A Series of Questions About Some Series: Primes, Waves, and Factors

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June 9, 2016

Abstract

A Trigonometric Series is crafted that gives the number of factors of \( x \) for all \( x \). It is then manipulated to list all those factors. It is then used to give the exact distribution of the primes. It is then changed into a recursive sequence that generates the \( n \)th prime. Finally, a Product Polynomial Series is suggested as an improved alternative to the original. There is a series of questions in the second section regarding the functions.

1 Introduction

This function, \( F(x) \), tells you the number of factors of a number \( x \), not including the number 1. You can add 1 to its value, or run the index \( k \) from 1 instead, to get the exact value.

\[
F(x) = \sum_{k=2}^{j} \frac{\cos \frac{2\pi x}{k} - \cos \frac{1}{k} + |\cos \frac{2\pi x}{k} - \cos \frac{1}{k}|}{2 \times (1 - \cos \frac{1}{k})} \text{ for } j \geq x
\] (1)

This function, \( T(x) \), outputs a Binary Looking Base 10 digit, whose 1s positions are all the factors of the number.

\[
T(x) = \sum_{k=1}^{j} \frac{10^{k-1} \times (\cos \frac{2\pi x}{k} - \cos \frac{1}{k} + |\cos \frac{2\pi x}{k} - \cos \frac{1}{k}|)}{2 \times (1 - \cos \frac{1}{k})} \text{ for } j \geq x
\] (2)

This next function, \( G(x) \), equals 0 for all primes, and the (number of factors of \( x \)) - 2 for all composites.

\[
G(x) = \frac{F(x) - 1 + |F(x) - 1|}{2}
\] (3)

\( H(x) \) is 0 for all primes and 1 for all composites.

\[
H(x) = \frac{G(x) + 1 - |1 - G(x)|}{2}
\] (4)
This is \( H(x) \) in terms of \( F(x) \).

\[
H(x) = \frac{F(x) + 1 + |F(x) - 1| - |3 - F(x) - |F(x) - 1||}{4}
\]  

(5)

This is the exact prime counting function.

\[
\pi(x) = x - 1 - \sum_{n=1}^{x} H(n)
\]  

(6)

This is the full form for the counting function.

\[
\pi(x) = x - 1 - \sum_{n=1}^{x} \frac{\sum_{j=2}^{k} \cos \frac{2\pi n}{k} \cos \frac{1}{2} \left( |\cos \frac{2\pi n}{k} - \cos \frac{1}{2} | \right)}{2 + \left( 1 - \cos \frac{1}{2} \right)} + 1 + \left( \frac{\sum_{k=2}^{n} \cos \frac{2\pi n}{k} - \cos \frac{1}{2} \left( |\cos \frac{2\pi n}{k} - \cos \frac{1}{2} | \right)}{2 + \left( 1 - \cos \frac{1}{2} \right)} \right) - 1 - \left( \frac{3 - \left( \frac{\sum_{j=2}^{k} \cos \frac{2\pi n}{k} - \cos \frac{1}{2} \left( |\cos \frac{2\pi n}{k} - \cos \frac{1}{2} | \right)}{2 + \left( 1 - \cos \frac{1}{2} \right)} \right)}{4} \right)
\]  

(7)

This is a Prime Terminal Recursive Sequence that always ends on the \( n \)th prime, in less iterations than \( n \), where \( Q_0 = 0 \), \( Q_1 = x \), and \( \pi \) is the counting function.

\[
Q_s(n) = n + Q_{s-1} - \pi(Q_{s-1})
\]  

(8)

2 Questions

Can \( F(x) \) be simplified?

Is it better to use the square root of the squared rather than the absolute value?

What pros and cons are there for starting with other periodic functions, and in general what other ways could \( F(x) \) be adapted?

\( F(x) \) does not converge, but rather oscillates between 0 and the number of factors of \( x \). However, it is bounded. What is the closest bound you can find?

In other words, what is the bound of the Number of Factors of \( x \) in terms of \( x \)?

By treating the Binary Looking Base 10 digit output of \( T(x) \) as an actual Base 2 binary digit, and converting it into base 10, do you always get a prime? If so, it only spans a subset of the primes.

Are there more useful tags for \( T(x) \) than \( 10^{k-1} \)?

What are the efficiencies and Time Complexity Spaces of \( F(x) \), \( \pi(x) \), and \( Q_s(n) \)?

Which open problems can \( F(x) \) be used to help solve? For example, showing whether the system \( F(x) = F(x + 2) = 1 \) has infinite solutions, confirms or disproves the Twin Prime Conjecture, and similarly, so does \( F(2^m - 1) = 1 \) for the Mersenne Conjecture.

Does the Series of Product Polynomials, \( F(x) \), mentioned in the next section, show that the primes can be calculated in polynomial time?
3 Attempted Work and Other Stuffs

I have tried many methods of simplifying and or changing F(x), and while I’ve come up with plenty of variants, not many seem as concise. A simpler version would be interesting to see. One variation that stood out as most likely being more computationally efficient was this one.

\[
F(x) = \sum_{k=1}^{j} \left( (-1)^{\left\lfloor \frac{k}{x} \right\rfloor} * (4x + k - 2k * \left\lfloor \frac{2x}{k} \right\rfloor) \right) + \frac{1}{2} \left[ ((-1)^{\left\lfloor \frac{k}{x} \right\rfloor} * (4x + k - 2k * \left\lfloor \frac{2x}{k} \right\rfloor)) + (1 - k) \right]
\]

Another rather intriguing candidate, with much that could be said about it, is this one.

\[
F(x) = \sum_{k=2}^{x} \frac{1 - \left( \prod_{n=1}^{b} (x - kn) \right)^2}{2} + \left| 1 - \left( \prod_{n=1}^{b} (x - kn) \right)^2 \right|
\]

Which is the same as this, where \( \theta(x) \) is the Unit Step Function, the subscript is Pochhammer Notation, and \( b \geq \left\lfloor \frac{x}{2} \right\rfloor \).

\[
F(x) = \sum_{k=2}^{x} \theta \left( -(-k)^{2b} * \left( \left( 1 - \frac{x}{k} \right)_{b} \right)^2 \right)
\]

Those last 2 equations show that the number of factors of a number can be calculated from a finite polynomial. This also means that the distribution of primes and the nth prime can both be calculated from a finite polynomial. Note that each nth prime is not coming from the same polynomial, just rather a related family of polynomials. Also, those 2 equations lead to some interesting Primality tests such as:

\[
Signum \prod_{k=2}^{j} \left( 1 - \frac{x}{k} \right)_{b} = 1
\]

for \( \frac{x}{2} \leq b \leq (x - 1) \) and \( \frac{x}{2} \leq j \leq (x - 1) \).

And this next one, is where if it’s 0 it’s not prime and if it’s > 0 it’s prime.

\[
\prod_{k=2}^{\left\lfloor \frac{x}{2} \right\rfloor} \left( 1 - \frac{x}{k} \right)_{b} > 0
\]

And this one, is where 0 equals composite, and < or > 0 it equals a prime.

\[
\sum_{k=2}^{j} \left( 1 - \frac{x}{k} \right)_{b} < > 0
\]

Another nice construction is this, as a Primes = 1 Wave.
\[ P(x) = \frac{1 - |F(x) - 1| + |1 - |F(x) - 1||}{2} \] (15)

As for the question about the bound of \( F(x) \), it’s easy to bound by say, \( y = x \), or closer by \( 2\sqrt{x} \cos^2(\pi x) \), but even better bounds would likely be more valuable as they could lead to a simplification of the series. I think at one point, I found a bound that just touched all the composite peaks, but that function is lost in my notes somewhere at the moment.

As for better tags for \( T(x) \), the main requirement is that the tag makes each value add to the series in some linearly independent fashion. The tag used does this by placing each value in its own decimal place to the left of the decimal. An easy alternative would be the right side decimal places, but besides that at the moment, I don’t know any other options or what might be better.

As for the efficiency and time complexity questions, those topics are new to me. I’ve been slowly learning about time complexity when I have the chance, but I have not found enough charts or equations showing specific time complexities for certain functions, such that I can say what class these are. Therefore, I have yet to get a handle on or understand the efficiency spaces of these equations.

As to other problems that can be viewed through the lens of these equations, there are many on Wikipedia’s list of related prime problems. I have found many alternate restatements for problems such as the Twin Prime Conjecture, but have not included them all here. Often, the condition requires showing an infinite number of roots. If you can see any poignant restatements using these tools, it would be interesting to see.

As to the Series of Product Polynomials, I do think it shows primes can be calculated in polynomial time. The product is finite, and the finite sum of that product is finite. The following changes that take place during \( H(x) \), also leave it finite, so that it’s still finite entering \( \pi(x) \). The sum in \( \pi(x) \) also still leaves it finite. Lastly, the recursive sequence is just more adding, so it should still be a finite polynomial. Any comments on this would be appreciated.

4 Conclusion

Many other interesting equations related to this topic have not been posted, but I may add some if I see some more that stand out or are relevant. There are some connecting \( F(x) \) to Bernoulli Numbers and the Riemann Function but they were not full developed yet and are hiding somewhere in some notes.

In conclusion, these are useful formulae for analyzing the primes and can be used to answer previously unanswered questions about the topic.