The Canonical Commutation Relation is Unitary due to Scaling between Complementary Variables

Homogeneity of Space is Non-unitary

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27th December 2017

Abstract Textbook theory says that the Canonical Commutation Relation derives from the homogeneity of space. This paper shows that additionally, an a dual of *accidental coincident scalings* is needed, as extra information, without which the Canonical Commutation Relation is left non-unitary and broken. This single counter-example removes symmetry, as intrinsic ontological reason, for axiomatically imposing unitarity (or self-adjointness) — *by Postulate* — on quantum mechanical systems.

Keywords foundations of quantum theory, quantum mechanics, wave mechanics, Canonical Commutation Relation, symmetry, homogeneity of space, unitary.

1 Homogeneity of Space and Wave Mechanics

The *Canonical Commutation Relation:*

$$\mathbf{px} - \mathbf{xp} = -i\hbar$$

embodies core algebra at the heart of wave mechanics. With general acceptance amongst quantum theorists, the professed significance of this relation is that it derives from the *homogeneity of space* — and is *unitary.* In this paper, I re-examine the Canonical Relation's derivation and establish that the homogeneity symmetry is of itself *not* unitary. And in consequence establish that the Canonical Commutation Relation does not, itself, faithfully represent homogeneity, but contains extra unitary information also.

Imposing homogeneity on a system is identical to imposing a null physical or geometrical effect under arbitrary translation of reference frame. To formulate this arbitrary translation, resulting in null effect, the principle we invoke is *Form Invariance.* This is the concept from relativity that symmetry transformations leave formulae fixed in *form*, though *values* may alter [3]. In the case at hand, the relevant formula whose form is held fixed is the eigenvalue equation for position:

$$\mathbf{x}\,|f_{\mathsf{x}}(x)\rangle = \mathsf{x}\,|f_{\mathsf{x}}(x)\rangle\,. \tag{1}$$

In this, the san-serif x denotes the eigenvalue and labels its eigenvector f_{x}; the variable x (curly) is the function domain. The use of two different variables here, may seem unusual and pointless. In fact, logically they are different; x is quantified *existentially* but x is quantified *universally.*

With form held fixed as the reference system is displaced, variation in the position operator $\mathbf{x}$ determines a group relation, representing the homogeneity symmetry. Under arbitrarily small displacements, this group corresponds to a linear algebra representing homogeneity locally. These are a Lie group and Lie algebra. To maintain the form of (1) under translation, the basis $|f_{\mathsf{x}}\rangle$ is cleverly managed: whilst the translation transforms the basis from $|f_{\mathsf{x}}\rangle$ to $|f_{\mathsf{x}-\epsilon}\rangle$, a similarity transformation is also applied, chosen to revert $|f_{\mathsf{x}-\epsilon}\rangle$ back to $|f_{\mathsf{x}}\rangle$. In this way $|f_{\mathsf{x}}\rangle$ is held static. Actually, similarity transforms can be found only for a certain class of functions: $\left\{\psi_{\mathsf{x}} \in L^1\right\} \subset \{f_{\mathsf{x}}\}$. These are the functions in Banach space — having no inner product. Hilbert space is not needed at this point.

The similarity transformations are the one-parameter subgroup of the general

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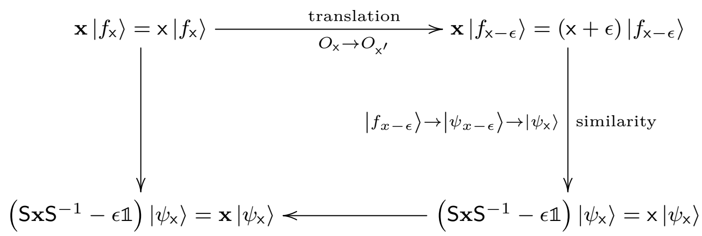

Figure 1 Scheme of transformations. The bottom left hand formula is the resulting group relation.

linear group, $\mathsf{S}(\epsilon) \subset \mathsf{S} \in \mathsf{GL}(\mathbb{F})$, with the transformation parameter ϵ coinciding with the displacement parameter, and where $\mathbb{F}$ is any infinite field. The overall scheme of transformations is depicted in Figure 1.

In standard theory, textbook understanding is that $\mathsf{S}(\epsilon)$ is intrinsically and necessarily unitary [1, p.109][2, p.34], and it is in *that* unitarity where the Canonical Commutation Relation finds its unitary origins. And so, because its presence is thought *intrinsically necessary*, unitarity is imposed axiomatically on the theory, *by Postulate*. This imposed unitarity is added information, extra to the information of homogeneity. In consequence, the underlying symmetry beneath wave mechanics is not homogeneity of space, but instead, a unitary subgroup of it.

As an experiment, I proceed by treating unitarity as a purely separate issue from homogeneity, allowing $\mathsf{S}(\epsilon)$ it's widest generality, so that the *whole information* of homogeneity (upto the general linear similarity transformation) is faithfully and genuinely conveyed through the theory.

The experiment begins with the position eigenvalue equation (1) being rewritten, in the form of a quantified proposition (2). From here on, all informal assumptions are to be shed, with the Dirac notation dropped to avoid any inference that vectors are intended as orthogonal, in Hilbert space, or equipped with a scalar product; none of these is implied.

Consider the eigenformula for position operator $\mathbf{x}$, eigenfunctions f_{x} and eigenvalues x, seen from the reference frame O_{x}:

$$\forall x \exists \mathbf{x} \exists \mathsf{x} \exists f_{\mathsf{x}} \mid \quad \mathbf{x} f_{\mathsf{x}}(x) = \mathsf{x} f_{\mathsf{x}}(x) \tag{2}$$

Translation: Applying the translation first. Under translation, homogeneity demands existence of an equally relevant reference frame $\mathsf{O}_{\mathsf{x}'}$ displaced arbitrarily through ϵ. See Figure 2. *Form Invariance* guarantees a formula for $\mathsf{O}_{\mathsf{x}'}$ of the same form as that for O_{x} in (2), thus:

$$\forall x' \exists \mathbf{x} \exists \mathsf{x}' \exists f'_{\mathsf{x}} \mid \quad \mathbf{x} f'_{\mathsf{x}}(x') = \mathsf{x}' f'_{\mathsf{x}}(x') \tag{3}$$

A relation for $\mathbf{x}$ is to be evaluated, so $\mathbf{x}$ is held static for all reference frames. The translation transforms position, thus:

$$\forall \epsilon \forall \mathsf{x}' \exists \mathsf{x} \mid \quad \mathsf{x} \mapsto \mathsf{x}' = \mathsf{x} + \epsilon \tag{4}$$

and transforms the function, thus:

$$\forall \epsilon \forall x' \forall f'_{\mathsf{x}} \exists f_{\mathsf{x}} \exists x \mid \quad f_{\mathsf{x}}(x) \mapsto f'_{\mathsf{x}}(x') = f_{\mathsf{x}-\epsilon}(x - \epsilon) \tag{5}$$

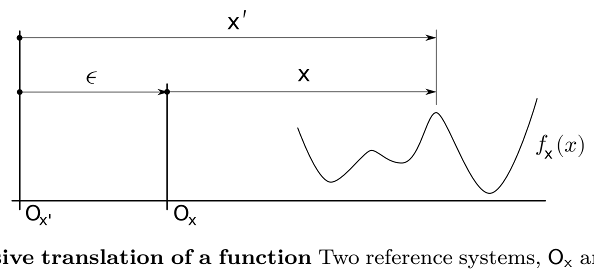

Figure 2 Passive translation of a function Two reference systems, O_{x} and $\mathsf{O}_{\mathsf{x}'}$, arbitrarily displaced by ϵ, individually act as reference systems for position of a function f_{x}. If the x-space is homogeneous, then regardless of the value of ϵ, physics concerning this function is described by formulae whose form remains invariant, though values may change. **Note:** The function and reference frames are not epistemic; f_{x} is non-observable and O_{x} and $\mathsf{O}_{\mathsf{x}'}$ are not observers.

Substituting (4) and (5) into (3) gives the translated formula:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists f_{\mathsf{x}} \mid \quad \mathbf{x} f_{\mathsf{x}-\epsilon}(x-\epsilon) = (\mathsf{x}+\epsilon) f_{\mathsf{x}-\epsilon}(x-\epsilon)\,. \tag{6}$$

Similarity: Applying the similarity transformation. This involves the one-parameter linear operator $\mathsf{S}_{(\epsilon)}$. Any such transformation would be invalid if it were to result in an unbounded f_{x}. Valid transformations $\mathsf{S}_{(\epsilon)}$ exist only if there exists a function space $\{\psi_{\mathsf{x}}\}$, which is complete, normalisable, not restricted to separable[1] functions, and of course, also be a subset of the translatable functions f_{x}.

Such function spaces are well-known; they are the normed L^1 spaces, known as Banach spaces. See Figure 3. Hilbert space L^2 is a particular class of Banach space whose norm is determined by an *inner product*. For the purpose of our transformation, Hilbert space is extra unnecessary conditionality. Hilbert space materialises incidentally and downstream of this point, arising through circumstances independent of homogeneity. Significantly, the operator $\mathsf{S}_{(\epsilon)}$ and functions ψ_{x} can be real, so the following proposition is valid for non-unitary, real operators $\mathsf{S}_{(\epsilon)}$ and real functions ψ_{x}:

$$\forall x \forall \epsilon \forall \psi_{\mathsf{x}-\epsilon} \exists \mathsf{S} \exists \psi_{\mathsf{x}} \mid \quad \mathsf{S}^{-1}_{(\epsilon)} \psi_{\mathsf{x}}(x) = \psi_{\mathsf{x}-\epsilon}(x-\epsilon)\,. \tag{7}$$

In standard theory, $\mathsf{S}_{(\epsilon)}$ would be set unitary by the mathematician. Doing that restricts the space of functions ψ_{x} to the Hilbert space L^2 without homogeneity demanding it.

The similarity transformation is formed, thus:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}^{-1}_{(\epsilon)} \psi_{\mathsf{x}}(x) = (\mathsf{x}+\epsilon)\, \psi_{\mathsf{x}}(x)\,.$$

Introducing the trivial eigenformula: $\forall \psi_{\mathsf{x}} \forall x \forall \epsilon \mid \epsilon \mathbb{1} \psi_{\mathsf{x}}(x) = \epsilon \psi_{\mathsf{x}}(x)$ and subtracting:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \mathsf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \left(\mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}^{-1}_{(\epsilon)} - \epsilon \mathbb{1} \right) \psi_{\mathsf{x}}(x) = \mathsf{x} \psi_{\mathsf{x}}(x)\,. \tag{8}$$

Now comparing the original position eigenformula (2) against the transformed one (8), we deduce the group relation for similarity transformed homogeneity:

$$\forall x \forall \epsilon \exists \mathbf{x} \exists \psi_{\mathsf{x}} \exists \mathsf{S} \mid \quad \mathbf{x} \psi_{\mathsf{x}}(x) = \left(\mathsf{S}_{(\epsilon)} \mathbf{x} \mathsf{S}^{-1}_{(\epsilon)} - \epsilon \mathbb{1} \right) \psi_{\mathsf{x}}(x)\,. \tag{9}$$

From this group relation, the commutator for the *Lie algebra* is now computed. Because $\mathsf{S}_{(\epsilon)}$ is a one-parameter subgroup of $\mathsf{GL}(\mathbb{F})$, there exists a unique linear operator $\mathbf{g}$ for real parameters ϵ, such that:

$$\forall \mathsf{S} \exists \mathbf{g} \mid \quad \mathsf{S}_{(\epsilon)} = \mathrm{e}^{\epsilon \mathbf{g}} \tag{10}$$

Noting that homogeneity is totally independent of scale, an arbitrary scale factor η is extracted, thus: $\forall \mathbf{g} \forall \eta \exists \mathbf{k} : \ \mathbf{g} = \eta \mathbf{k}$, implying:

$$\forall \eta \forall \mathsf{S} \exists \mathbf{k} \mid \quad \mathsf{S}_{(\epsilon)} = \mathrm{e}^{\eta \epsilon \mathbf{k}} \tag{11}$$

$$\forall \eta \forall \mathsf{S} \exists \mathbf{k} \mid \quad \mathsf{S}^{-1}_{(\epsilon)} = \mathsf{S}_{(-\epsilon)} = \mathrm{e}^{-\eta \epsilon \mathbf{k}} \tag{12}$$

Substitution of (11) and (12) into (9) gives:

$$\begin{aligned}
&\forall x \forall \eta \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \exp(+\eta\epsilon\mathbf{k})\, \mathbf{x} \exp(-\eta\epsilon\mathbf{k})\, \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon\mathbb{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow\ &\forall x \forall \eta \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \left[\mathbb{1} + \eta\epsilon\mathbf{k} + \mathcal{O}\left(\epsilon^2\right)\right] \mathbf{x} \left[\mathbb{1} - \eta\epsilon\mathbf{k} + \mathcal{O}\left(\epsilon^2\right)\right] \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon\mathbb{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow\ &\forall x \forall \eta \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \left[\mathbf{x} + \eta\epsilon\mathbf{k}\mathbf{x} + \mathcal{O}\left(\epsilon^2\right)\right] \left[\mathbb{1} - \eta\epsilon\mathbf{k} + \mathcal{O}\left(\epsilon^2\right)\right] \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon\mathbb{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow\ &\forall x \forall \eta \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && \left[\mathbf{x} + \eta\epsilon\mathbf{k}\mathbf{x} - \eta\epsilon\mathbf{x}\mathbf{k} + \mathcal{O}\left(\epsilon^2\right)\right] \psi_{\mathsf{x}}(x) = [\mathbf{x} + \epsilon\mathbb{1}]\, \psi_{\mathsf{x}}(x) \\
\Rightarrow\ &\forall x \forall \eta \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid && [\mathbf{k}\mathbf{x} - \mathbf{x}\mathbf{k}]\, \psi_{\mathsf{x}}(x) = \left[\eta^{-1}\mathbb{1} - \mathcal{O}(\epsilon)\right] \psi_{\mathsf{x}}(x)
\end{aligned}$$

At the limit, as $\epsilon \to 0$, we have:

$$\forall x \forall \eta \exists \psi_{\mathsf{x}} \exists \mathbf{x} \exists \mathbf{k} \mid \quad [\mathbf{k}, \mathbf{x}]\, \psi_{\mathsf{x}}(x) = \eta^{-1} \mathbb{1} \psi_{\mathsf{x}}(x) \tag{13}$$

And by an analogous proof, similar to all that above, but conditional upon the existence of eigenfunctions $\chi_{\mathsf{k}}(k)$ of $\mathbf{k}$:

$$\forall k \forall \zeta \exists \chi_{\mathsf{k}} \exists \mathbf{x} \exists \mathbf{k} \mid \quad [\mathbf{x}, \mathbf{k}]\, \chi_{\mathsf{k}}(k) = \zeta^{-1} \mathbb{1} \chi_{\mathsf{k}}(k)\,. \tag{14}$$

Individually, each of the formulae (13) and (14) are separate consequences of the homogeneity symmetry, and yet they are not the Canonical Commutation Relation; and there is no assurance they offer complementarity.

[1] Separable means countable, as are the integers, as opposed to continuous, like the reals.

Substitution involving quantified variables

$$\begin{aligned}
\forall \beta \underline{\forall \gamma} \exists \alpha \mid &\quad \alpha = \beta + \gamma \\
\forall \beta \underline{\exists \gamma} \mid &\quad \gamma = \beta + \beta \\
\Rightarrow\ \forall \beta \exists \alpha \mid &\quad \alpha = \beta + \beta + \beta
\end{aligned}$$

For logically dependent substitution, an existential quantifier of one proposition should be matched with a universal quantifier of the other. This is because, for this type of substitution coincidence is certain and not accidental. Matching quantifiers are underlined.

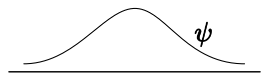

Figure 3 The linear transformations S exist only for bounded ψ_{x}, maximally, the Banach space L^1. These are the Lebesgue integrable functions: $\int |\psi_{\mathsf{x}}|$ is finite.

2 New *logically independent* information

If homogeneity is to imply the Canonical Commutation Relation, new information is needed, in addition to (13) and (14). For one thing, quantifiers $\forall\eta$ in (13) and $\forall\zeta$ in (14) contradict the Canonical Commutation Relation. Hence, some extra condition that restricts these is necessary information. It should be noted that this extra condition will be new information that is *logically independent* of homogeneity.

I proceed by making the assumption that the extra information needed is for both these formulae to be valid — simultaneously. As they appear, there is no guarantee of that. Note that (13) is quantified $\exists\psi_{\mathsf{x}}$, and (14) quantified $\exists\chi_{\mathsf{k}}$. And so their combined quantification is $\exists\psi_{\mathsf{x}}\exists\chi_{\mathsf{k}}$; it is not $\forall\psi_{\mathsf{x}}\exists\chi_{\mathsf{k}}$ or $\forall\chi_{\mathsf{k}}\exists\psi_{\mathsf{x}}$. Hence, non-contradictory values for ψ_{x} and χ_{k} are not guaranteed; any happy coincidence between them would be accidental.

In precise terms, to uncover the extra information that guarantees simultaneity, I pose the assumed simultaneity formally as an hypothesis, then proceed to deduce conditionality implied by it. Essentially, the hypothesis is an experiment needing guesswork, and it seems likely that, vectors ψ_{x} and χ_{k} must be particular parallel scalings of one another.

Hypothesised coincidence:

$$\forall\chi_{\mathsf{k}}\forall\zeta\forall\eta\exists\psi_{\mathsf{x}} \;\wedge\; \forall x\exists k \;|\quad \chi_{\mathsf{k}}(k) = \zeta\eta\psi_{\mathsf{x}}(x) \tag{15}$$

Taking (13) and the negative of (14) gives us the pair:

$$\forall x\forall\eta\exists\psi_{\mathsf{x}}\exists\mathbf{x}\exists\mathbf{k} \;|\quad [\mathbf{k},\mathbf{x}]\,\psi_{\mathsf{x}}(x) = +\eta^{-1}\mathbb{1}\psi_{\mathsf{x}}(x) \tag{16}$$
$$\forall k\forall\zeta\exists\chi_{\mathsf{k}}\exists\mathbf{x}\exists\mathbf{k} \;|\quad [\mathbf{k},\mathbf{x}]\,\chi_{\mathsf{k}}(k) = -\zeta^{-1}\mathbb{1}\chi_{\mathsf{k}}(k) \tag{17}$$

Substuting the **Hypothesised coincidence** (15) into (17) gives the pair:

$$\forall x\forall\eta\exists\psi_{\mathsf{x}}\exists\mathbf{x}\exists\mathbf{k} \;|\quad [\mathbf{k},\mathbf{x}]\,\psi_{\mathsf{x}}(x) = +\eta^{-1}\mathbb{1}\psi_{\mathsf{x}}(x) \tag{18}$$
$$\forall x\forall\zeta\forall\eta\exists\psi_{\mathsf{x}}\exists\mathbf{x}\exists\mathbf{k} \;|\quad \zeta\eta\,[\mathbf{k},\mathbf{x}]\,\psi_{\mathsf{x}}(x) = -\eta^{+1}\mathbb{1}\psi_{\mathsf{x}}(x) \tag{19}$$

Subtracting (18) and (19):

$$\forall x\forall\zeta\forall\eta\exists\psi_{\mathsf{x}}\exists\mathbf{x}\exists\mathbf{k} \;|\quad \left\{(\zeta\eta-1)\,[\mathbf{k},\mathbf{x}] + \left(\eta+\eta^{-1}\right)\mathbb{1}\right\}\psi_{\mathsf{x}}(x) = \mathbb{0} \tag{20}$$

The formula (20) is self-contradictory, because it cannot be true for all values of ζ and η. In truth, (20) is valid only for values:

$$\zeta = \pm\mathrm{i} \qquad\qquad \eta = \mp\mathrm{i} \tag{21}$$

This confirms there is something invalid about the **Hypothesis** (15). Nonetheless, an **Adjusted Hypothesis** (22), in which quantifiers $\forall\zeta\forall\eta$ are replaced by $\exists\zeta\exists\eta$, thus:

$$\forall\chi_{\mathsf{k}}\exists\zeta\exists\eta\exists\psi_{\mathsf{x}} \;\wedge\; \forall x\exists k \;|\quad \chi_{\mathsf{k}}(k) = \zeta\eta\psi_{\mathsf{x}}(x) \tag{22}$$

eliminates the self-contradiction, thus:

$$\forall x\exists\zeta\exists\eta\exists\psi_{\mathsf{x}}\exists\mathbf{x}\exists\mathbf{k} \;|\quad \left\{(\zeta\eta-1)\,[\mathbf{k},\mathbf{x}] + \left(\eta+\eta^{-1}\right)\mathbb{1}\right\}\psi_{\mathsf{x}}(x) = \mathbb{0} \tag{23}$$

Summarising

On top of homogeneity, logically independent, extra new information is needed in constructing the Canonical Commutation Relation:

$$[\mathbf{k},\mathbf{x}] = -i\mathbb{1} \quad \text{or} \quad [\mathbf{p},\mathbf{x}] = -i\hbar\mathbb{1} \tag{24}$$

That information is represented in the steps taken in going from the non-unitary (13) and (14) to the unitary (24). Precisely, the Canonical Commutation Relation does not represent the homogeneity of space; it represents homogeneity for a particular scaling between position space and wave-number space (momentum space).

Conclusion

The above establishes that the homogeneity of space, or indeed, the homogeneity symmetry is not the source of unitary information in wave mechanics. That is to say, the foundational symmetry we suppose to be the fundamental ontology of this quantum system is not unitary. Rather, unitarity is separate, logically independent of the underlying ontology, and a condition implied within complementarity.

And therefore, if the reason given for postulating that quantum theory should be unitary or self-adjoint, is that symmetries in Nature are intrinsically, unavoidably and ontologically unitary, then this one counter-example requires that a different reason be found, or otherwise, the *Postulate* be withdrawn.

This does not mean Quantum Theories are not unitary, because certainly they are; it means that unitarity may not be imposed by the mathematician, for the reason she believes unitarity to be a Fundamental Physical Principle.

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