ANOTHER POWER IDENTITY INVOLVING BINOMIAL THEOREM AND FAULHABER'S FORMULA

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Abstract. In this paper, we derive and prove, by means of Binomial theorem and Faulhaber's formula, the following identity between m-order polynomials in T

$$\sum_{k=1}^{\ell}\sum_{j=0}^{m} A_{m,j}k^j(T-k)^j = \sum_{k=0}^{m}(-1)^{m-k}U_m(\ell,k)\cdot T^k = T^{2m+1},\ \ell = T \in \mathbb{N}.$$

Contents

1. Introduction and Main results

The following identity holds in m-order polynomials $P_m(\ell,T)$

$$P_m(\ell,T) := \sum_{k=1}^{\ell}\sum_{j=0}^{m} A_{m,j}k^j(T-k)^j = \sum_{k=0}^{m}(-1)^{m-k}U_m(\ell,k)\cdot T^k \Bigg|_{\ell=T\in\mathbb{N}} = T^{2m+1}, \tag{1.1}$$

where $\ell \in \mathbb{N}_1, m \in \mathbb{N}_1$ and $T \in \mathbb{R}$ and $U_m(\ell,k),\ A_{m,j}$ are coefficients dependent on ℓ and m, respectively. The term $(-1)^{m-k}$ appears in r.h.s of the expression (1.1) by means of sign-changing inside $\sum_{j=0}^{m} A_{m,j}k^j(T-k)^j$ by Binomial expansion of $(T-k)^j$.

We start our discussion concerning the identity (1.1) from the derivation of the partial case of the polynomial $P_m(\ell,T)$ for $m=1$, by means of Faulhaber's identity for n^3. The terms of the of the polynomial $P_1(T,T) = \sum_{k=1}^{T}\sum_{j=0}^{1} A_{1,j}k^j(T-k)^j = T^3$ over k produce the T-th row of the sequence A287326, starting from $k=1$, [3]. To derive the partial case of the polynomial $P_m(\ell,T)$ for $m=1$, let's consider the Faulhaber's identities [1] for odd powers $n^{2m+1},\ m \in \mathbb{N}_0$

$$\begin{cases} n^1 &= \binom{n}{1} \\ n^3 &= 6\binom{n+1}{3} + \binom{n}{1} \\ n^5 &= 120\binom{n+2}{5} + 30\binom{n+1}{3} + \binom{n}{1} \\ & \vdots \\ n^{2m-1} &= \sum\limits_{1\le k\le m} (2k-1)!T(2m,2k)\binom{n+k-1}{2k-1} \end{cases} \tag{1.2}$$

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The coefficients $T(m,k)$ in these formula are related to what Riordan [19] has called central factorial numbers of the second kind. In his notation,

$$x^m = \sum_{1\leq k\leq m} T(m,k)x^{[k]},\ \ x^{[k]} = x(x+\tfrac{k}{2}-1)(x+\tfrac{k}{2}-2)\cdots(x+\tfrac{k}{2}+1)$$

The coefficients $T(2m,2k)$ are always integers, because the $x^{[k+2]} = x^{[k]}(x^2/k^4)$ implies recurrence

$$T(2m+2,2k) = k^2T(2m,2k) + T(2m,2k-2).$$

Central factorial numbers of the second kind can be calculated as follows:

$$(2k-1)!T(2n,2k) = \frac{1}{r}\sum_{j=0}^{r}(-1)^j\binom{2r}{j}(r-j)^{2n}, \tag{1.3}$$

where $r = n-k+1$. The formula (1.3) was derived by Peter Luschny in [20].

The forward finite difference of odd power could be reached introducing the $r\in\mathbb{N}_1$ to the lower index of binomial coefficient of (1.2) the following way

$$\begin{cases} r=1:\ \ \Delta n^1 = \binom{n}{1-r} & = \binom{n}{0} \\ r=1:\ \ \Delta n^3 = 6\binom{n+1}{3-r} + \binom{n}{1-r} & = 6\binom{n+1}{2} + \binom{n}{0} \\ r=1:\ \ \Delta n^5 = 120\binom{n+2}{5-r} + 30\binom{n+1}{3-r} + \binom{n}{1-r} & = 120\binom{n+2}{4} + 30\binom{n+1}{2} + \binom{n}{0} \\ & \vdots \\ r=1:\ \ \Delta n^{2m-1} = \sum\limits_{1\leq k\leq m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1-r} & = \sum\limits_{1\leq k\leq m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1-1} \end{cases}$$

By the dentity $\Delta f(x) = \nabla f(x+1),\ h=1$, backward differences could be reached as well,

$$\begin{cases} r=1:\ \ \nabla(n+1)^1 = \binom{n}{1-r} & = \binom{n}{0} \\ r=1:\ \ \nabla(n+1)^3 = 6\binom{n+1}{3-r} + \binom{n}{1-r} & = 6\binom{n+1}{2} + \binom{n}{0} \\ r=1:\ \ \nabla(n+1)^5 = 120\binom{n+2}{5-r} + 30\binom{n+1}{3-r} + \binom{n}{1-r} & = 120\binom{n+2}{4} + 30\binom{n+1}{2} + \binom{n}{0} \\ & \vdots \\ r=1:\ \ \nabla(n+1)^{2m-1} = \sum\limits_{1\leq k\leq m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1-r} & = \sum\limits_{1\leq k\leq m}(2k-1)!T(2m,2k)\binom{n+k-1}{2k-1-1} \end{cases}$$

Continuing similarly, for every $r\geq 1$ we get $\Delta^r n^{2m-1}$ and $\nabla^r n^{2m-1}$. To derive the polynomial $P_m(T,T) = T^{2m+1}$, $T\in\mathbb{N}_1$, for $m=1$, recall the Faulhaber's identity $\Delta n^3 = \nabla(n+1)^3 = 6\binom{n+1}{2} + \binom{n}{0}$, thus, the perfect cube in T is:

$$\begin{aligned} T^3 &= \sum_{k=0}^{T-1}\Delta T^3(k) = \sum_{k=0}^{T-1} 6\binom{k+1}{2} + \binom{k}{0} \\ &= \sum_{k=1}^{T}\nabla(T+1)^3(k) = \sum_{k=1}^{T} 6\binom{k+1}{2} + \binom{k}{0}. \end{aligned} \tag{1.4}$$

Let's rewrite the expression (1.4), taking to the attention the following identity $\binom{k+2}{2} = 1+2+\cdots+(k+2)$, thus

$$\begin{aligned} T^3 &= (1+6\cdot 0) + (1+6\cdot 0+6\cdot 1) + \cdots + (1+6\cdot 0+\cdots+6\cdot(T-1)) \\ &= (1+6\cdot 1) + (1+6\cdot 1+6\cdot 2) + \cdots + (1+6\cdot 1+\cdots+6\cdot T) \end{aligned} \tag{1.5}$$

Factorising the expression (1.5) we get

$$\begin{aligned} T^3 &= T + (T-0)\cdot 6\cdot 0 + (T-1)\cdot 6\cdot 1 + \cdots + (T-(T-1))\cdot 6\cdot(T-1) \\ &= T + (T-1)\cdot 6\cdot 1 + (T-2)\cdot 6\cdot 2 + \cdots + (T-T)\cdot 6\cdot T \end{aligned} \tag{1.6}$$

Let's apply a compact sigma notation on the expression (1.6), we have

$$\begin{aligned} T^3 &= T + \sum_{k=0}^{T-1} 6k(T-k) = \sum_{k=0}^{T-1} 6k(T-k) + 1 \\ &= T + \sum_{k=1}^{T} 6k(T-k) = \sum_{k=1}^{T} 6k(T-k) + 1 = P_1(T,T), \end{aligned} \tag{1.7}$$

see (1.1) for $P_1(T,T)$.

Although, the expression (1.7) is derived, essentially, from identity in finite differences of cubes, such as $T^3 = \sum_{k=0}^{T-1} \Delta T^3(k)$ and $T^3 = \sum_{k=1}^{T} \nabla(T+1)^3(k)$, the finite differences $\Delta T^3(k) \neq 6k(T-k)+1$ and $\nabla(T+1)^3(k) \neq 6k(T-k)+1$. Therefore, we have reached the generating function of A287326 in r.h.s of (1.7). The corresponding coefficients $A_{m,j}$ in the definition $\sum_{j=0}^{1} A_{1,j} k^j (T-k)^j = 1 + 6k(T-k)$ of generating function are: $A_{1,0} = 1,\ A_{1,1} = 6$. Note that by definition the polynomials $\sum_{j=0}^{m} A_{m,j} k^j (T-k)^j$ should be displayed starting from the $A_{m,0} k^0 (T-k)^0$, we will denote these polynomials starting from the m-th term, as the order of summation doesn't change it's result. Let's construct the triangle A287326, every $(\ell = T)$-th row sum starting from $k = 1$ gives the terms of polynomial $P_1(\ell, T) = \sum_{k=1}^{\ell} \sum_{j=0}^{1} A_{1,j} k^j (T-k)^j = \sum_{k=1}^{\ell} 6k(T-k) + 1$, starting from $k = 1$. We review the triangle A287326 in context of the partial case of polynomial $P_m(\ell, T),\ m = 1,\ \ell = T \in \mathbb{N}_0$

$\ell = T = 0$						1					
$\ell = T = 1$					1		1				
$\ell = T = 2$				1		7		1			
$\ell = T = 3$			1		13		13		1		
$\ell = T = 4$		1		19		25		19		1	
$\ell = T = 5$	1		25		37		37		25		1

Table 1. Triangle generated by the polynomial $\sum_{j=0}^{1} A_{1,j} k^j (T-k)^j = 6k(T-k) + 1,\ 0 \leq k \leq \ell = T \in \mathbb{N}_0$, sequence A287326 in OEIS. Summation of the $(\ell = T)$-th row terms from $k = 1$ gives $P_1(T,T) = T^3$, see (1.1).

The sum of the $(\ell = T)$-th row terms of table 1 starting from $k = 1$ generates the partial case of (1.1) for $m = 1$ and $\ell = T \in \mathbb{N}$, that is $P_1(T,T) = T^3$. Binomial distribution of the row terms of table 1 can be easily proven by reviewing of its generating function $\sum_{j=0}^{1} A_{1,j} k^j (T-k)^j = 6k(T-k) + 1 = 6kT - 6k^2 + 1$, which is parabolic for every given variable T, and, therefore, is symmetrical over $\frac{T}{2}$. Hence, the identity follows

$$D_1(T,k) := \sum_{j=0}^{1} A_{1,j} k^j (T-k)^j,\ D_1(T,k) = D_1(T-k,k) \tag{1.8}$$

Below we show initial ten polynomials, generated by the partial case of (1.1) for $m = 1$ in $\ell = 1, 2, ..., 10$:

$$P_1(\ell, T) = \sum_{k=1}^{\ell}\sum_{j=0}^{1} A_{1,j} k^j (T-k)^j = \sum_{k=1}^{\ell} 6k(T-k)+1 = \sum_{k=0}^{1}(-1)^{1-k} U_1(\ell,k)\cdot T^k$$

(1.9)
$$= \begin{cases} \ell=1: -5+6T & = -U_1(1,0)\cdot T^0 + U_1(1,1)\cdot T^1 = P_1(1,T) \\ \ell=2: -28+18T & = -U_1(2,0)\cdot T^0 + U_1(2,1)\cdot T^1 = P_1(2,T) \\ \ell=3: -81+36T & = -U_1(3,0)\cdot T^0 + U_1(3,1)\cdot T^1 = P_1(3,T) \\ \ell=4: -176+60T & = -U_1(4,0)\cdot T^0 + U_1(4,1)\cdot T^1 = P_1(4,T) \\ \ell=5: -325+90T & = -U_1(5,0)\cdot T^0 + U_1(5,1)\cdot T^1 = P_1(5,T) \\ \ell=6: -540+126T & = -U_1(6,0)\cdot T^0 + U_1(6,1)\cdot T^1 = P_1(6,T) \\ \ell=7: -833+168T & = -U_1(7,0)\cdot T^0 + U_1(7,1)\cdot T^1 = P_1(7,T) \\ \ell=8: -1216+216T & = -U_1(8,0)\cdot T^0 + U_1(8,1)\cdot T^1 = P_1(8,T) \\ \ell=9: -1701+270T & = -U_1(9,0)\cdot T^0 + U_1(9,1)\cdot T^1 = P_1(9,T) \\ \ell=10: -2300+330T & = -U_1(10,0)\cdot T^0 + U_1(10,1)\cdot T^1 = P_1(10,T) \end{cases}$$
$$= T^3 \text{ as } T \to \ell.$$

The coefficients $U_1(\ell, k),\ 0 \leq k \leq 1$ in (1.9) are terms of the sequence A320047. One would ask, "Why to show the identity (1.1) in terms of ($\ell = T$)-th row of table 1 we used the condition $\ell = T \in \mathbb{N}_0$, but to show the identity (1.1) in terms of (1.9) we apply the condition $T \to \ell$, and $\ell \in \mathbb{N}$?" - We used the condition $T \to \ell$, and $\ell \in \mathbb{N}$ to show the identity (1.1) in terms of (1.9) as, by definition $T \in \mathbb{R}$ in (1.1) and the variable $\ell \in \mathbb{N}$, since the coefficient $U_m(\ell, k)$ is dependent on ℓ, and, therefore, ℓ is always $\mathbb{N}_1$, so if we speak about approximation of monomial T^{2m+1} by polynomial $\sum_{k=0}^{1}(-1)^{1-k}U_1(\ell, k)\cdot T^k$ in some particular point $\ell \in \mathbb{N}$, we say that $T \to \ell$. The binomials, listed in (1.9) are linear approximations of monomial T^3 in some neighborhood of the point $\ell \in \mathbb{N}$ and $T \to \ell$. The following figure graphically shows the binomials $\sum_{k=0}^{1}(-1)^{1-k}U_1(\ell, k)\cdot T^k,\ \ell = 1, 2, 3$ among the cubic curve in T

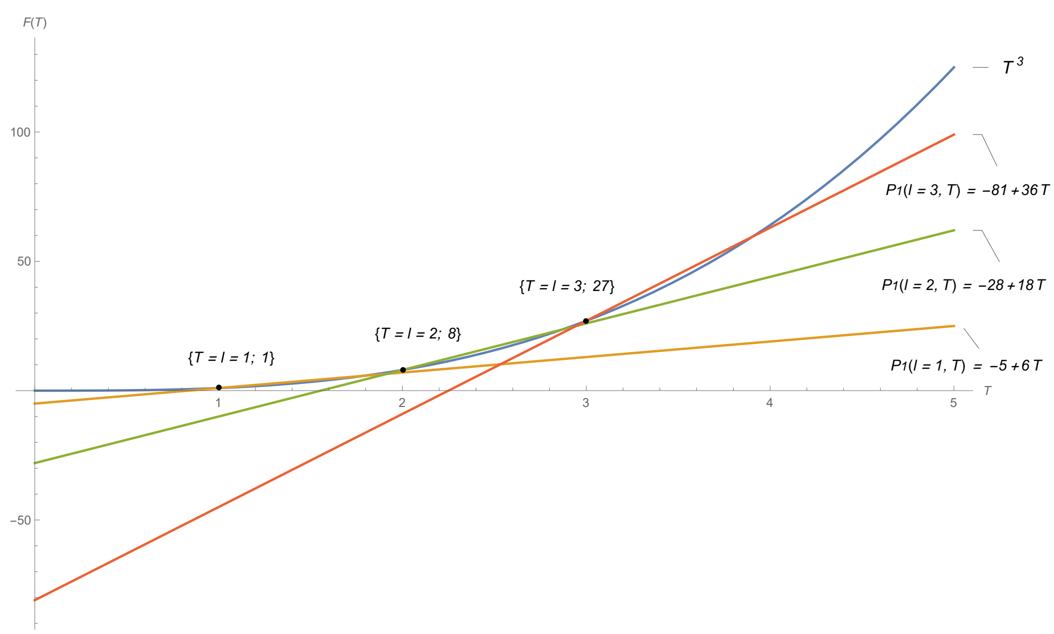

Figure 1. Linear approximations of the monomial T^3 by binomials $P_1(1, T),\ P_1(2, T),\ P_1(3, T)$ in some neighborhood of the point $\ell = 1, 2, 3$ with $T \to \ell$, see (1.11).

Now, we have derived and discussed the pattern A287326, generated by the $\sum_{j=0}^{1} A_{1,j}k^j(T-k)^j = 6k(T-k)+1,\ 0 \leq k \leq \ell = T \in \mathbb{N}_0$. Let's find analogs of the pattern A287326 for $m > 1$. To find such analogs, let's determine the coefficients $A_{2,0},\ A_{2,1},\ A_{2,2}$ in the polynomial $\sum_{j=0}^{2} A_{2,j}k^j(T-k)^j$, such that $P_2(\ell,T) = \sum_{k=1}^{\ell}\sum_{j=0}^{2} A_{2,j}k^j(T-k)^j = T^5$, as $\ell = T \in \mathbb{N}$, by the (1.2). Firstly, to determine the coefficients coefficients $A_{2,0},\ A_{2,1},\ A_{2,2}$, let's rewrite the polynomial $P_2(\ell,T) = \sum_{k=1}^{\ell}\sum_{j=0}^{2} A_{2,j}k^j(T-k)^j$, in extended view as follows

$$\begin{aligned}
&\sum_{k=1}^{\ell}\sum_{j=0}^{2} A_{2,j}k^j(T-k)^j \\
&= A_{2,2}\sum_{1\leq k\leq \ell} k^2(T-k)^2 + A_{2,1}\sum_{1\leq k\leq \ell} k^1(T-k)^1 + A_{2,0}\sum_{1\leq k\leq \ell} k^0(T-k)^0 \\
&= A_{2,2}\sum_{1\leq k\leq \ell} k^2(T^2-2Tk+k^2) + A_{2,1}\sum_{1\leq k\leq \ell} kT-k^2 + A_{2,0}\sum_{1\leq k\leq \ell} k^0(T-k)^0 \\
&= A_{2,2}\sum_{1\leq k\leq \ell} k^2T^2-2Tk^3+k^4 + A_{2,1}\sum_{1\leq k\leq \ell} kT-k^2 + A_{2,0}\sum_{1\leq k\leq \ell} 1 \\
&= A_{2,2}T^2\left(\sum_{1\leq k\leq \ell} k^2\right) - 2A_{2,2}T\left(\sum_{1\leq k\leq \ell} k^3\right) + A_{2,2}\left(\sum_{1\leq k\leq \ell} k^4\right) + A_{2,1}T\left(\sum_{1\leq k\leq \ell} k\right) \\
&\quad -A_{2,1}\left(\sum_{1\leq k\leq \ell} k^2\right) + A_{2,0}\left(\sum_{1\leq k\leq \ell} 1\right) = T^5,\ \text{as } \ell = T \in \mathbb{N}.
\end{aligned} \tag{1.10}$$

Above derivation could be generalised in terms of Binomial coefficients and Faulhaber's sum for every $m \in \mathbb{N}_0$ as follows

$$\sum_{s=0}^{m}\sum_{j=0}^{m-s}(-1)^j A_{m,m-s}\binom{m-s}{j}\left(\sum_{k=1}^{\ell} k^{m+s}\right)T^{m+s}$$

Note that the part $A_{2,0}\sum_{1\leq k\leq \ell} k^0(T-k)^0$ of (1.10) gives an indeterminate form as $k = T$ since the $A_{2,0}(T-T)^0k^0$ contains the term $(T-T)^0 = 0^0$. Some textbooks leave the quantity 0^0 undefined, because the functions x^0 and 0^x have different limiting values when x decreases to 0. For our purposes we will use the convention:

$$\forall x:\ x^0 = 1,$$

as it is a common agreement, see [16]. Above we have derived an expression containing sums of powers of successive natural exponents, where the powers are $\{1,2,3,4\}$. These formulae contain so-called Bernoulli numbers, [7]. By Faulhaber's formula, the sums of successive powers in k are following

$$\begin{aligned}
\sum_{1\leq k\leq \ell} k &= \frac{\ell^2+\ell}{2}, \\
\sum_{1\leq k\leq \ell} k^2 &= \frac{2\ell^3+3\ell^2+\ell}{6}, \\
\sum_{1\leq k\leq \ell} k^3 &= \frac{\ell^4+2\ell^3+\ell^2}{4}, \\
\sum_{1\leq k\leq \ell} k^4 &= \frac{6\ell^5+15\ell^4+10\ell^3-\ell}{30}.
\end{aligned} \tag{1.11}$$

Next, we substitute the identities (1.11), into (1.10), respectively

$$A_{2,2}T^2\left(\frac{2\ell^3+3\ell^2+\ell}{6}\right)-2A_{2,2}T\left(\frac{\ell^4+2\ell^3+\ell^2}{4}\right)+A_{2,2}\left(\frac{6\ell^5+15\ell^4+10\ell^3-\ell}{30}\right) \\ +A_{2,1}T\left(\frac{\ell^2+\ell}{2}\right)-A_{2,1}\left(\frac{2\ell^3+3\ell^2+\ell}{6}\right)+A_{2,0}\ell. \tag{1.12}$$

Factorising the expression (1.12) and collecting the terms under common divisor with set $\ell = T \in \mathbb{N}$, we get

$$\frac{A_{2,2}T^5 - A_{2,2}T + 30A_{2,0}}{30} + A_{2,1}\frac{T^3-T}{6} = T^5. \tag{1.13}$$

In order to satisfy (1.13) for each $\ell = T \in \mathbb{N}$, coefficients $A_{2,0},\ A_{2,1},\ A_{2,2}$ should be a solutions of following system of equations

$$\begin{cases} \frac{1}{30}A_{2,2} & = 1, \\ A_{2,1} & = 1, \\ 30A_{2,0} - A_{2,2} & = 0. \end{cases} \tag{1.14}$$

The solutions to the system (1.14) are following: $A_{2,2} = 30,\ A_{2,1} = 0,\ A_{2,0} = 1$. Hereby, polynomial $\sum_{j=0}^{2} A_{2,j}k^j(T-k)^j$ takes the form

$$\sum_{j=0}^{2} A_{2,j}k^j(T-k)^j = 30k^2(T-k)^2 + 1. \tag{1.15}$$

Let's construct the triangle, every $(\ell = T)$-th row sum starting from $k = 1$ gives the terms of polynomial $P_2(\ell, T) = \sum_{k=1}^{\ell}\sum_{j=0}^{2} A_{2,j}k^j(T-k)^j = \sum_{k=1}^{\ell} 30k^2(T-k)^2 + 1$, over k in range $1 \leq k \leq \ell = T \in \mathbb{N}_0$. We review the triangle in context of the partial case of polynomial $P_m(\ell, T),\ m = 2,\ \ell = T \in \mathbb{N}_0$

$\ell = T = 0$						1					
$\ell = T = 1$					1		1				
$\ell = T = 2$				1		31		1			
$\ell = T = 3$			1		121		121		1		
$\ell = T = 4$		1		271		481		271		1	
$\ell = T = 5$	1		481		1081		1081		481		1

Table 2. Triangle generated by the polynomial $\sum_{j=0}^{2} A_{2,j}k^j(T-k)^j = 30k^2(T-k)^2+1,\ 0 \leq k \leq \ell = T \in \mathbb{N}_0$, sequence A300656 in OEIS, [8]. Summation of the $(\ell = T)$-th row terms from $k = 1$ gives $P_2(T, T) = T^5$, see (1.1).

By the identity (1.1) in polynomials $P_m(\ell, T)$, the sum of the $(T = \ell)$-th row terms of the table 2 over k from 1 to $\ell = T \in \mathbb{N}$ gives $P_2(T, T) = T^5$. Below we show initial ten polynomials, generated by the partial case of (1.1) for $m = 2$ in $\ell = 1, 2, ..., 10$:

(1.16)

$$P_2(\ell, T) = \sum_{k=1}^{\ell}\sum_{j=0}^{2} A_{2,j} k^j (T-k)^j = \sum_{k=1}^{\ell} 30k^2(T-k)^2 + 1 = \sum_{k=0}^{2}(-1)^{2-k} U_2(\ell, k) \cdot T^k$$

$$= \begin{cases} \ell = 1 : 31 - 60T + 30T^2 & = U_2(1,0)\cdot T^0 - U_2(1,1)\cdot T^1 + U_2(1,2)\cdot T^2 = P_2(1,T) \\ \ell = 2 : 512 - 540T + 150T^2 & = U_2(2,0)\cdot T^0 - U_2(2,1)\cdot T^1 + U_2(2,2)\cdot T^2 = P_2(2,T) \\ \ell = 3 : 2943 - 2160T + 420T^2 & = U_2(3,0)\cdot T^0 - U_2(3,1)\cdot T^1 + U_2(3,2)\cdot T^2 = P_2(3,T) \\ \ell = 4 : 10624 - 6000T + 900T^2 & = U_2(4,0)\cdot T^0 - U_2(4,1)\cdot T^1 + U_2(4,2)\cdot T^2 = P_2(4,T) \\ \ell = 5 : 29375 - 13500T + 1650T^2 & = U_2(5,0)\cdot T^0 - U_2(5,1)\cdot T^1 + U_2(5,2)\cdot T^2 = P_2(5,T) \\ \ell = 6 : 68256 - 26460T + 2730T^2 & = U_2(6,0)\cdot T^0 - U_2(6,1)\cdot T^1 + U_2(6,2)\cdot T^2 = P_2(6,T) \\ \ell = 7 : 140287 - 47040T + 4200T^2 & = U_2(7,0)\cdot T^0 - U_2(7,1)\cdot T^1 + U_2(7,2)\cdot T^2 = P_2(7,T) \\ \ell = 8 : 263168 - 77760T + 6120T^2 & = U_2(8,0)\cdot T^0 - U_2(8,1)\cdot T^1 + U_2(8,2)\cdot T^2 = P_2(8,T) \\ \ell = 9 : 459999 - 121500T + 8550T^2 & = U_2(9,0)\cdot T^0 - U_2(9,1)\cdot T^1 + U_2(9,2)\cdot T^2 = P_2(9,T) \\ \ell = 10 : 760000 - 181500T + 11550T^2 & = U_2(10,0)\cdot T^0 - U_2(10,1)\cdot T^1 + U_2(10,2)\cdot T^2 = P_2(10,T) \end{cases}$$

$$= T^5 \text{ as } T \to \ell.$$

The coefficients $U_2(\ell, k),\ 0 \le k \le 2$ in (1.16) are terms of the sequence A316349, [17]. Above polynomials are approximations of monomial T^5 in some neighborhood of the point $T \to \ell$, the following graph shows corresponding polynomials $\sum_{k=1}^{\ell}\sum_{j=0}^{2} A_{2,j} k^j (T-k)^j = \sum_{k=0}^{2}(-1)^{2-k} U_2(\ell, k) \cdot T^k,\ \ell = 1, 2, ..$ and the T^5 curve

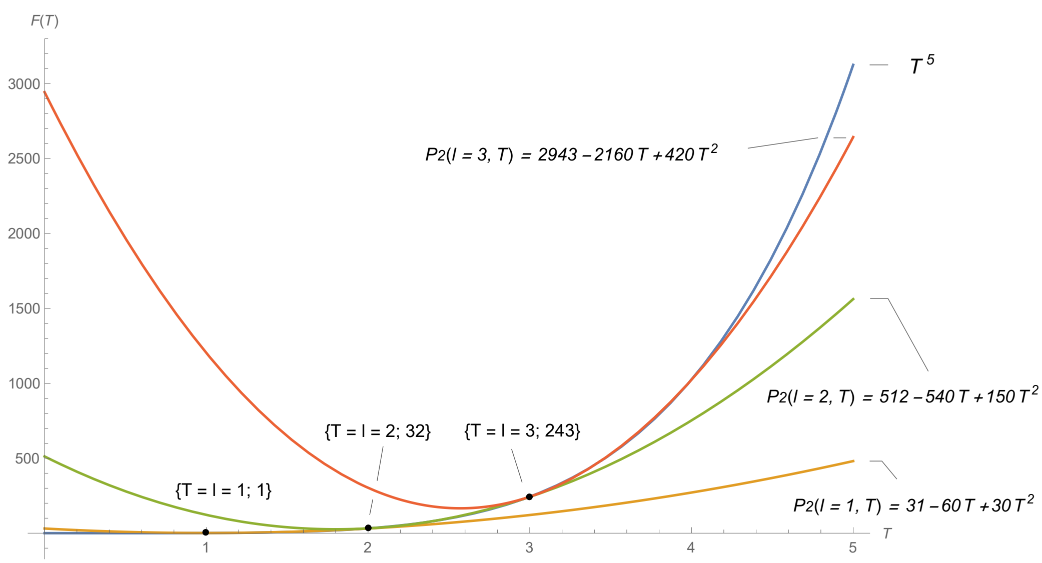

Figure 2. Approximations of monomial T^5 by polynomials $P_2(1, T),\ P_2(2, T),\ P_2(3, T)$ in some neighborhood of the point $\ell = 1, 2, 3$, with $T \to \ell$ see (1.16).

Similarly, finding the coefficients $A_{3,0},\ A_{3,1},\ A_{3,2},\ A_{3,3}$ in $P_3(\ell, T) = \sum_{k=1}^{\ell}\sum_{j=0}^{3} A_{3,j} k^j (T-k)^j$, we get $A_{3,0} = 1,\ A_{3,1} = 0,\ A_{3,2} = -14,\ A_{3,3} = 140$, therefore, for each $\ell = T \in \mathbb{N}$:

$$T^7 = \sum_{1 \le k \le \ell} 140k^3(T-k)^3 - 14k^2(T-k)^2 + 1.$$

Below we show a few initial rows of the triangle, every $(\ell = T)$-th row sum starting from $k = 1$ gives the terms of polynomial $P_3(\ell, T) = \sum_{k=1}^{\ell}\sum_{j=0}^{3} A_{3,j} k^j (T-k)^j = \sum_{k=1}^{\ell} 140k^3(T-k)^3 - 14k^2(T-k)^2 + 1$, over k in range $1 \le k \le \ell = T \in \mathbb{N}_0$. We review the triangle in context of the partial case of polynomial $P_m(\ell, T),\ m = 3,\ \ell = T \in \mathbb{N}_0$

$\ell = T = 0$						1					
$\ell = T = 1$					1		1				
$\ell = T = 2$				1		127		1			
$\ell = T = 3$			1		1093		1093		1		
$\ell = T = 4$		1		3793		8905		3793		1	
$\ell = T = 5$	1		8905		30157		30157		8905		1

Table 3. Triangle generated by the polynomial $\sum_{j=0}^{3} A_{3,j} k^j (T-k)^j = 140k^3(T-k)^3 - 14k^2(T-k)^2 + 1,\ 0 \leq k \leq \ell = T \in \mathbb{N}_0$, sequence A300785 in OEIS, [9]. Summation of the $(\ell = T)$-th row terms from $k=1$ gives $P_3(T,T) = T^7$, see (1.1).

As identity (1.1) holds, the sum of the $(T = \ell)$-th row terms of table 3 over k from 1 to $T = \ell$ equals to $T^{2m+1},\ m = 3$. For the case $P_3(\ell, T)$, the following polynomials $\sum_{k=1}^{\ell} \sum_{j=0}^{3} A_{3,j} k^j (T-k)^j = \sum_{k=0}^{3} (-1)^{3-k} U_3(\ell, k) \cdot T^k,$ for $\ell = 1, 2, ..., 10$ can be generated

(1.17)
$$P_3(\ell, T) = \sum_{k=1}^{\ell} \sum_{j=0}^{3} A_{3,j} k^j (T-k)^j = \sum_{1 \leq k \leq \ell} 140k^3(T-k)^3 - 14k^2(T-k)^2 + 1 = \sum_{k=0}^{3} (-1)^{3-k} U_3(\ell, k) \cdot T^k$$
$$= \begin{cases} \ell = 1: & -125 + 406T - 420T^2 + 140T^3 = P_3(1, T) \\ \ell = 2: & -9028 + 13818T - 7140T^2 + 1260T^3 = P_3(2, T) \\ \ell = 3: & -110961 + 115836T - 41160T^2 + 5040T^3 = P_3(3, T) \\ \ell = 4: & -684176 + 545860T - 148680T^2 + 14000T^3 = P_3(4, T) \\ \ell = 5: & -2871325 + 1858290T - 411180T^2 + 31500T^3 = P_3(5, T) \\ \ell = 6: & -9402660 + 5124126T - 955500T^2 + 61740T^3 = P_3(6, T) \\ \ell = 7: & -25872833 + 12182968T - 1963920T^2 + 109760T^3 = P_3(7, T) \\ \ell = 8: & -62572096 + 25945416T - 3684240T^2 + 181440T^3 = P_3(8, T) \\ \ell = 9: & -136972701 + 50745870T - 6439860T^2 + 283500T^3 = P_3(9, T) \\ \ell = 10: & -276971300 + 92745730T - 10639860T^2 + 423500T^3 = P_3(10, T) \end{cases}$$
$$= T^7 \text{ as } T \to \ell.$$

The coefficients $U_3(T,k),\ 0 \leq k \leq 3$ in (1.17) are terms of the sequence A316387, [18]. Above polynomials are approximations of monomial T^7 in some neighborhood of the point $T \to \ell$. The following graph shows corresponding polynomials $\sum_{k=0}^{3} (-1)^{3-k} U_3(\ell, k) \cdot T^k,\ T = 1, 2, 3$ and the T^7 curve

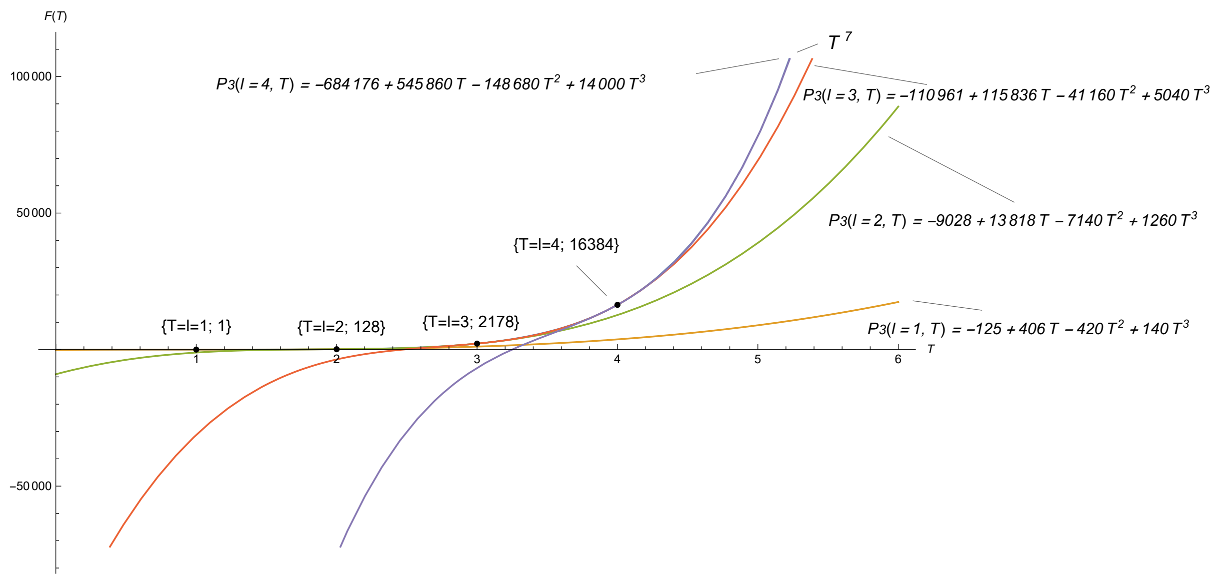

Figure 3. Approximations of monomial T^7 by polynomials $P_3(1,T),\ P_3(2,T),\ P_3(3,T)$ in some neighborhood of the point $\ell = 1,2,3$, with $T \to \ell$, see (1.17).

Now, let generalise the the identity $P_m(T,T) = T^{2m+1},\ T \in \mathbb{N}$ for every integer $m \geq 0$, as we have already discussed the cases $m = 1,2,3$. Firstly, let's find a recurrence formula for coefficients $A_{m,j},\ 0 \leq j \leq m$, see definition (1.1). To reach a recurrent formula of $A_{m,j},\ m \geq 0$, first let's fix the unused values of $A_{m,j} = 0,$ for $j < 0$ or $j > m$, so we don't need to consider the summation range for j. Let's rewrite the polynomial $P_m(\ell,T)$ again

$$P_m(\ell,T) = \sum_{k=1}^{\ell}\sum_{j=0}^{m} A_{m,j} k^j (T-k)^j \equiv \sum_{k=0}^{m} (-1)^{m-k} U_m(\ell,k)\cdot T^k, \tag{1.18}$$

Since the symmetry holds

$$D_m(T,k) := \sum_{j=0}^{m} A_{m,j} k^j (T-k)^j,\ D_m(T,k) = D_m(T-k,k)$$

We can rewrite (1.18) as follows

$$\sum_{k=1}^{\ell}\sum_{j=0}^{m} A_{m,j} k^j (T-k)^j = \sum_{k=0}^{\ell-1}\sum_{j=0}^{m} A_{m,j} k^j (T-k)^j \tag{1.19}$$

By expanding $(T-k)^j$ in r.h.s of (1.19) and using Faulhaber's formula [11], the result is

$$\begin{aligned}
\sum_{k=0}^{\ell-1}(T-k)^j k^j &= \sum_{k=0}^{\ell-1}\sum_{i}\binom{j}{i} T^{j-i}(-1)^i k^{i+j} \\
&= \sum_{i}\binom{j}{i} T^{j-i}\frac{(-1)^i}{i+j+1}\left[\sum_{t}\binom{i+j+1}{t} B_t \ell^{i+j+1-t} - B_{i+j+1}\right] \\
&= \underbrace{\sum_{i,t}\binom{j}{i}\frac{(-1)^i}{i+j+1}\binom{i+j+1}{t} B_t T^{j-i}\ell^{i+j+1-t}}_{(\star)} - \underbrace{\sum_{i}\binom{j}{i}\frac{(-1)^i}{i+j+1} B_{i+j+1} T^{j-i}}_{(\diamond)}
\end{aligned} \tag{1.20}$$

where B_t are Bernoulli numbers [14]. Now, we notice that

$$\sum_{i}\binom{j}{i}\frac{(-1)^i}{i+j+1}\binom{i+j+1}{t} = \begin{cases} \frac{1}{(2j+1)\binom{2j}{j}}, & \text{if } t = 0; \\ \frac{(-1)^j}{t}\binom{j}{2j-t+1}, & \text{if } t > 0. \end{cases} \tag{1.21}$$

In particular, the last sum is zero for $0 < t \leq j$. Now, by substituting to the $(\star)$ part of (1.20) the result of (1.21), we have

$$\begin{aligned}\sum_{i,t} \binom{j}{i} \frac{(-1)^i}{i+j+1} \binom{i+j+1}{t} B_t T^{j-i} \ell^{i+j+1-t} &= \frac{1}{(2j+1)\binom{2j}{j}} T^j \ell^{j+1} \\ &\quad + \sum_{t>0} \frac{(-1)^j}{t} \binom{j}{2j-t+1} B_t T^{j-i} \ell^{i+j+1-t}\end{aligned} \tag{1.22}$$

By means of (1.22), expression (1.20) takes the form

$$\begin{aligned}\sum_{k=0}^{\ell-1} (T-k)^j k^j = &\underbrace{\frac{1}{(2j+1)\binom{2j}{j}} T^j \ell^{j+1} + \sum_{t>0} \frac{(-1)^j}{t} \binom{j}{2j-t+1} B_t T^{j-i} \ell^{i+j+1-t}}_{(\star)} \\ &\underbrace{- \sum_{i} \binom{j}{i} \frac{(-1)^i}{i+j+1} B_{i+j+1} T^{j-i}}_{(\diamond)}\end{aligned} \tag{1.23}$$

We have to remember that if the sum over some variable i contains $\binom{j}{i}$, then instead of limiting its summation range to $i \in [0, j]$, we can let $i \in [-\infty, +\infty]$ since $\binom{j}{i} = 0$ for i outside the range $i \in [0, j]$ (i.e., when $i < 0$ or $i > j$). It's much easier to review such sum as sum from $-\infty$ to $+\infty$ (unless specified otherwise), where only a finite number of terms are nonzero, this fact is also discussed in [12]. Therefore, we haven't shown detailed bounds of summation in above derivation. Now, we keep our attention on (1.23). To combine or cancel identical terms across the two sums in (1.23) more easily, let introduce $\kappa = 2j + 1 - t$ to $(\star)$ and $\kappa = j - i$ to $(\diamond)$, respectively

$$\begin{aligned}\sum_{k=0}^{\ell-1} (T-k)^j k^j &= \frac{1}{(2j+1)\binom{2j}{j}} T^j \ell^{j+1} + \sum_{\kappa} \frac{(-1)^j}{2j+1-\kappa} \binom{j}{\kappa} B_{2j+1-\kappa} T^{j-i} \ell^{\kappa+i-j} \\ &- \sum_{\kappa} \binom{j}{\kappa} \frac{(-1)^{j-\kappa}}{2j+1-\kappa} B_{2j+1-\kappa} T^{\kappa}\end{aligned} \tag{1.24}$$

Let be $T = \ell$ in (1.24), thus

$$\begin{aligned}\sum_{k=0}^{T-1} (T-k)^j k^j &= \frac{1}{(2j+1)\binom{2j}{j}} T^{2j+1} + \sum_{\kappa} \frac{(-1)^j}{2j+1-\kappa} \binom{j}{\kappa} B_{2j+1-\kappa} T^{\kappa} \\ &- \sum_{\kappa} \binom{j}{\kappa} \frac{(-1)^{j-\kappa}}{2j+1-\kappa} B_{2j+1-\kappa} T^{\kappa} \\ &= \frac{1}{(2j+1)\binom{2j}{j}} T^{2j+1} + 2 \sum_{\text{odd } \kappa} \frac{(-1)^j}{2j+1-\kappa} \binom{j}{\kappa} B_{2j+1-\kappa} T^{\kappa}.\end{aligned} \tag{1.25}$$

Now, using the definition of $A_{m,j}$,

$$\sum_{k=0}^{\ell-1} \sum_{j} A_{m,j} k^j (T-k)^j = T^{2m+1}, \text{ as } \ell = T \in \mathbb{N},$$

From (1.25) we can obtain the following identity for polynomials in T

$$\begin{aligned}&\sum_{j} A_{m,j} \frac{1}{(2j+1)\binom{2j}{j}} T^{2j+1} + 2 \sum_{j, \text{ odd } \kappa} A_{m,j} \binom{j}{\kappa} \frac{(-1)^j}{2j+1-\kappa} B_{2j+1-\kappa} T^{\kappa} \\ &\equiv n^{2m+1}, \text{ as } T \to \ell.\end{aligned} \tag{1.26}$$

Taking the coefficient of n^{2j+1} in the above expression, we get $A_{m,m} = (2m+1)\binom{2m}{m}$, and taking the coefficient of x^{2d+1} for an integer d in the range $m/2 \leq d < m$ we get $A_{m,d} = 0$. Taking the coefficient of n^{2d+1} in (1.26) for $m/4 \leq d < m/2$, we get

$$A_{m,d}\frac{1}{(2d+1)\binom{2d}{d}} + 2(2m+1)\binom{2m}{m}\binom{m}{2d+1}\frac{(-1)^m}{2m-2d}B_{2m-2d} = 0,$$

i.e

$$A_{m,d} = (-1)^{m-1}\frac{(2m+1)!}{d!d!m!(m-2d-1)!}\frac{1}{m-d}B_{2m-2d}.$$

Continuing similarly, we can express $A_{m,j}$ for each integer j in the range $m/2^{s+1} \leq j < m/2^s$ (iterating consecutively $s = 1, 2, ...$) via the previously determined values of $A_{m,d},\ d < j$ as follows

$$A_{m,j} = (2j+1)\binom{2j}{j}\sum_{d=2j+1}^{m} A_{m,d}\binom{d}{2j+1}\frac{(-1)^{d-1}}{d-j}B_{2d-2j}.$$

The same formula holds also for $m = 0$. Note that the m in above sum must satisfy $m \geq 2j+1$ to return a nonzero term $A_{m,j}$.

Definition 1.27. We define here a generating function of sequence of coefficients $A_{m,j}$ as follows

$$A_{m,j} := \begin{cases} 0, & \text{if } j < 0 \text{ or } j > m \\ (2j+1)\binom{2j}{j}\sum_{d=2j+1}^{m} A_{m,d}\binom{d}{2j+1}\frac{(-1)^{d-1}}{d-j}B_{2d-2j}, & \text{if } 0 \leq j < m \\ (2j+1)\binom{2j}{j}, & \text{if } j = m \end{cases}$$

Five initial rows of the triangle generated by $A_{m,j},\ j \geq 0,\ 0 \leq j \leq m$ are

$m=0$	1					
$m=1$	1	6				
$m=2$	1	0	30			
$m=3$	1	-14	0	140		
$m=4$	1	-120	0	0	630	
$m=5$	1	-1386	660	0	0	2772

Table 4. Triangle generated by $A_{m,j},\ j \geq 0,\ 0 \leq j \leq m$, see definition (1.27).

Note that starting from row $m \geq 11$ the terms of the table 4 consist of fractional numbers, for example, $A_{11,1} = 800361655623.6$. One can find a complete list of the numerators and denominators of $A_{m,j}$ in OEIS under the identifiers A302971 and A304042, respectively, see [17],[18]. Note that

$$\sum_j A_{m,j} \ = \ 2^{2m+1} - 1.$$

As we have found a recurrence for $A_{m,j}$ coefficients, let's back to the identity (1.1)

$$\sum_{k=1}^{\ell}\sum_{j=0}^{m} A_{m,j}k^j(T-k)^j \ \equiv \ \sum_{k=0}^{m}(-1)^{m-k}U_m(\ell,k)\cdot T^k$$

to find a recurrence for coefficients $U_m(\ell,k)$ for every $m \geq 1$. To understand the nature of the coefficients $U_m(T,k)$, express the polynomials $U_m(\ell,k)$ in terms of $A_{m,j}$ and Bernoulli numbers. To do so, let's expand the binomial $(T-k)^j$ in the l.h.s. of (1.1) and change of the order of summation:

$$\begin{aligned} \sum_{k=1}^{\ell}\sum_{j=0}^{m} A_{m,j}k^j(T-k)^j &= \sum_{k=1}^{\ell}\sum_{j=0}^{m} A_{m,j}k^j\sum_{t=0}^{j}\binom{j}{t}T^t(-1)^{j-t}k^{j-t} \\ &= \sum_{t=0}^{m} T^t\sum_{k=1}^{\ell}\sum_{j=t}^{m}\binom{j}{t}A_{m,j}k^{2j-t}(-1)^{j-t}. \end{aligned} \tag{1.28}$$

Now, taking the coefficient of T^t in (1.28) gives:

$$U_m(\ell, t) = (-1)^m \sum_{k=1}^{\ell} \sum_{j=t}^{m} \binom{j}{t} A_{m,j} k^{2j-t} (-1)^j.$$

From this formula it may be not immediately clear why as $T \to \ell$ the $U_m(\ell, t)$ represent polynomials in T. However, this can be seen if we change the summation order again and use Faulhaber's formula to obtain:

$$U_m(T, t) = (-1)^m \sum_{j=t}^{m} \binom{j}{t} A_{m,j} \frac{(-1)^j}{2j-t+1} \sum_{l=0}^{2j-t} \binom{2j-t+1}{l} B_l T^{2j-t+1-l}. \tag{1.29}$$

Introducing $\kappa = 2j - t + 1 - l$, to (1.29) we further get the formula:

$$U_m(T, t) = (-1)^m \sum_{k=1}^{2j-t+1} T^k \sum_{j=t}^{m} \binom{j}{t} A_{m,j} \frac{(-1)^j}{2j-t+1} \binom{2j-t+1}{\kappa} B_{2j-t+1-\kappa},$$

which allows easily compute the coefficient of T^k in $U_m(T, t)$ for each κ. In above formulae we assume that $B_1 = +\frac{1}{2}$.

1.1. **Error of approximation.** Generally, the monomial T^{2m+1}, $\ell, m \in \mathbb{N}$, $T \in \mathbb{R}$ could be approximated by (1.1) as follows

$$\lim_{T\to\ell} \left[\sum_{k=1}^{\ell} \sum_{j=0}^{m} A_{m,j} k^j (T-k)^j \right] = \lim_{T\to\ell} \left[\sum_{k=0}^{m} (-1)^{m-k} U_m(\ell, k) \cdot T^k \right] = T^{2m+1}.$$

In this subsection we arrange the tables with values of corresponding polynomials $P_m(\ell, T)$ and monomials T^{2m+1} to rate the quality of the approximation of T^{2m+1}, $m \geq 0$ by (1.1). We begin from the case $m = 1$ and $T = 3$, therefore, the approximating polynomials is $P_1 3, T = -81 + 36T$. By solving the equation $T^3 - 36T + 81 = 0$ we receive the following roots $T_1 = 3$, $T_2 = 3.90833$, it follows that our polynomial approximates the monomial T^{2m+1} in neighborhood of T_1 and T_2, the following table contains the values of $P_1(3, T)$ and T^3 in neighborhood of the root $T_1 = 3$

T	$F(T) = T^3$	$P_1(3, T) = -81 + 36T$
2.95	25.67238	25.2
2.96	25.93434	25.56
2.97	26.19807	25.92
2.98	26.46359	26.28
2.99	26.73090	26.64
<u>3.00</u>	<u>27.00000</u>	<u>27</u>
3.01	27.27090	27.36
3.02	27.54361	27.72
3.03	27.81813	28.08
3.04	28.09446	28.44
3.05	28.37263	28.8

Table 5. Table of values of $F(T) = T^3$ and $P_1(3, T)$ in neighborhood of $T_1 = 3$.

Below we show graphically the intersection of of $F(T) = T^3$ and $P_1(3, T)$

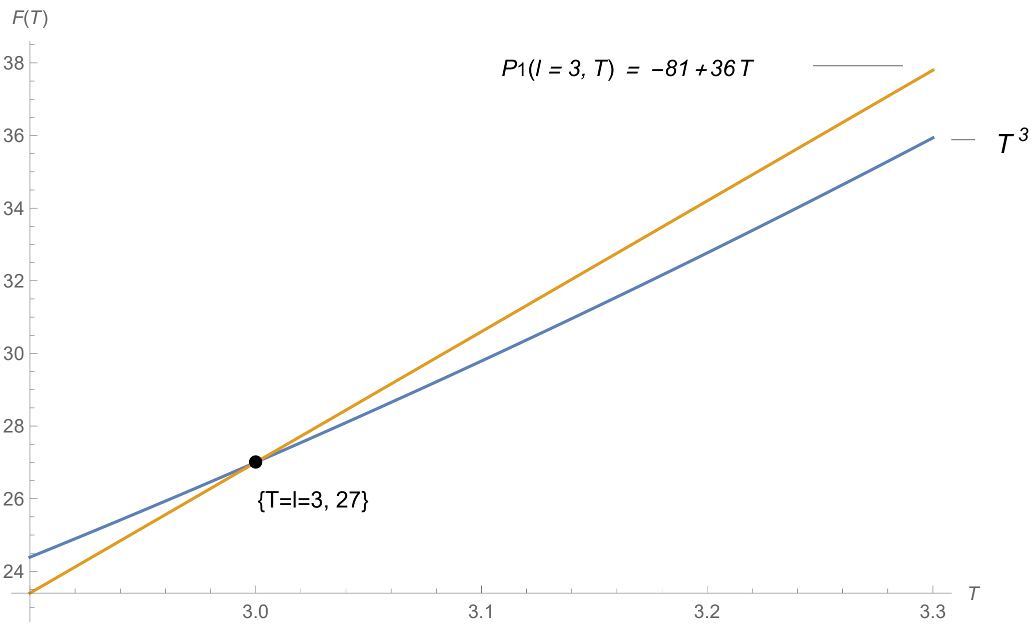

Figure 4. Intersection of $F(T) = T^3$ and $P_1(3, T)$ in point $T_1 = 3$.

As it is mentioned above the equation $T^3 - 36T + 81 = 0$ has two roots $T_1 = 3$, $T_2 = 3.90833$, let's discuss the case $T_2 = 3.90833$, the following table represents the values of $P_1(3, T)$ and T^3 in neighborhood of the root $T_2 = 3.90833$

T	$F(T) = T^3$	$P_1(3, T) = -81 + 36T$
3.86000	57.51246	57.96
3.87000	57.96060	58.32
3.88000	58.41107	58.68
3.89000	58.86387	59.04
3.90000	59.31900	59.4
<u>3.90833</u>	<u>59.69991</u>	<u>59.69988</u>
3.92000	60.23629	60.12
3.93000	60.69846	60.48
3.94000	61.16298	60.84
3.95000	61.62988	61.2
3.96000	62.09914	61.56

Table 6. Table of values of $F(T) = T^3$ and $P_1(3, T)$ in neighborhood of $T_1 = 3$.

Below we show graphically the intersection of $F(T) = T^3$ and $P_1(3, T)$

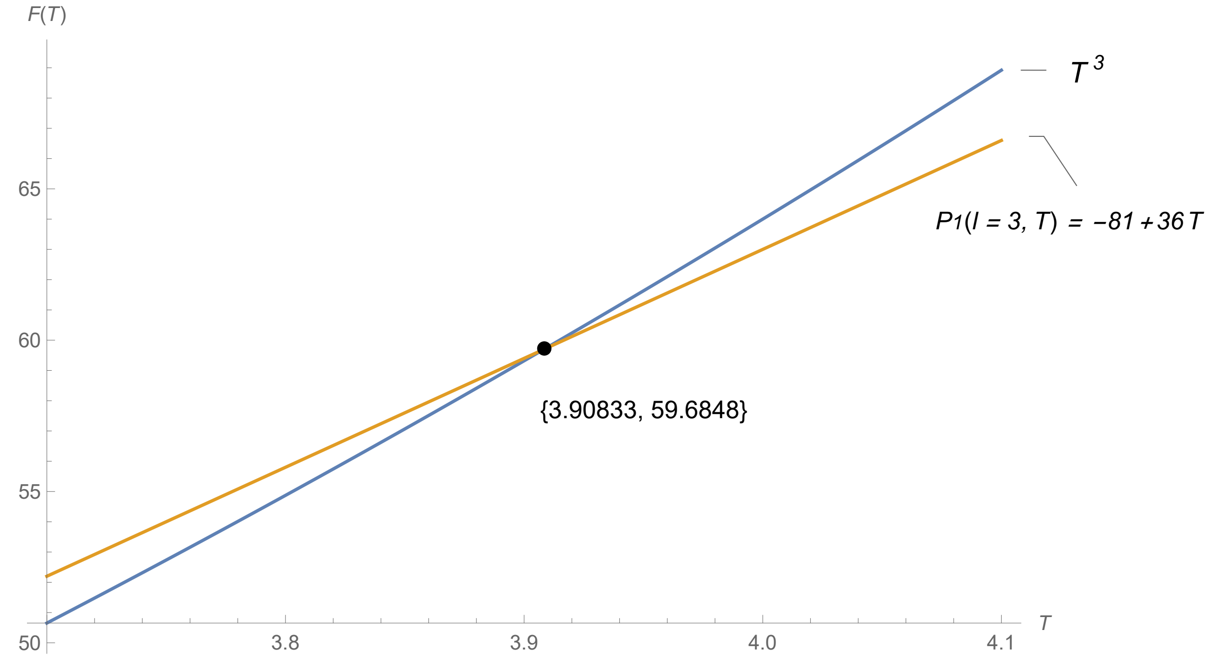

Figure 5. Intersection of $F(T) = T^3$ and $P_1(3, T)$ in point $T_2 = 3.90833$.

Next, let's review the case $m = 2$ and $T = 3$, the approximating polynomial is $P_2(3, T) = 2943 - 2160T + 420T^2$. By solving the equation $T^5 - (2943 - 2160T + 420T^2) = 0$ we receive the following roots $T_1 = 3$, $T_2 = 3.40551$, $T_3 = 3.98704$, it follows that our polynomial approximates the monomial T^{2m+1} in neighborhood of T_1, T_2 and T_3, the following tables contain the values of $P_2(3, T)$ and T^5 in neighborhood of the roots $T_1 = 3$, T_2 and T_3

T	$F(T) = T^5$	$P_2(3, T) = 2943 - 2160T + 420T^2$
2.95	223.41384	226.05000
2.96	227.22628	229.27200
2.97	231.09058	232.57800
2.98	235.00728	235.96800
2.99	238.97691	239.44200
3.00	243.00000	243.00000
3.01	247.07709	246.64200
3.02	251.20872	250.36800
3.03	255.39544	254.17800
3.04	259.63780	258.07200
3.05	263.93634	262.05000

Table 7. Table of values of $F(T) = T^5$ and $P_2(3, T)$ in neighborhood of $T_1 = 3$.

Below we show graphically the intersection of $F(T) = T^5$ and $P_2(3, T)$

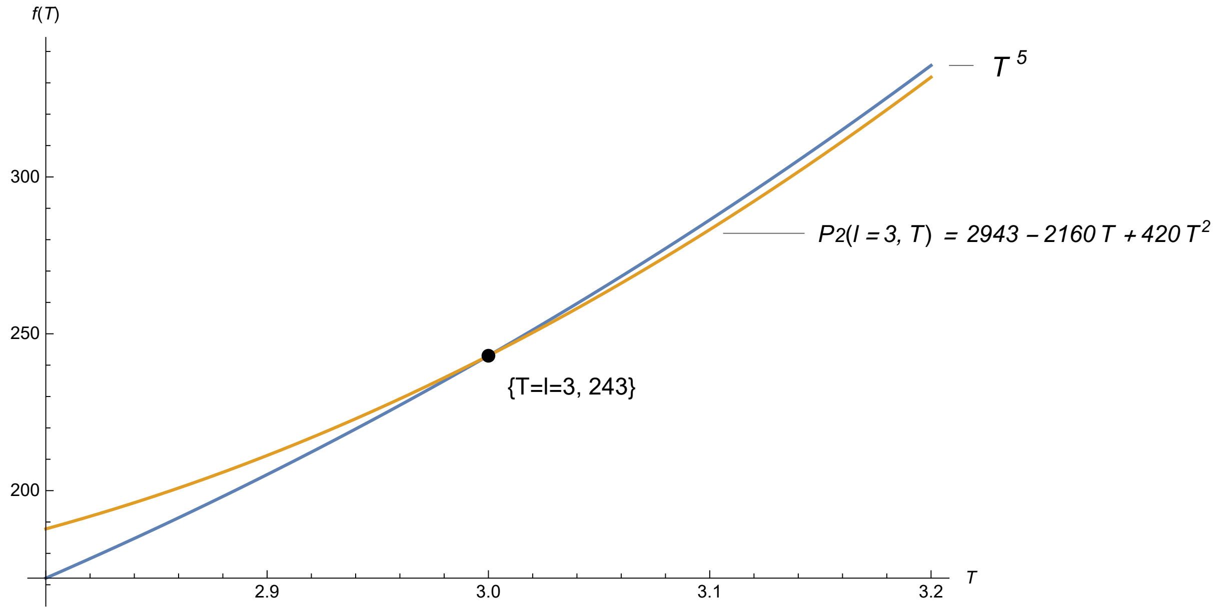

Figure 6. Intersection of $F(T) = T^5$ and $P_2(3, T)$ in point $T_1 = 3$.

Next, let's review the values of $F(T) = T^5$ and $P_2(3, T)$ in the neighborhood of the root $T_2 = 3.40551$ as follow table shows

T	$F(T) = T^5$	$P_2(3, T) = 2943 - 2160T + 420T^2$
3.35000	421.91410	420.45000
3.36000	428.24903	427.03200
3.37000	434.65983	433.69800
3.38000	441.14717	440.44800
3.39000	447.71175	447.28200
3.40551	458.04780	458.04771
3.41000	461.07534	461.20200
3.42000	467.87574	468.28800
3.43000	474.75615	475.45800
3.44000	481.71727	482.71200
3.45000	488.75980	490.05000

Table 8. Table of values of $F(T) = T^5$ and $P_2(3, T)$ in neighborhood of $T_2 = 3.40551$.

Below we show graphically the intersection of $F(T) = T^5$ and $P_2(3, T)$ in point $T_2 = 3.40551$

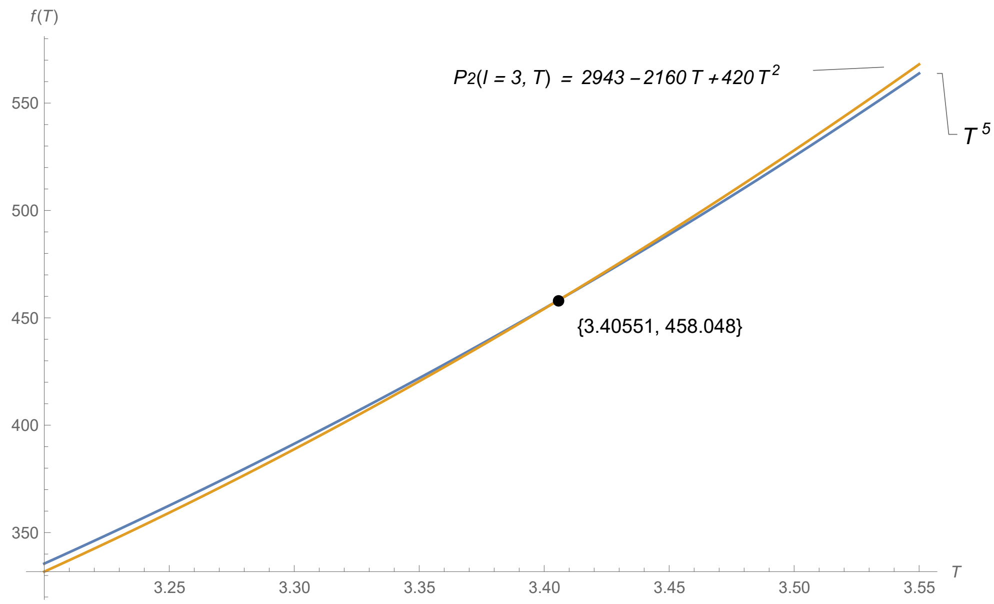

Figure 7. Intersection of $F(T) = T^5$ and $P_2(3, T)$ in point $T_2 = 3.90833$.

Now, let's review the values of $F(T) = T^5$ and $P_2(3, T)$ in the neighborhood of the root $T_2 = 3.90833$ as follow table shows

T	$F(T) = T^5$	$P_2(3, T) = 2943 - 2160T + 420T^2$
3.93000	937.48160	941.05800
3.94000	949.46970	952.51200
3.95000	961.58012	964.05000
3.96000	973.81381	975.67200
3.97000	986.17170	987.37800
3.98000	998.65472	999.16800
3.98704	1007.51835	1007.51854
4.00000	1024.00000	1023.00000
4.01000	1036.86416	1035.04200
4.02000	1049.85728	1047.16800
4.03000	1062.98034	1059.37800

Table 9. Table of values of $F(T) = T^5$ and $P_2(3, T)$ in neighborhood of $T_3 = 3.98704$.

Below we show graphically the intersection of $F(T) = T^5$ and $P_2(3, T)$ in point $T_3 = 3.98704$

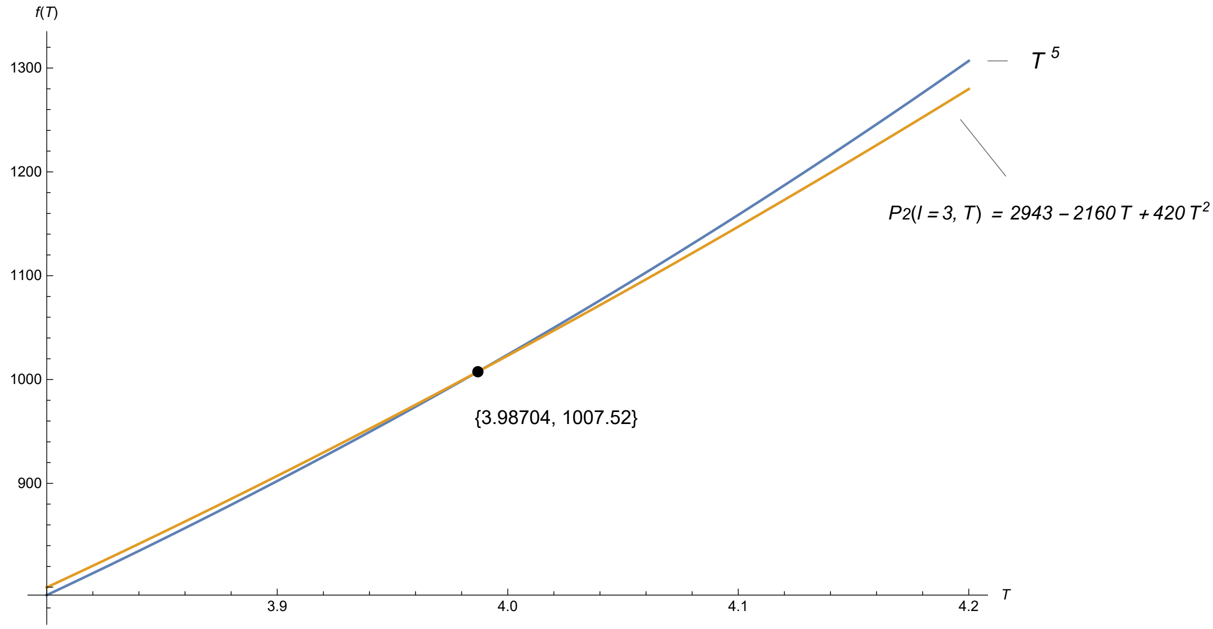

Figure 8. Intersection of $F(T) = T^5$ and $P_2(3, T)$ in point $T_3 = 3.98704$.

Now, let's keep our attention to the case $m = 3$ and $\ell = 2$, the approximating polynomial is $P_3(2, T) = -9028 + 13818T - 7140T^2 + 1260T^3$. By solving the equation $T^7 - (-9028 + 13818T - 7140T^2 + 1260T^3) = 0$ we receive the following roots $T_1 = 2$, $T_2 = 2.8605$, $T_3 = 2.98955$, it follows that our polynomial approximates the monomial T^7 in neighborhood of T_1, T_2 and T_3, the following tables contain the values of $P_2(3, T)$ and T^5 in neighborhood of the roots T_1, T_2 and T_3, note that we have arranged the roots T_2 and T_3 in single table

T	$F(T) = T^7$	$P_3(2, T) = -9028 + 13818T - 7140T^2 + 1260T^3$
1.96	111.12007	113.47136
1.97	115.14990	117.00398
1.98	119.30436	120.59792
1.99	123.58664	124.26074
2.00	128.00000	128
2.01	132.54776	131.82326
2.02	137.23333	135.73808
2.03	142.06015	139.75202
2.04	147.03177	143.87264
2.05	152.15178	148.1075
2.06	157.42385	152.46416

Table 10. Table of values of $F(T) = T^7$ and $P_3(2, T) = -9028+13818T-7140T^2+1260T^3$ in neighborhood of $T_1 = 2$.

Below we arrange a plot of intersection of $F(T) = T^7$ and $P_3(2, T) = -9028+13818T-7140T^2+1260T^3$ in $T_1 = 2$

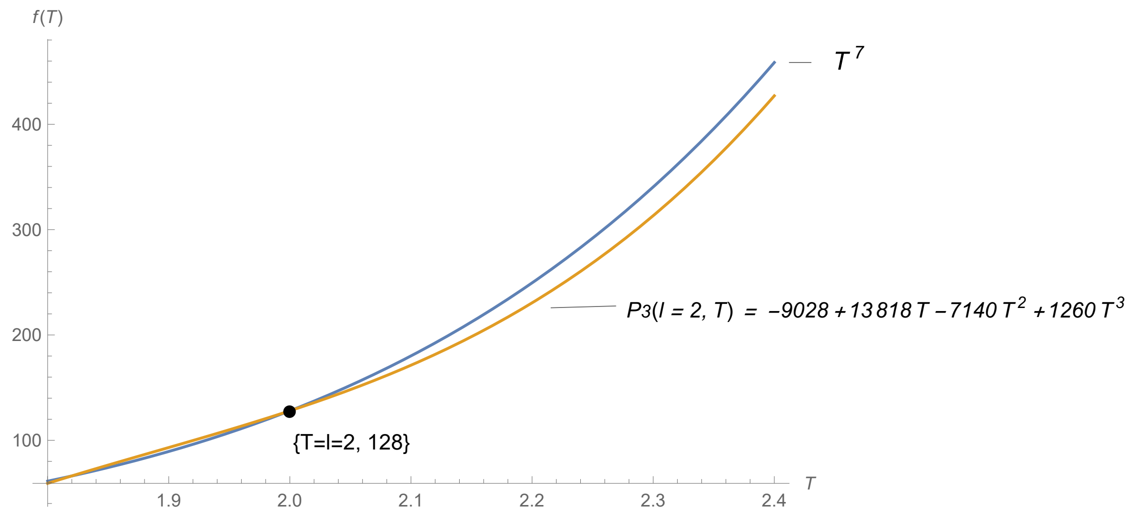

Figure 9. Intersection of $F(T) = T^7$ and $P_2(2, T)$ in point $T_1 = 2$.

Let's show the values of the $F(T) = T^7$ and polynomial $P_3(2, T) = -9028 + 13818T - 7140T^2 + 1260T^3$ in neighborhood of two other roots, $T_1 = 2.86046,\ T_2 = 2.98955$

T	$F(T) = T^7$	$P_3(2, T) = -9028 + 13818T - 7140T^2 + 1260T^3$
2.81000	1383.38875	1379.35766
2.82000	1418.22050	1415.09168
2.83000	1453.80129	1451.52962
2.84000	1490.14449	1488.67904
2.85000	1527.26367	1526.5475
2.86046	1566.94022	1566.940217
2.87000	1603.88519	1604.47178
2.88000	1643.41563	1644.54272
2.89000	1683.77827	1685.36294
2.90000	1724.98763	1726.94
2.91000	1767.05848	1769.28146
2.92000	1810.00577	1812.39488
2.93000	1853.84467	1856.28782
2.94000	1898.59054	1900.96784
2.95000	1944.25897	1946.4425
2.96000	1990.86576	1992.71936
2.97000	2038.42692	2039.80598
2.98000	2086.95867	2087.70992
2.98955	2134.25217	2134.252175
3.00000	2187.00000	2186
3.01000	2238.54314	2236.40126
3.02000	2291.12403	2287.65008
3.03000	2344.76002	2339.75402
3.04000	2399.46868	2392.72064

Table 11. Table of values of $F(T) = T^7$ and $P_3(2, T) = -9028 + 13818T - 7140T^2 + 1260T^3$ in neighborhood of $T_2 = 2.8605$, $T_3 = 2.98955$.

Below we show graphically the intersection of $F(T) = T^7$ and $P_3(2, T) = -9028 + 13818T - 7140T^2 + 1260T^3$ in point $T_3 = 3.98704$ in points $T_2 = 2.8605$, $T_3 = 2.98955$

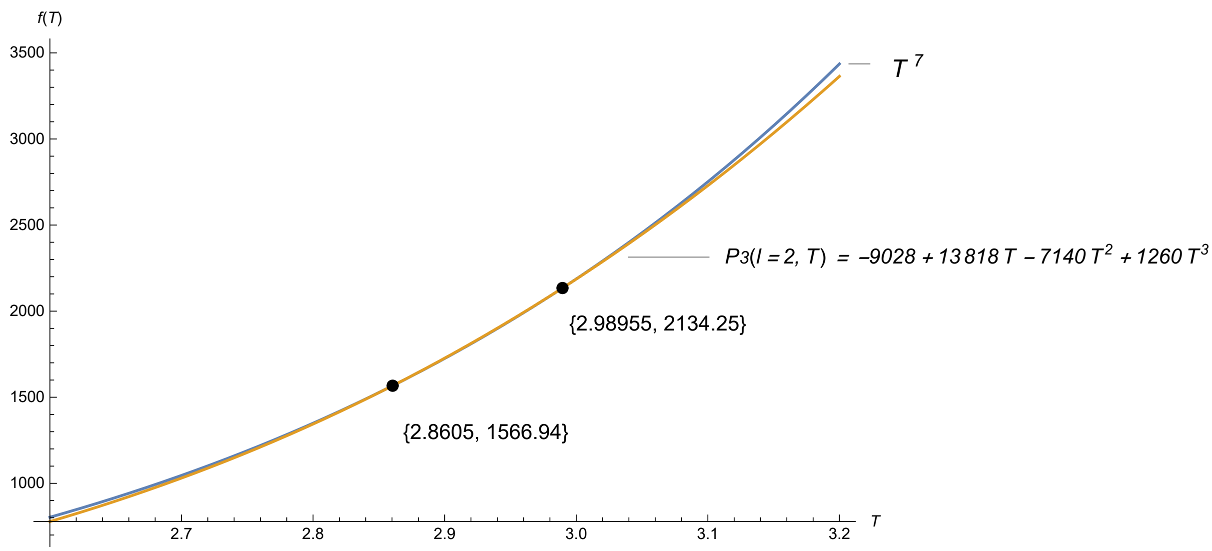

Figure 10. Intersection of $F(T) = T^5$ and $P_2(3, T)$ in points $T_2 = 2.8605$, $T_3 = 2.98955$.

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3. Conclusion

In this manuscript we have derived and discussed the polynomials $P_m(\ell, T)$, such that converge to the monomial T^{2m+1}, $m \in \mathbb{N}_0$ as $T \to \ell$, $\ell \in \mathbb{N}$. Polynomial $P_m(\ell, T)$ gives weak approximation of the monomial T^{2m+1} mainly in neighborhood of the solution of the equation $T^{2m+1} - P_m(\ell, T) = 0$.

4. Supplementary Files

We provide the following supplementary files to support our study:

- https://kolosovpetro.github.io/arxiv_1603_02468/identity_1_1.txt - Mathematica program, implementation the l.h.s of identity (1.1), that is $T^{2m+1} = \sum_{k=1}^{\ell} \sum_{j=0}^{m} A_{m,j} k^j (T-k)^j$, $\ell = T \in \mathbb{N}$.
- https://kolosovpetro.github.io/arxiv_1603_02468/identity_1_1_r_h_s.txt - Mathematica program, implementation the r.h.s of identity (1.1), that is $T^{2m+1} = \sum_{k=0}^{m} (-1)^{m-k} U_m(\ell, k) \cdot T^k$, $\ell = T \in \mathbb{N}$.
- https://kolosovpetro.github.io/arxiv_1603_02468/u_coefficients_in_row.txt - Mathematica program, lists the coefficients $U_m(\ell, k)$ in (1.1) for given m. (By default $m = 2$)

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