# SERIES REPRESENTATION OF POWER FUNCTION

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ABSTRACT. In this paper described numerical expansion of natural-valued power function $x^n$, in point $x = x_0$ where $n,\ x_0$ - natural numbers. Applying numerical methods, that is calculus of finite differences, namely, discrete case of Binomial expansion is reached. Received results were compared with solutions according to Newton's Binomial theorem and MacMillan Double Binomial sum. Additionally, in section 4 exponential function's $e^x$ representation is shown.

**Keywords.** Power function, Binomial coefficient, Binomial Theorem, Finite difference, Perfect cube, Exponential function, Pascal's triangle, Series representation, Binomial Sum, Multinomial theorem, Multinomial coefficient, Binomial distribution, Hypercube

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## 1. INTRODUCTION AND MAIN RESULTS

In this paper coefficient $\left\llbracket {n \atop k} \right\rrbracket$ is introduced and its properties are shown, one of which is power's expansion. We will generalize the partial case of power $n = 3$, received using $\left\llbracket {n \atop k} \right\rrbracket$ to high power $u > 3$. First, let basically describe Newton's

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Binomial Theorem, since our coefficient $\left[\!\left[ {n \atop k} \right]\!\right]$ is derived from finite difference of perfect cubes, which is taken regarding Binomial expansion. In elementary algebra, the Binomial theorem describes the algebraic expansion of powers of a binomial. The theorem describes expanding of the power of $(x+y)^n$ into a sum involving terms of the form $ax^by^c$ where the exponents $b$ and $c$ are nonnegative integers with $b+c=n$, and the coefficient $a$ of each term is a specific positive integer depending on $n$ and $b$. The coefficient $a$ in the term of $ax^by^c$ is known as the Binomial coefficient. The main properties of the Binomial Theorem are next

**Properties 1.1.** *Binomial Theorem properties*

(1) *The powers of $x$ go down until it reaches $x_0 = 1$ starting value is $n$ (the $n$ in $(x+y)^n$)*
(2) *The powers of $y$ go up from 0 ($y^0 = 1$) until it reaches $n$ (also $n$ in $(x+y)^n$)*
(3) *The $n$-th row of the Pascal's Triangle (see* [1], [13]*) will be the coefficients of the expanded binomial.*
(4) *For each line, the number of products (i.e. the sum of the coefficients) is equal to $x+1$*
(5) *For each line, the number of product groups is equal to $2^n$*

According to the Binomial theorem, it is possible to expand any power of $x+y$ into a sum of the form (see [2], [4])

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{1.2}$$

Let expand monomial $x^n$ such that $(x,\ n) \in \mathbb{N}$ applying finite differences, that are reached by means of Binomial theorem (1.2)

**Lemma 1.3.** *Power function could be represented as discrete integral of its first order finite difference*

$$\begin{aligned} x^n &= ⨍_0^{x-1} \Delta_h[x^n]h \\ &= \sum_{k=0}^{x-1} \underbrace{nk^{n-1}h + \binom{n}{2}k^{n-2}h^2 + \cdots + \binom{n}{n-1}kh^{n-1} + h^n}_{\Delta_h[x^n]=(x+h)^n - x^n} \\ &= \sum_{j=0}^{x-1}\sum_{k=1}^{n} \binom{n}{k} j^{n-k}h^k,\ h \in \mathbb{R} \end{aligned} \tag{1.4}$$

We can reach the same result using Fundamental Theorem of Calculus

$$x^n = \int_0^x nt^{n-1}dt = \sum_{k=0}^{x-1}\int_k^{k+1} nt^{n-1}dt = \sum_{k=0}^{x-1}(k+1)^n - k^n \tag{1.5}$$

**Lemma 1.6.** *From lemma (1.4) follows that finite difference of power $x^n,\ n \in \mathbb{N}$ could be reached by Binomial expansion of the form*

$$\Delta_h[x^n] = (x+h)^n - x^n = \sum_{k=1}^{n} \binom{n}{k} x^{n-k}h^k \tag{1.7}$$

Otherwise, let be a difference table of perfect cubes (see also [6], eq. 7) $\Delta^k[x^3],\ x \in \mathbb{N},\ k \in [1,3] \subseteq \mathbb{N}$ over $x$ from $[0,7] \subseteq \mathbb{N}$

(1.8)

| $x$ | $x^3$ | $\Delta[x^3]$ | $\Delta^2[x^3]$ | $\Delta^3[x^3]$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 6 | 6 |
| 1 | 1 | 7 | 12 | 6 |
| 2 | 8 | 19 | 18 | 6 |
| 3 | 27 | 37 | 24 | 6 |
| 4 | 64 | 61 | 30 | 6 |
| 5 | 125 | 91 | 36 | |
| 6 | 216 | 127 | | |
| 7 | 343 | | | |

Figure 1: Difference table of $x^3,\ x \in \mathbb{N}$ up to $3^{\text{rd}}$ order, [6], eq. 7

Note that increment $h$ is set to be $h = 1$ and $k > 2$-order difference is taken regarding to [10]. From Figure 1 the regularity concerning sequence A008458, [8] in OEIS is observed, [9], that is the first order finite difference of consequent perfect cubes equals to $1 + a(n)$, where $a(n)$ is generating function of sequence A008458. Let show, that

$$\Delta_{h=1}[x^3] = (x+1)^3 - x^3 = 1 + 3! \cdot 0 + 3! \cdot 1 + \cdots + 3! \cdot x \tag{1.9}$$

Applying compact sigma notation on (1.9), by means of identities,

$$\begin{aligned} \Delta_{h=1}[x^3] &= 1 + 3! \cdot 1 + 3! \cdot 2 + \cdots + 3! \cdot x \\ \nabla_{h=1}[x^3] &= 1 + 3! \cdot 0 + 3! \cdot 1 + \cdots + 3! \cdot (x-1) \end{aligned} \tag{1.10}$$

the first order forward, as well as backward finite difference of consequent perfect cubes could be represented as

$$\begin{aligned} \Delta_{h=1}[x^3] &= \sum_{k=1}^{x} 3! \cdot k + \frac{1}{x} = 3!\binom{x+1}{2} + 1 \\ &= \sum_{k=1}^{x} 3!\binom{k}{k-1} + \frac{1}{x}, \quad x = x_0 \in \mathbb{N},\ x \neq 0 \end{aligned} \tag{1.11}$$

$$\begin{aligned} \nabla_{h=1}[x^3] &= \sum_{k=0}^{x-1} 3! \cdot k + \frac{1}{x} \\ &= \sum_{k=0}^{x-1} 3!\binom{k}{k-1} + \frac{1}{x}, \quad x = x_0 \in \mathbb{N},\ x \neq 0 \end{aligned} \tag{1.12}$$

By lemma (1.3) discrete integral of first difference is used to reach expansion of monomial $x^n,\ h \in \mathbb{R}$ and we have an equality between (1.7) and (1.11)

$$\Delta_{h=1}[x^3] = \sum_{k=1}^{3} \binom{3}{k} x^{3-k} = \sum_{k=1}^{x} 3! \cdot k + \frac{1}{x},\ x = x_0 \in \mathbb{N},\ x \neq 0 \tag{1.13}$$

Then we have right to substitute (1.11) into (1.4) instead binomial expansion and represent perfect cube $x^3$ as summation over $k$ from 0 to $x - 1$. Let be extended example of expansion of $x^3,\ x \in \mathbb{N}$

$$
\begin{aligned}
(1.14)\quad x^3 &= (1 + 3! \cdot 0) + (1 + 3! \cdot 0 + 3! \cdot 1) + (1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2) \\
&\quad \cdots + (1 + 3! \cdot 0 + 3! \cdot 1 + 3! \cdot 2 + \cdots + 3! \cdot (x-1)) \\
&= x + (x-0) \cdot 3! \cdot 0 + (x-1) \cdot 3! \cdot 1 + (x-2) \cdot 3! \cdot 2 \\
&\quad \cdots + (x-(x-1)) \cdot 3! \cdot (x-1)
\end{aligned}
$$

Provided that $x$ is natural. Note that increment $h$ in equation (1.14) is set to be $h = 1$ and not displayed. Using compact sigma notation on (1.14), we received

$$
\begin{aligned}
(1.15)\qquad x^3 &= x + 3! \cdot \sum_{m=0}^{x-1} mx - m^2 = x + 3! \cdot \sum_{m=0}^{x} mx - m^2 \\
&= x + 3! \cdot \sum_{m=1}^{x} mx - m^2 = \sum_{m=0}^{x-1} 3! \cdot mx - 3! \cdot m^2 + 1 \\
&= \sum_{m=1}^{x} 3! \cdot mx - 3! \cdot m^2 + 1, \quad x \in \mathbb{N},\ n \in \mathbb{N}
\end{aligned}
$$

**Property 1.16.** *Let be a sets $\mathfrak{S}(x) := \{1,\ 2,\ \ldots,\ x\} \subseteq \mathbb{N}$, $\mathfrak{C}(x) := \{0,\ 1,\ \ldots,\ x\} \subseteq \mathbb{N}$, $\mathfrak{U}(x) := \{0,\ 1,\ \ldots,\ x-1\} \subseteq \mathbb{N}$, let be right part of (1.15) written as*

$$T(x,\ \mathfrak{U}(x)) := \sum_{m \in \mathfrak{U}(x)} 3! \cdot mx - 3! \cdot m^2 + 1$$

*where $x \in \mathbb{N}$ is variable and $\mathfrak{U}(x)$ is iteration set of (1.15), then we have equality*

$$(1.17)\qquad T(x,\ \mathfrak{U}(x)) = T(x,\ \mathfrak{S}(x)),\ x \in \mathbb{N}$$

*Let be central part of (1.15) denoted as*

$$U(x,\ \mathfrak{C}(x)) := x + 3! \cdot \sum_{m \in \mathfrak{C}(x)} mx - m^2\ ,$$

*then*

$$(1.18)\qquad U(x,\ \mathfrak{C}(x)) = U(x,\ \mathfrak{S}(x)) = U(x,\ \mathfrak{U}(x))$$

*Other words, change of iteration sets of central and right parts of (1.15) by $\mathfrak{C}(x),\ \mathfrak{S}(x),\ \mathfrak{U}(x)$ and $\mathfrak{U}(x),\ \mathfrak{S}(x)$ respectively, doesn't change resulting value of (1.15) given $x \in \mathbb{N}$.*

*Proof.* Let be a plot of $y = 3! \cdot kx - 3! \cdot k^2 + 1$ by $k$ over $\mathbb{R}^+_{\leq 10}$, given $x = 10$

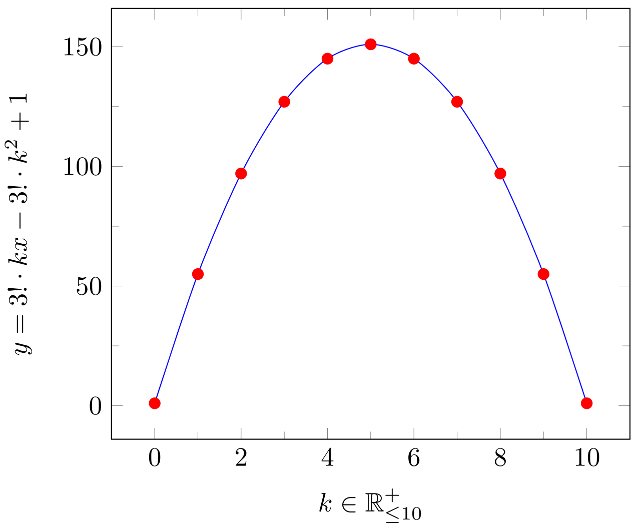


Figure 2. Plot of $y = 3! \cdot kx - 3! \cdot k^2 + 1$ by $k$ over $\mathbb{R}^{+}_{\leq 10}$, given $x = 10$

Obviously, being a parabolic function, it's symmetrical over $\frac{x}{2}$, hence equivalent $T(x,\ \mathfrak{U}(x)) = T(x,\ \mathfrak{S}(x)),\ x \in \mathbb{N}$ follows. Reviewing central part of (1.15) and denote $u(t) = tx^{n-2} - t^2x^{n-3}$, we can conclude, that $u(0) = u(x)$, then equality of $U(x,\ \mathfrak{C}(x)) = U(x,\ \mathfrak{S}(x)) = U(x,\ \mathfrak{U}(x))$ immediately follows.
This completes the proof. $\square$

Note that for right part of (1.15) property (1.16) holds only in case (1.17). Let analyse property (1.16). Let be an example of triangle[1] built using right part of expression (1.15) over $x$ from 0 up to $x = 4$. We can see that for each row corresponded to variable $x = 0,\ 1,\ 2,\ 3,\ 4, \ldots$, we have Binomial distribution of row items, more detailed

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 7 & & 1 & & \\
 & 1 & & 13 & & 13 & & 1 & \\
1 & & 19 & & 25 & & 19 & & 1
\end{array} \tag{1.19}$$

Figure 3. Triangle generated by $3! \cdot kx - 3! \cdot k^2 + 1$, that is right part of (1.15) from 0 up $x = 4$ over $k$ from 0 to $x$, sequence A287326 in OEIS, [9], [12].

One could compare Figure (1.19) with Pascal's triangle [1], [13]

[1]Note that each $x = 0,\ 1,\ 2,\ 3, \ldots,\ \mathbb{N}$ row's item of triangle is generated by $3! \cdot kx - 3! \cdot k^2 + 1$ over $k \in [0,\ x] \subseteq \mathbb{N}$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Figure 4. Pascal's triangle built up given $n = 4$ over $k$ from 0 to $n$, sequence A007318 in OEIS, [9], [1], [13].

Review the triangle (1.19), define the $k$-th, $0 \geq k \leq n$, item of $n$-th row of triangle as

**Definition 1.20.**

$$\left\llbracket \begin{matrix} n \\ k \end{matrix} \right\rrbracket := 3! \cdot nk - 3! \cdot k^2 + 1, \quad 0 \leq k \leq n \tag{1.21}$$

Note that definition (1.20) also could be rewritten as

$$\left\llbracket \begin{matrix} n \\ k \end{matrix} \right\rrbracket = 3! \cdot nk - 3! \cdot n^0 k^2 + n^0, \quad 0 \leq k \leq n \tag{1.22}$$

Let us approach to show a few properties of triangle (1.19)

**Properties 1.23.** *Properties of triangle (1.19).*

(1) *Summation of items* $\left\llbracket \begin{matrix} n \\ k \end{matrix} \right\rrbracket$ *of $n$-th row of triangle (1.19) over $k$ from 0 to $n-1$ returns perfect cube $n^3$ as binomial of the form*

$$\sum_{k=0}^{n-1} \left\llbracket \begin{matrix} n \\ k \end{matrix} \right\rrbracket = A_{0,n} n - B_{0,n} = n^3, \tag{1.24}$$

*Since the property (1.17) holds, (1.24) could be rewritten as*

$$\sum_{k=1}^{n} \left\llbracket \begin{matrix} n \\ k \end{matrix} \right\rrbracket = A_{1,n} n - B_{1,n} = n^3, \tag{1.25}$$

*where $A_{\overline{0,1},n}$ and $B_{\overline{0,1},n}$ - integers **depending** on variable $n \in \mathbb{N}$ and on sets $\mathfrak{U}(n)$, $\mathfrak{S}(n)$, respectively.*

(2) *Recurrence relation between $A_{0,n}$ and $A_{1,n}$*

$$A_{0,n+1} = A_{1,n}, \; n \geq 1$$

(3) *Induction by power. Summation of items of $n$-th row of triangle (1.19), multiplied by $n^{m-3}$, from 0 to $n-1$ returns $n^m$*

(4) *Summation of items* $\left\llbracket \begin{matrix} n \\ k \end{matrix} \right\rrbracket$ *of $n$-th row of triangle (1.19) over $k$ from 0 to $n$ returns $n^3 + n^0$*

$$\sum_{k=0}^{n} \left\llbracket \begin{matrix} n \\ k \end{matrix} \right\rrbracket = n^3 + 1 \tag{1.26}$$

(5) *Induction by power. Summation of each $n$-th row of triangle (1.19) multiplied by $n^{m-3}$ from 0 to $n$ returns $n^m + n^{m-3}$*

(6) *First item of each row's number corresponding to central polygonal numbers sequence $a(n)$ (sequence A000124 in OEIS,* [9], [14]*) returns finite difference $\Delta[n^3]$ of consequent perfect cubes. For example, let be the $k$-th row of triangle (1.19), such that $k$ is central polygonal number, i.e $k = \frac{n^2+n+2}{2}$, $n = 0, 1, 2, ..., \mathbb{N}$, then item*

$$\left[\!\!\left[ \begin{matrix} \frac{n^2+n+2}{2} \\ 1 \end{matrix} \right]\!\!\right] = \Delta_h[n^3], \; h = 1 \tag{1.27}$$

(7) *Items of (1.19) have Binomial distribution over rows*

(8) *The linear recurrence, for any $k$ and $n > 0$*

$$2\left[\!\!\left[ \begin{matrix} n \\ k \end{matrix} \right]\!\!\right] = \left[\!\!\left[ \begin{matrix} n+1 \\ k \end{matrix} \right]\!\!\right] + \left[\!\!\left[ \begin{matrix} n-1 \\ k \end{matrix} \right]\!\!\right] \tag{1.28}$$

(9) *Linear recurrence, for each $n > k$*

$$2\left[\!\!\left[ \begin{matrix} n \\ k \end{matrix} \right]\!\!\right] = \left[\!\!\left[ \begin{matrix} 2n-k \\ k \end{matrix} \right]\!\!\right] + \left[\!\!\left[ \begin{matrix} 2n-k \\ 0 \end{matrix} \right]\!\!\right] \tag{1.29}$$

(10) *From (1.27) follows that*

$$n^3 = \sum_{k=0}^{n-1} \left[\!\!\left[ \begin{matrix} n \\ k \end{matrix} \right]\!\!\right] = \sum_{k=0}^{n-1} \left[\!\!\left[ \begin{matrix} \frac{k^2+k+2}{2} \\ 1 \end{matrix} \right]\!\!\right] \tag{1.30}$$

(11) *Triangle is symmetric, i.e*

$$\left[\!\!\left[ \begin{matrix} n \\ k \end{matrix} \right]\!\!\right] = \left[\!\!\left[ \begin{matrix} n \\ n-k \end{matrix} \right]\!\!\right]$$

(12) *A relation between $n$-th diagonal and corresponding column of triangle (1.19), note that it holds for any binomial distributed triangle as well*

$$\left[\!\!\left[ \begin{matrix} n+k \\ k \end{matrix} \right]\!\!\right] = \left[\!\!\left[ \begin{matrix} n+k \\ n \end{matrix} \right]\!\!\right]$$

In (1.24) is noticed, that summation of each $n$-th row of Triangle (1.19) from 0 to $n-1$ returns perfect cube $n^3$, then, by properties (1.27), (1.28), (1.29), for each given number $x \in \mathbb{N}$ the $x^n$ could be easy found via multiplication of each term of (1.24) by $x^{n-3}$

$$\begin{aligned} x^n &= \sum_{k=0}^{x-1} \left[\!\!\left[ \begin{matrix} x \\ k \end{matrix} \right]\!\!\right] x^{n-3} = \frac{1}{2} \sum_{k=0}^{x-1} \left[ \left[\!\!\left[ \begin{matrix} x+1 \\ k \end{matrix} \right]\!\!\right] + \left[\!\!\left[ \begin{matrix} x-1 \\ k \end{matrix} \right]\!\!\right] \right] x^{n-3} \\ &= \sum_{k=0}^{x-1} \frac{1}{2} \left[ \left[\!\!\left[ \begin{matrix} 2x-k \\ k \end{matrix} \right]\!\!\right] + \left[\!\!\left[ \begin{matrix} 2x-k \\ 0 \end{matrix} \right]\!\!\right] \right] x^{n-3} \\ &= \sum_{k=0}^{x-1} \frac{1}{2} \left[\!\!\left[ \begin{matrix} \frac{x^2+x+2}{2} \\ 1 \end{matrix} \right]\!\!\right] x^{n-3} = \sum_{k=0}^{x-1} \frac{1}{2} \left[ \left[\!\!\left[ \begin{matrix} \frac{x^2+x}{2} \\ 1 \end{matrix} \right]\!\!\right] + \left[\!\!\left[ \begin{matrix} \frac{x^2+x+4}{2} \\ 1 \end{matrix} \right]\!\!\right] \right] x^{n-3} \\ &= \sum_{k=0}^{x-1} \frac{1}{2} \left[ \left[\!\!\left[ \begin{matrix} \binom{n+1}{2} \\ 1 \end{matrix} \right]\!\!\right] + \left[\!\!\left[ \begin{matrix} \binom{n+1}{2} + \binom{2}{1} \\ 1 \end{matrix} \right]\!\!\right] \right] x^{n-3} \end{aligned} \tag{1.31}$$

To show other way of representation of power, let move the $x$ from (1.15), $x + 3! \sum mx - m^2$, under the sum operator and change iteration set from $\{0, \; x -$

$1\}$ to $\{1,\ x-1\}$, then we get

$$x^3 = \sum_{m=1}^{x-1} 3! \cdot mx - 3! \cdot m^2 + \frac{x}{(x-1)}, \qquad x \neq 1,\ x \in \mathbb{N} \tag{1.32}$$

Review right part of (1.32), let be item $\frac{x}{x-1}$ written as $\frac{x}{x-1} = \frac{x+1-1}{x-1} = 1 + \frac{1}{x-1}$, given the power $n > 3$, multiplying each term of (1.32) by $x^{n-3}$ we can observe that

$$x^n - 1 = \sum_{m=1}^{x-1} \left\lceil\!\left\lceil \begin{matrix} x \\ m \end{matrix} \right\rfloor\!\right\rfloor x^{n-3} + x^{n-4} + x^{n-5} + \cdots + x + 1 \tag{1.33}$$

Applying properties (1.27), (1.29), (1.30), we can rewrite (1.22) as

$$\begin{aligned} x^n - 1 &= \sum_{k=1}^{x-1} \frac{1}{2} \left[ \left\lceil\!\left\lceil \begin{matrix} 2x-k \\ k \end{matrix} \right\rfloor\!\right\rfloor + \left\lceil\!\left\lceil \begin{matrix} 2x-k \\ 0 \end{matrix} \right\rfloor\!\right\rfloor \right] x^{n-3} + x^{n-4} + \cdots + x + 1 \\ &= \sum_{k=1}^{x-1} \frac{1}{2} \left[ \left\lceil\!\left\lceil \begin{matrix} x+1 \\ k \end{matrix} \right\rfloor\!\right\rfloor + \left\lceil\!\left\lceil \begin{matrix} x-1 \\ k \end{matrix} \right\rfloor\!\right\rfloor \right] x^{n-3} + x^{n-4} + \cdots + x + 1 \\ &= \sum_{m=0}^{x-1} \frac{1}{2} \left[ \left\lceil\!\left\lceil \begin{matrix} \frac{m^2+m}{2} \\ 1 \end{matrix} \right\rfloor\!\right\rfloor + \left\lceil\!\left\lceil \begin{matrix} \frac{m^2+m+4}{2} \\ 1 \end{matrix} \right\rfloor\!\right\rfloor \right] x^{n-3} + x^{n-4} + \cdots + x + 1 \\ &= \sum_{k=1}^{x-1} \left\lceil\!\left\lceil \begin{matrix} \frac{k^2+k+2}{2} \\ 1 \end{matrix} \right\rfloor\!\right\rfloor x^{n-3} + x^{n-4} + \cdots + x + 1 \end{aligned} \tag{1.34}$$

Moving 1 from left part of (1.33) under sum operator, we add a term $\frac{1}{x-1}$ to initial function $\left\lceil\!\left\lceil \begin{smallmatrix} x \\ m \end{smallmatrix} \right\rfloor\!\right\rfloor x^{n-3} + x^{n-4} + x^{n-5} + \cdots + x + 1$. By means of expansion $\frac{1}{1-x} = -\frac{1}{x-1} = 1 + x + x^2 + x^3 + \cdots$, the (1.33) could be rewritten accordingly

$$x^n = \sum_{m=1}^{x-1} \left\lceil\!\left\lceil \begin{matrix} x \\ m \end{matrix} \right\rfloor\!\right\rfloor x^{n-3} + x^{n-4} + \cdots + x + 1 - 1 - x^2 - x^3 - \cdots \tag{1.35}$$

Generalizing (1.35) we have

$$\begin{aligned} x^n &= \sum_{m=1}^{x-1} \left[ \left\lceil\!\left\lceil \begin{matrix} x \\ m \end{matrix} \right\rfloor\!\right\rfloor - 1 \right] x^{n-3} - x^{n-2} - x^{n-1} - \cdots \\ &= \sum_{m=1}^{x-1} \left[ \left\lceil\!\left\lceil \begin{matrix} \frac{m^2+m+2}{2} \\ 1 \end{matrix} \right\rfloor\!\right\rfloor - 1 \right] x^{n-3} - x^{n-2} - x^{n-1} - \cdots \end{aligned} \tag{1.36}$$

A product generalization of even and odd power for each $x \in \mathbb{N}$, in case we define $\left\lceil\!\left\lceil \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor\!\right\rfloor$ as (1.22)

$$x^n = \begin{cases} \displaystyle\prod_{i=0}^{j} \sum_{k=0}^{x-1} \left\lceil\!\left\lceil \begin{matrix} x^0 \\ k \end{matrix} \right\rfloor\!\right\rfloor, \ n = 2j,\ j > 0,\ n \in \mathbb{N} \\ \displaystyle\prod_{i=0}^{j} \sum_{k=0}^{x-1} \left\lceil\!\left\lceil \begin{matrix} x^0 \\ k \end{matrix} \right\rfloor\!\right\rfloor x^{\frac{1}{j}}, \ n = 2j+1,\ j > 0,\ n \in \mathbb{N} \end{cases} \tag{1.37}$$

1.1. **Binomial Theorem based on definition (1.20).** Let be two positive integers $x,\ y$ raised to power $n \in \mathbb{N}$, then, using definition 1.20, we receive

$$
\begin{aligned}
(x+y)^n &= \sum_{k=0}^{x+y-1} \sum_{j=0}^{n-3} \left\llbracket \begin{matrix} x+y \\ k \end{matrix} \right\rrbracket \binom{n-3}{j} x^{n-3-j} y^j \qquad (1.38) \\
&= \frac{1}{2} \sum_{k=0}^{x+y-1} \sum_{j=0}^{n-3} \left[ \left\llbracket \begin{matrix} x+y+1 \\ k \end{matrix} \right\rrbracket + \left\llbracket \begin{matrix} x+y-1 \\ k \end{matrix} \right\rrbracket \right] \binom{n-3}{j} x^{n-3-j} y^j \\
&= \frac{1}{2} \sum_{k=0}^{x+y-1} \sum_{j=0}^{n-3} \left[ \left\llbracket \begin{matrix} 2(x+y)-k \\ k \end{matrix} \right\rrbracket + \left\llbracket \begin{matrix} 2(x+y)-k \\ 0 \end{matrix} \right\rrbracket \right] \binom{n-3}{j} x^{n-3-j} y^j \\
&= \sum_{k=0}^{x+y-1} \sum_{j=0}^{n-3} \left\llbracket \begin{matrix} \frac{k^2+k+2}{2} \\ 1 \end{matrix} \right\rrbracket \binom{n-3}{j} x^{n-3-j} y^j \\
&= \sum_{k=0}^{x+y-1} \frac{1}{2} \sum_{j=0}^{n-3} \left[ \left\llbracket \begin{matrix} \frac{k^2+k}{2} \\ 1 \end{matrix} \right\rrbracket + \left\llbracket \begin{matrix} \frac{k^2+k+4}{2} \\ 1 \end{matrix} \right\rrbracket \right] \binom{n-3}{j} x^{n-3-j} y^j
\end{aligned}
$$

Mathematica code of above expression is attached in Application 1.

1.2. **Multinomial case for $n=4$.** . For each $u, j, \dots, m \in \mathbb{N}$ and $n=4$ holds

$$
\begin{aligned}
(u+j+\cdots+m)^4 \quad &= \sum_{k=0}^{u+j+\cdots+m-1} \left\llbracket \begin{matrix} u+j+\cdots+m \\ k \end{matrix} \right\rrbracket u + \left\llbracket \begin{matrix} u+j+\cdots+m \\ k \end{matrix} \right\rrbracket j + \cdots \\
&\cdots + \left\llbracket \begin{matrix} u+j+\cdots+m \\ k \end{matrix} \right\rrbracket m
\end{aligned}
$$

1.3. **Generalized Multinomial Theorem based on definition (1.20).** Expression (1.21) could be generalized accordingly

$$
(1.39) \qquad (u+j+\cdots+m)^n
$$

$$
\begin{aligned}
= & \sum_{v=0}^{u+j+\cdots+m-1} \sum_{k_u+k_j+\cdots+k_m=n-4} \binom{n-4}{k_u, k_j, \dots, k_m} \left\llbracket \begin{matrix} u+j+\cdots+m \\ v \end{matrix} \right\rrbracket u^{k_u+1} j^{k_j} \cdots m^{k_m} \\
+ & \sum_{k_u+k_j+\cdots+k_m=n-4} \binom{n-4}{k_u, k_j, \dots, k_m} \left\llbracket \begin{matrix} u+j+\cdots+m \\ v \end{matrix} \right\rrbracket u^{k_u} j^{k_j+1} \cdots m^{k_m} + \cdots \\
\cdots & \sum_{k_u+k_j+\cdots+k_m=n-4} \binom{n-4}{k_u, k_j, \dots, k_m} \left\llbracket \begin{matrix} u+j+\cdots+m \\ v \end{matrix} \right\rrbracket u^{k_u} j^{k_j} \cdots m^{k_m+1}
\end{aligned}
$$

where

$$
(1.40) \qquad \binom{n}{k_u, k_j, \dots, k_m} = \frac{n!}{k_u!\, k_j! \cdots k_m!}
$$

is multinomial coefficient.

1.4. **Generalized Binomial Series by means of properties (1.24), (1.25).** Reviewing properties (1.24), (1.25) we can say that

$$x^n = A_{\overline{0,1},x} x^{n-2} - B_{\overline{0,1},x} x^{n-3},\ x = x_0 \in \mathbb{N} \tag{1.41}$$

Rewrite the right part of (1.42) regarding to itself as recursion

$$\begin{aligned}(1.42) x^n &= A_{\overline{0,1},x}(A_{\overline{0,1},x} x^{n-4} - B_{\overline{0,1},x} x^{n-5}) - B_{\overline{0,1},x}(A_{\overline{0,1},x} x^{n-5} - B_{\overline{0,1},x} x^{n-6}) \\ &= A^2_{\overline{0,1},x} x^{n-4} - 2A_{\overline{0,1},x} B_{\overline{0,1},x} x^{n-5} + B^2_{\overline{0,1},x} x^{n-6}\end{aligned}$$

Reviewing (1.43) we can observe binomial coefficients before each $A_{\overline{0,1},x} \cdot B_{\overline{0,1},x}$. Continuous $j$-times recursion of right part of (1.42) gives us

$$x^n = \sum_{k=0}^{j} (-1)^k \binom{j}{k} A^{j-k}_{\overline{0,1},x} B^k_{\overline{0,1},x} x^{n-2j-k} \tag{1.43}$$

Suppose that we want to repeat action (1.43) infinite-many times, then

$$\begin{aligned} x^n &= \binom{j}{0} A^j_{\overline{0,1},x} B^0_{\overline{0,1},x} x^{n-2j} - \binom{j}{1} A^{j-1}_{\overline{0,1},x} B^1_{\overline{0,1},x} x^{n-2j-1} + \cdots \\ &+ \binom{j}{2} A^{j-2}_{\overline{0,1},x} B^2_{\overline{0,1},x} x^{n-2j-2} - \cdots + \sum_{k}^{\infty} (-1)^k \binom{j}{k} A^{j-k}_{\overline{0,1},x} B^k_{\overline{0,1},x} x^{n-2j-k} + \cdots \end{aligned}$$

We know the solutions of above equation (1.42) for all $A_{\overline{0,1},x},\ B_{\overline{0,1},x}$ that present in follow table. The table arranged next way

| $x =$ | $A_{0,x} =$ | $B_{0,x} =$ | $A_{1,x} =$ | $B_{1,x} =$ |
|---|---|---|---|---|
| 1 | 1 | 0 | 6 | 5 |
| 2 | 6 | 4 | 18 | 28 |
| 3 | 18 | 25 | 36 | 81 |
| 4 | 36 | 80 | 60 | 176 |
| 5 | 60 | 175 | 90 | 325 |
| 6 | 90 | 324 | 126 | 540 |
| 7 | 126 | 539 | 168 | 833 |
| 8 | 168 | 832 | 216 | 1216 |
| 9 | 216 | 1215 | 270 | 1701 |
| 10 | 270 | 1700 | 330 | 2300 |

Table 1. Array of coefficients $A_{\overline{0,1},x},\ B_{\overline{0,1},x}$ over $x$ from 1 to 10.

Sequence $B_{1,x}$ is generated by $2x^3 + 3x^2,\ x \in \mathbb{N}$, sequence A275709 in OEIS, [9]. Sequence $A_{1,x}$ is generated by $3x^2 + 3x,\ x \in \mathbb{N}$, sequence A028896 in OEIS, [9].

1.5. **Comparison of (1.31) to Binomial Theorem and MacMillan Double Binomial sum.** In this subsection let compare (1.31) to Binomial Theorem and MacMillan Double Binomial sum[2] (see [3], eq. 12) hereby follow changes hold

[2]MacMillan Double Binomial sum is the other one numerical expansion of the form

$$\sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^{k-j} j^n \binom{k}{j} \binom{x}{k} = x^n$$

(1) While expansion is taken by Binomial theorem or MacMillan Double Binomial sum, the summation (1.2) is done over $k = 0, 1, 2..., \mathbb{N}$ from 0 and 1 to $n$, respectively, where $n \in \mathbb{N}$ - power. Reviewing (1.31) we can observe that summation is taken over $k$ from 0 to $x$, where $x$ - variable, provided that natural-valued.
(2) According to property (1.16) the iteration sets of right and central parts (1.15) could be changed to (1.17), (1.18) without changes in result.
(3) We observed that expansion (1.31) returns the power of each particular number $x \in \mathbb{N}$ as binomial of the form

$$A_{\overline{0,1},x} x^{n-2} - B_{\overline{0,1},x} x^{n-3}, \tag{1.44}$$

which holds in particular case $x = x_0 \in \mathbb{N}$.

## 2. $e^x$ Representation

Since the exponential function $f(x) = e^x,\ x \in \mathbb{N}$ is defined as infinite summation of $\frac{x^n}{n!},\ n = 0,\ 1,\ 2,\ ...,\ \infty$ over $n$ (see [5]). Then (1.31) could be applied, hereby

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=1}^{x-1} \left[ \left\| \begin{matrix} x \\ m \end{matrix} \right\| - 1 \right] x^{n-3} - x^{n-2} - x^{n-1} - \cdots \qquad (2.1) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=1}^{x-1} \left[ \left\| \begin{matrix} \frac{m^2+m+2}{2} \\ 1 \end{matrix} \right\| - 1 \right] x^{n-3} - x^{n-2} - x^{n-1} - \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=1}^{x-1} \left[ \frac{1}{2} \left[ \left\| \begin{matrix} x+1 \\ k \end{matrix} \right\| + \left\| \begin{matrix} x-1 \\ k \end{matrix} \right\| \right] - 1 \right] x^{n-3} - x^{n-2} - x^{n-1} - \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=1}^{x-1} \left[ \frac{1}{2} \left[ \left\| \begin{matrix} 2x-k \\ k \end{matrix} \right\| + \left\| \begin{matrix} 2x-k \\ 0 \end{matrix} \right\| \right] - 1 \right] x^{n-3} - x^{n-2} - x^{n-1} - \cdots \end{aligned}$$

From (2.1) we get

$$\begin{aligned} e^x - e &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=1}^{x-1} \left\| \begin{matrix} x \\ m \end{matrix} \right\| x^{n-3} + x^{n-4} + \cdots + x + 1 \qquad (2.2) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=1}^{x-1} \frac{1}{2} \left[ \left\| \begin{matrix} x+1 \\ m \end{matrix} \right\| + \left\| \begin{matrix} x-1 \\ m \end{matrix} \right\| \right] x^{n-3} + x^{n-4} + \cdots + x + 1 \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=1}^{x-1} \left\| \begin{matrix} \frac{m^2+m+2}{2} \\ 1 \end{matrix} \right\| x^{n-3} + x^{n-4} + \cdots + x + 1 \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{m=1}^{x-1} \frac{1}{2} \left[ \left\| \begin{matrix} \frac{m^2+m}{2} \\ 1 \end{matrix} \right\| + \left\| \begin{matrix} \frac{m^2+m+4}{2} \\ 1 \end{matrix} \right\| \right] x^{n-3} + x^{n-4} + \cdots + x + 1 \end{aligned}$$

## 3. Relation between Pascal's Triangle and hypercubes

In this section let review and generalize well known fact about connection between row sums of Pascal triangle and $n = 2$ hypercube, recall property

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{3.1}$$

Review a triangle

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 2 & & & \\
 & & 1 & & 4 & & 4 & & \\
 & 1 & & 6 & & 12 & & 8 & \\
1 & & 8 & & 24 & & 32 & & 16
\end{array}
$$

Figure 5. Triangle built by $\binom{n}{k} \cdot 2^k,\ 0 < k < n$.

We can find that

$$\sum_{k=0}^{n} \binom{n}{k} \cdot 2^k = 3^n \tag{3.2}$$

Hereby, let be theorem

**Theorem 3.3.** *Volume of n-dimension hypercube with length m could be calculated as*

$$m^n = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \tag{3.4}$$

*where m - positive integer.*

*Proof.* Recall induction over $m$, in (3.1) is shown a well-known example for $m = 1$. Review (3.2) and suppose that

$$\sum_{k=0}^{n} \binom{n}{k} \cdot \underbrace{(2+1)^k}_{m=3} = \underbrace{(3+1)^n}_{m+1} \tag{3.5}$$

And, obviously, this statement holds by means of Newton's Binomial Theorem given $m = 3$. Next, let show example for each $m \in \mathbb{N}$. Recall Binomial theorem to show that

$$\sum_{k=0}^{n} \binom{n}{k} \cdot m^k = (m+1)^n \tag{3.6}$$

Hereby, for $m + 1$ as well holds

$$\sum_{k=0}^{n} \binom{n}{k} \cdot (m+1)^k = (m+2)^n \tag{3.7}$$

Particularizing this result, we receive

$$\sum_{k=0}^{n} \binom{n}{k} \cdot (m-1)^k = m^n \tag{3.8}$$

Substituting $\sum_{j=0}^{k} \binom{k}{j} (-m)^j$ instead $(m-1)^k$ and rearrange it, we receive desired result

$$m^n = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} m^j \tag{3.9}$$

This completes the proof. $\square$

## 4. Conclusion and future research

In this paper the coefficient $\left\llbracket \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rrbracket$ was introduced in definition (1.20), its properties are shown in (1.23). Power function's $x^n,\ x, n \in \mathbb{N}$ expansion firstly shown in (1.20) and other versions are shown below. Going from (1.31) Power's expansion applying (5) from properties (1.23) is shown below the subsection (1.3), "Generalized Multinomial Theorem based definition (1.20)". In subsection (1.1) Binomial Theorem based on definition (1.20) is shown. In section 2 we show a various representations of exponential function $e^x,\ x \in \mathbb{N}$. In subsection (1.2) we put example of Multinomial theorem for $n = 4$, based on $\left\llbracket \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rrbracket$, and generalized it in subsection (1.3). We attach a Wolfram Mathematica codes of most important equations in Application 1. Extended version of Triangle (1.19) is attached in Application 2. Future research should be done in $\left\llbracket \begin{smallmatrix} n^m \\ k^j \end{smallmatrix} \right\rrbracket,\ m, j \in \mathbb{N}$ to verify its properties. A research on finding difference and consequently derivative using (1.31) and extend it over real functions applying Taylor's theorem also could be done.

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## 5. Application 1. Wolfram Mathematica 11 codes of some expressions

In this section Wolfram Mathematica codes of most expressions are shown. Note that Mathematica `.cdf`-file of all mentioned expressions is available for download at this link. The `.txt`-file reader could find here.

Define coefficient $\left\llbracket \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rrbracket$, definition (1.20)

```
In[1]:= K[n_, k_] := 3!*n*k − 3!*k^2 + 1
```

Check of property (1.27)

```
In[1]:= K[(n^2 + n + 2)/2, 1]
```

Check of expression (1.31), $x^n$

```
In[1]:= 1/2*Sum[(K[2*x − k, k] + K[2*x − k, 0])*x^(n − 3), {k, 0, x − 1}]

        1/2*Sum[(K[x + 1, k] + K[x − 1, k])*x^(n − 3), {k, 0, x − 1}]

        1/2*Sum[(K[(k^2 + k)/2, 1] + K[(k^2 + k + 4)/2, 1])*x^(n − 3),
                {k, 0, x − 1}]

            Sum[K[x, k]*x^(n−3), {k, 0, x − 1}]

            Sum[K[(k^2 + k + 2)/2, 1]*x^(n − 3), {k, 0, x − 1}]
```

Generating formula of Triangle (1.17), Figure 3

```
In[1]:= Column[Table[K[n, k], {n, 0, 5}, {k, 0, n}], Center]
```

Expression (1.33), $x^n - 1$

```
In[1]:= Sum[K[x, m]*x^(n − 3) + Sum[x^(n − t), {t, 4, n}], {m, 1, x − 1}]
```

Expression (1.34), $x^n - 1$ using properties (1.27), (1.29), (1.30)

```
In[1]:= Sum[1/2*(K[2*x − k, k] + K[2*x − k, 0])*x^(n − 3) +
        Sum[x^(n − t), {t, 4, n}], {k, 1, x − 1}]

        Sum[1/2*(K[x + 1, k] + K[x − 1, k])*x^(n − 3) +
        Sum[x^(n − t), {t, 4, n}], {k, 1, x − 1}]

        Sum[1/2*(K[(k^2 + k)/2, 1] + K[(k^2 + k + 4)/2, 1])*x^(n − 3) +
        Sum[x^(n − t), {t, 4, n}], {k, 1, x − 1}]

        Sum[K[(k^2 + k + 2)/2, 1]*x^(n − 3) + Sum[x^(n − t), {t, 4, n}],
```

```
         {k, 1, x − 1}]
```

Expression (1.35), $x^n$

```
In[1]:= Sum[K[x, m]*x^(n − 3) + Sum[x^(n − t), {t, 4, n}] −
        Sum[x^j, {j, 0, Infinity}], {m, 1, x − 1}]
```

Expression (1.36), generalized version of (1.35), $x^n$

```
In[1]:= Sum[(K[x, m] − 1)*x^(n − 3) −
        Sum[x^(n − 3 + j), {j, 1, Infinity}], {m, 1, x − 1}]

        Sum[(K[(m^2 + m + 2)/2, 1] − 1)*x^(n − 3) −
        Sum[x^(n − 3 + j), {j, 1, Infinity}], {m, 1, x − 1}]
```

Expression (1.21), Multinomial theorem for $n = 4$

```
In[1]:= Sum[K[u + j + m, k]*u + K[u + j + m, k]*j + K[u + j + m, k]*m,
        {k, 0, u + j + m − 1}]
```

Section 2, Expression (2.1), $e^x$ representation

```
In[1]:= Sum[1/n!*Sum[(K[x, m] − 1)*x^(n − 3) − Sum[x^(n − 3 + j),
        {j, 1, Infinity}], {m, 1, x − 1}], {n, 0, Infinity}]

        Sum[1/n!*Sum[(K[(m^2 + m + 2)/2, 1] − 1)*x^(n − 3) −
        Sum[x^(n − 3 + j), {j, 1, Infinity}], {m, 1, x − 1}],
        {n, 0, Infinity}]

        Sum[1/n!*Sum[(1/2*(K[x + 1, m] + K[x − 1, m]) − 1)*x^(n − 3) −
        Sum[x^(n − 3 + j), {j, 1, Infinity}], {m, 1, x − 1}],
        {n, 0, Infinity}]

        Sum[1/n!*Sum[(1/2*(K[2x−m, m] + K[2x−m, 0]) − 1)*x^(n − 3) −
        Sum[x^(n − 3 + j), {j, 1, Infinity}], {m, 1, x − 1}],
        {n, 0, Infinity}]
```

Section 2, Expression (2.2), $e^x - e$ representation

```
In[1]:= Sum[1/n!*Sum[K[x, m]*x^(n − 3) +
        Sum[x^(n − t), {t, 4, n}], {m, 1, x − 1}], {n, 0, Infinity}]

        Sum[1/n!*Sum[1/2*(K[x + 1, m] + K[x − 1, m])*x^(n − 3) +
        Sum[x^(n − t), {t, 4, n}], {m, 1, x − 1}], {n, 0, Infinity}]

        Sum[1/n!*Sum[K[(m^2 + m + 2)/2, 1]*x^(n − 3) +
        Sum[x^(n − t), {t, 4, n}], {m, 1, x − 1}], {n, 0, Infinity}]

        Sum[1/(n!)*
        Sum[1/2*(K[(m^2+m+1)/2, 1] + K[(m^2+m+3)/2, 1])*x^(n − 3) +
        Sum[x^(n − t), {t, 4, n}], {m, 1, x − 1}], {n, 0, Infinity}]
```

Binomial Theorem based on definition (1.20), subsection (1.1)

```
In[1]:= Sum[Sum[K[x + y, k]*Binomial[n − 3, j]*x^(n − 3 − j)*y^j,
        {j, 0, n − 3}], {k, 0, x + y − 1}]

        Sum[1/2*Sum[(K[x+y+1, k] + K[x+y−1, k])*Binomial[n − 3, j]*
        x^(n − 3 − j)*y^j, {j, 0, n − 3}], {k, 0, x + y − 1}]

        Sum[1/2*Sum[(K[2(x+y) − k, k] + K[2(x+y) − k, 0])*
        Binomial[n−3, j]*x^(n−3−j)*y^j, {j, 0, n−3}], {k, 0, x+y−1}]

        Sum[Sum[K[(k^2+k+2)/2, 1]*Binomial[n−3, j]*x^(n−3−j)*y^j,
        {j, 0, n − 3}], {k, 0, x + y − 1}]
```

## 6. Application 2. An extended version of triangle (3)

1

1 1

1 7 1

1 13 13 1

1 19 25 19 1

1 25 37 37 25 1

1 31 49 55 49 31 1

1 37 61 73 73 61 37 1

1 43 73 91 97 91 73 43 1

1 49 85 109 121 121 109 85 49 1

1 55 97 127 145 151 145 127 97 55 1

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 61 | 109 | 145 | 169 | 181 | 181 | 169 | 145 | 109 | 61 | 1 |

Figure 5. Extended version of Triangle (1.19) generated from (1.15) given $n = 3$ over $x$ from 0 to 11, sequence A287326 in OEIS, [9], [12].