reach an Earth year. This is naturally the same as we would get with Einstein's formula.

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