From Bernoulli to Laplace and Beyond

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Abstract. Reviewing Laplace's equation of gravitation from the perspective of D. Bernoulli, known as Poisson-equation, it will be shown that Laplace's equation tacitly assumes the temperature T of the mass system to be approximately $0^\circ K$. For temperatures greater zero, the gravitational field will have to be given an additive correctional field. Now, temperature is intimately related to the heat, and heat is known to be radiated as an electromagnetic field. It is shown to take two things in order to get at the gravitational field in the low temperature limit: the total square energy density of the source in space-time and a (massless) field, which defines interaction as quadratic, Lorentz-invariant and $U(4)$-symmetric form. This field not only necessarily must include electromagnetic interaction, it also will be seen to behave like it.

1. Problem Statement

A system of N particles in spacetime in Newtonian mechanics is a system that is to be defined by $3N$ location coordinates q_k as well as a common time coordinate and their associated $3N$ momentum coodinates p_k as a function of time. Mostly these systems are stably confined to a fixed region in space over time like a drop of water or a stone. So, there will be many equations of confinement, and to simplify the mathematical model, Bernoulli changed that model by replacing the particles' position with a spatial mass density $\rho(t) : \mathbb{R}^3 \ni \vec{x} \mapsto \rho(\vec{x}(t)) \geq 0$. Laplace then took over that model and showed that the gravitational force of a mass density ρ could be expressed as Poisson equation $\Delta\Phi = 4\pi G\rho$ of a potential function Φ, the gravitational field and the gravitational constant G, $\Delta := \partial_1^2 + \partial_2^2 + \partial_3^2$ being the Laplace operator. That marked the introduction of field as a concept into physics. What made it both bold and dubious, was that it said that the field was to be the sheer equivalent of the mass distribution. It was soon found out that the field was to be an harmonic function of the space coordinates, which meant that that field was completely determined by its values on a mass enclosing closed (smooth) surface. The problem here is, that it needs 3 dimensions to define the composite mass distribution of the system. Yet, only 2 dimensions are necessary to define the field. How then can both, field an mass, be equivalent?

2. Even more problems

Both, Bernoulli and Laplace took it as evident that a (smooth) mass distribution $\rho(x)$ of N particles, which is confined to a bounded region $K \in \mathbb{R}^3$ (for all times t), could be resolved at each given time t into N disjoint bounded regions $K_1, \ldots, K_N$, containing a unique particle, if only the particles would stay apart from eachother. With that, it should be possible to replace ρ with the sum $\sum_k \rho_k$ of smooth, non-negative functions ρ_k of disjoint support and compact support, each (which means, they all vanish outside a bounded set, e.g. K, and if one is greater zero at some point x, then all the others must vanish at this point x). If so, the above Poisson equation could be rewritten as a sum $\sum_k \Delta\Phi_k = \sum_k 4\pi G\rho_k$ of N independent gravitational equations for each and every particle.
And indeed, mathematics proved this to be possible, now known as the partion of unity (see e.g. [6, Ch.16]). That, on one side, means that even if all particles are pointwise in nature, we can approximate these particles through Bernoulli's ingenious replacement of mass position by smooth mass densities. On the downside, that shows that Laplace's theory of gravitation must lack generality, because in it, all the particles of a body are independent from eachother: they just add up individually!

3. Inspection of the Problems

In mechanics, there is a unique N-particle system for which the above condition of independency of its constituing particles holds: it is the free particle system, in which each particle moves in a straight line. But if they move relative to eachother, then the compound system will disperse and cannot be confined to a bounded region K for all t. So, assuming spatial confinement of the system, it follows that Laplace's gravitational theory demands all particles to be at rest with respect to eachother. So, there is an inertial coordinate system, in which the compound body as well as all its particles are rest. Equivalently put, the average kinetic energy of the particles is zero, which means that the system must have the absolute temperature $T = 0°K$.

For $T > 0$ then, the field Φ must be added a non-trivial corrective $\Phi_T \neq 0$, which should converge pointwise to zero as $T \to 0$, and it is to be hoped, that $\Delta\Phi_T$ likewise will converge to zero as $T \to 0$.
Now, we already have such a field, but it's the electromagnetic field, although this field is (surprisingly) not reserved for gravitational purposes, but interacts primarily with charges: apart from perhaps neutrinos and Higgs bosons,

we know that every system of temperature greater zero radiates an electromagnetic field and that every particle of mass $m > 0$ gets scattered by light as it crosses its way.

Next, given a charge being smeared out as a charge density in space (for each instance of time), be it appreciable, or be it just an infinetisimal little bit, there is no way to tell how the charges or particles will have to move: they may be jiggling in one, two, or three dimensions. But they could also move synchronously around a steady axis. If the particles move in the body, we only know two things: The motion must be confined to the boundary of that body, and we know their average kinetic energy. Consequently, there is no concept of temperature exclusively for jiggling, there is but a unique notion of temperature, which is proportional to the particles' kinetic energy, regardless of their kind of motion!
By the principle of relativity, the laws must be the same in all inertial coodinate systems. Hence, a system of particles (or charges) that doesn't radiate an electromagnetic field at $T = 0$, would never be able to radiate at a temperature $T > 0$. We know that this is wrong. That implies that the electromagnetic, corrective field Φ_T above, cannot not drop to zero as $T \to 0$!

Furthermore, it has been known even before Einstein's theory of special relativity, that the electromagnetic field $A = (A_0, \ldots, A_3)$, which satisfies $\Box A = (j_0, \ldots, j_3)$ with j as the 4-vector of charge $\rho = j_0$ and flux (j_1, j_2, j_3), converges to the Coulomb equation $\Delta A_0 = \rho$ as $T \to 0$: because the flux then converges to zero, and likewise $\Box = (1/c^2)\partial_0^2 - \Delta$ converges to $-\Delta$ as the speed of light c becomes large: $c \to \infty$.

Now, if that electromagnetic field would not be able to capture the entire gravitational field for $T \to 0$ somehow, then the difference of Φ minus the $(T \to 0)$-limit of that electrodynamic field would become a nontrivial field independent from the electromagnetic field. Given, that by the general principle of relativity, gravitation is the effect of a curvature in space-time, gravity must affect the electromagnetic field, because light and charges will be affected by that curvature. We'd therefore get an additional ghostly field to balance out the difference between gravitation and electromagnetism.

Let's see how one might fix the situation:

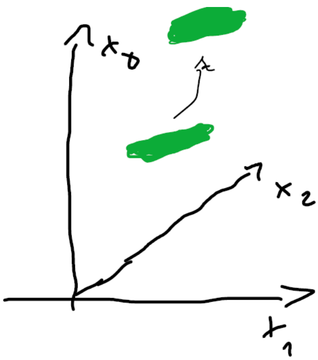
4. Examination

Under non-relative conditions (i.e. small temperature or large velocity of light), let S be an N-particle system of masses at rest, confined to a bounded region in space for all times. By Bernoulli, S is to be defined by the vector function $(t, \vec{x}) \mapsto (\rho_t(\vec{x}), \rho_t(\vec{x})\vec{v}_t(\vec{x}))$ of energy or mass density ρ_t and flux $\vec{j} := \rho_t \vec{v}_t$ for each time t, where $\vec{v}_t$ is the velocity density. Since we assume slow velocity, $t \mapsto \rho_t$ is approximately constant, and it consists of the energy of the rest masses of all N particles, as well as their mutual energy of interaction.

That makes

$$\mathcal{E}_{tot} = +\sqrt{\rho_t^2 c^4 + \rho_t^2 |\vec{v}_t|^2 c^2}$$

a good candidate for the total energy density of the system, from which $\mathcal{E}_{tot} \approx \rho c^2 + (1/2)\rho|\vec{v}|_t^2$ follows for $c \to \infty$. And, with the Poisson equation, $\rho\Delta\Phi_{T=0} = 4\pi G\rho^2$, this suggests, that Φ_T for $T > 0$ is to become a vector composition of four fields $(\Psi_0, \ldots, \Psi_3)$ for each of the four mass and flux components $(\rho = j_0, \ldots, j_3)$, such that $\sum_{0\leq\mu\leq 3} j_\mu \Delta\Psi_\mu = 4\pi j_\mu^2 = \mathcal{E}^2$ with $c \equiv 1$.
To extend these equations equations to relativistics, two measures must be taken: First, relativistic invariance of the fields Ψ_μ mandates Δ to be replaced by $-\Box$. The 2^{nd} is a more subtle one: For the system S to have a positive spatial extension, d, say, there is no unique time t to describe the particle densities as a function of t and $\vec{x}$: each single point in the system has its own eigentime, and these time values can utmost be synchronized up to a time difference $\delta t = d/c$. In a spatially smooth density function, we even loose control over spacelike and timelike regions on the whole. In other words, the localized (energy) density functions will have to become time-curves of packages of space-time densites, where integration has to be done over space-time $\mathbb{R}^4$ as sketched below:



Compare this with Maxwell's equations, which we write, dropping t, as:

$$j_0\Box A_0 + \cdots + j_3\Box A_3 = j_0^2 + \cdots + j_3^2,$$

where again $c \equiv 1$: We know that the charge (space-time-)density j_0 cannot be a scalar function, because generally, a negatively charged particle does not cancel against a positively charged one: both particles and charges will still be preserved: it's only that at large distances, the charges appear as a neutral composite, bare of charges. So, to keep track of these charges, their positive and negative charges should be kept on different components. (This kind of separation is already needed to get get at the notion of a flux $\vec{j}$ in a neutral wire.) Interaction of charges then must be expressed in terms of 2×2-matrix valued functions, Hamilton's quaternion-valued functions, which only later had been replaced by the notion of the 4-vector field A, only to be dug up again in the quantum mechanical concept of spin. And indeed, the algebraically complete square root of $\Box$ is $\sum_\mu \partial_\mu\gamma_\mu$, where the γ_μ are

4×4-matrices, known as Dirac-matrices.
With this, the j_μ and A_μ turn into quadruples, and the products $j_\mu A_\mu$ and j_μ^2 will be replaced by inner products. But the j_μ (along with the A_μ) now are complex quadruples, whereas the right hand side should be proportional to $\mathcal{E}^2$, which is to be always greater or equal to zero.
That condition however can be met by taking the left factors to be the (complex) adjoints $\bar{j}_\mu$ of the energy momentum-densities j_μ. In all, we arrive at:

$$\sum_\mu < j_\mu(x), \Box A_\mu(x) >= -4\pi G \sum_\mu \left| j_\mu(x) \right|^2 (= -\mathcal{E}^2(x)).$$

Did we miss out any (massy) particle on the right hand side?
It seems no: In all, we have 16 components $j_{\mu k}$, $k = 1, \ldots, 4$ and $\mu = 0, \ldots, 3$, and we sum up their absolute squares. That accounts for 4 dimensions, namely the energy-momentum density. Then we have 4 electric charge and flux components, which go into the total square. These four generate the symmetry group $U(2)$, which is isomorphic to the product $SU(2) \times U(1)$ and hence contains the Salam Weinberg group $SU(2)$. There are then 8 dimensions left over, and these can be identified with the 8 generators of the hadron group $SU(3)$. So, we are complete: all known sub-particles of mass greater zero will contribute to $\mathcal{E}^2$.

Now, as the equation above is in the square of the j_μ, it feels natural to integrate the square over space-time $\mathbb{R}^4$ to a square of energy, of which then the (positive) square root would be taken. That is currently hindered by the fact that the j_μ are based to be space-time densities of energy. However this is just a technical aspect: Remember that the $j_\mu \in \mathcal{C}_c^\infty(\mathbb{R}^4)$ are smooth functions of space-time with compact support, and as such they are in $L^p(\mathbb{R}^4)$, i.e. they are p-integrable for every $p = 1, \ldots, \infty$. We are therefore free to base the j_μ on the square root density, instead of density, which means that the $\left| j_\mu^2 \right|$ will now be densities of the square of energy. And this allows it to write the equation as a quadratic form:

$$\begin{aligned} < j, \Box A >:= \sum_\mu \int_{\mathbb{R}^4} \bar{j}_\mu(x) \Box A_\mu(x) d^4x = \\ -4\pi G < j, j >:= -4\pi G \sum_\mu \int_{\mathbb{R}^4} \left| j_\mu(x) \right|^2 d^4x (= -\mathcal{E}^2). \quad (4.1) \end{aligned}$$

Remark 4.1. What this equation says, is that a system of total square energy E^2 includes a mutual interaction through a massless gravitational field of a strength proportional to E^2.

For each μ, the mapping $\Theta_\mu : j_\mu \mapsto A_\mu$ defines a linear mapping from $j_\mu \in \mathcal{C}_c^\infty(\mathbb{R}^4)$ to a functional (see: below), which is defined "outside the support $supp(j_\mu)$ of j_μ":
For $x, y \in \mathbb{R}^4$ let $d(x-y) := (x-y)_\mu (x-x)^\mu \in \mathbb{R}$ be the Minkowksi distance

of x and y, and with $j_\mu \in \mathcal{C}_c^\infty(\mathbb{R}^4)$ and $x \in \mathbb{R}^4$ let

$$p(x, supp(j)) := \min_{0\leq\mu\leq 3} \inf_{y\in supp(j_\mu)} \left|d(x-y)\right| \in [0,\infty),$$

which defines a seminorm on $\mathbb{R}^4$. With it, given $j = (j_0, \dots, j_3)$ as above, let $\Omega(j) := \{x \in \mathbb{R}^4 \mid p(x, supp(j)) > 0\}$, which is open in $\mathbb{R}^4$. Then $\Theta = (\Theta_0, \dots, \Theta_3)$ maps j to a quadrupel of functionals on $\mathcal{C}_c^\infty(\Omega(j))$ (as shown subsequently).

$\Theta = S^2$ can be broken into a square of some operator S (see: below). So, $< j, \Box A >=< j, \Box S^2 j >=< S^* j, \Box S j >= -4\pi G < j, j >$, where Sj is simply the action function of j, that is traveling from its source j through space-time at the speed of light, and $S^* \bar{j}$ is its time inversion!

Let's define the functional spaces above and see what the seemingly undefined term $< j, A >=< j, S^2 j >$ gives in terms of distributions:

Let $K \subset \mathbb{R}^4$ be the (compact) closure of a non-empty, open, and bounded subset $K^o \subset \mathbb{R}^4$, and let $\Omega(K)$ as above be the set of all $x \in \mathbb{R}^4$ with $p(x, K) > 0$, which is an open, non-empty subset of $\mathbb{R}^4$. $\Omega(K)$ itself is the union of a sequence $X_1, X_2, \dots$ of compact regions of $\mathbb{R}^4$, which as K are the closures of nontrivial, open sets $X_l^o \subset \mathbb{R}^4$. Given such a compact region X, the set of all infinitely differentiable (complex-valued) functions with support in X is a vector space $\mathcal{C}_c^\infty(X)$, which becomes a complete locally convex, separable space, when equipping it with the sequence of supremum norms for all its n-th order partial derivatives (where $n \geq 0$ is understood), see e.g. [6]. Then the space $\mathcal{C}_c^\infty(X)^4 = \mathcal{C}_c^\infty(X) \oplus \dots \oplus \mathcal{C}_c^\infty(X)$ of quadruples $(j_1, \dots, j_4)$ is a (separable, complete) locally convex space, and so is its dual, $\mathcal{C}_c'^\infty(X)^4$, the space of continuous linear functionals on $\mathcal{C}_c^\infty(X)^4$ (see again: [6]). This then defines $\mathcal{C}_c'^\infty(\Omega(K))^4$ as the union $\bigcup_{l\in\mathbb{N}} \mathcal{C}_c'^\infty(X_l)^4$, giving it the finest locally convex topology, for which the embeddings $\iota : \mathcal{C}_c'^\infty(X_l)^4 \to \mathcal{C}_c'^\infty(\Omega(K))^4$ are continuous, which is called LF-space (see again: [6, Ch.13]).

Proposition 4.2. *S and S^2 are well-defined as linear mappings on $\mathcal{C}_c^\infty(K)^4$ into $\mathcal{C}_c'^\infty(\Omega(K))^4$, and $< j, Sj >=< j, S^2 j >= 0$ holds for each $j \in \mathcal{C}_c^\infty(K)^4$.*

Proof. Let $\delta : \mathcal{C}(\mathbb{R}^4) \ni f \mapsto f(0) \in \mathbb{C}$ be the Dirac-distribution (in 4 dimensions). Then $\Box f = \delta$ is solved by $f(x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{ix\cdot\xi} \frac{-1}{\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2} d^4\xi$, so for $x \in \Omega(K)$ and $j \in \mathcal{C}_c^\infty(K)^4$,

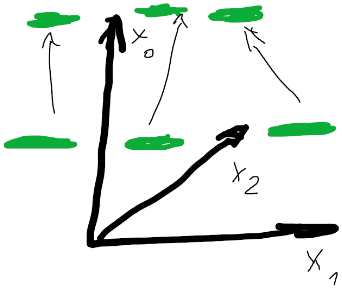
$$S^2 j(x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4\times\mathbb{R}^4} e^{(ix-y)\cdot\xi} \frac{-1}{(x_0-y_0)^2 - \dots - (x_3-y_3)^2} j(y) d^4y d^4\xi$$

is a well-defined complex functional on $\mathcal{C}_c^\infty(\Omega(K))^4$, since for $g \in \mathcal{C}_c^\infty(\Omega(K))^4$ $g(x) \cdot \int f(x-y) j(y) d^4y$ is integrable in x, due to $\inf_{x\in supp(g)} p(x, K) > 0$. And, since j is infinitely differentiable, $S^2 j$ is infinitely differentable on $\Omega(K)$. (Because the 4 components j_k of j satisfy $\int |j_k|\, d^4y \leq Vol(K) \sup_{y\in K} \left|j_k(y)\right|$, S^2 even defines a continuous mapping from $\mathcal{C}_c^\infty(K)^4$ into $\mathcal{C}_c'^\infty(\Omega(K))^4$.)
Along with S^2, all its partial derivatives are well-defined too.
Hence, $S = (\sum_{0\leq\mu\leq 3} \gamma_\mu \partial_\mu) S^2$ is a well-defined mapping from $\mathcal{C}_c^\infty(K)^4$ to

$\mathcal{C}'^{\infty}_c(\Omega(K))^4$.
Lastly, $< j, S^2 j >=< j, Sj >= 0$ follows from the fact that every $j_\mu \in \mathcal{C}^\infty_c(K)$ is equal to zero outside of K, so in particular vanishes on $\Omega(K)$. □

Remark 4.3. Physically, what the proposition tells, is that the field does not interact with its own source.

With it, we are finally able to deal with the gravitational interaction:
Let $j = \sum_{1 \leq k \leq N} j_k$ be the sum of N time-curves of smooth vector functions $t \mapsto j_1(t), \ldots, j_N(t) \in \mathcal{C}^\infty_c(\mathbb{R}^4)^4$ of disjoint support and of compact support at each instance of time as illustrated below:



That's what an external observer would e.g. see, as he looks at our solar system: at each time $t = x_0$, he sees planets and sun as chunks of energy-momentum distributions spatially staying apart of eachother. Dropping t again, equation 4.1 holds for the sum of energy momentum distributions $j = \sum_k j_k$, and as such it includes the interaction between all the N chunks j_k (at retarded times). If instead the N chunks were independently moving from eachother, we would see different distributions of energy-momentum $j_{free,1}, \ldots, j_{free,n}$, each moving in a straight line. What we want is an interaction defining field $V(j_{free,1}, \ldots, j_{free,N})$, which captures that interaction, i.e. such that:

$$< \sum_{1 \leq k \leq N} j_k, \sum_{1 \leq k \leq N} j_k >=< \sum_k j_{free,k}, \sum_k j_{free,k} > + V(j_{free,1}, \ldots, j_{free,N}).$$

This in mind, let's put $j_k = j_{free,k} + iSj_{free,k}$. Then,

$$\begin{aligned} < \sum_k j_k, \sum_k j_k >=< \sum_k j_{free,k}, \sum_k j_{free,k} > + \sum_k < S^* j_{free,k}, Sj_{free,k} > \\ + \sum_{1 \leq k,l \leq N} < S^* j_{free,k}, Sj_{free,l} > \\ =< \sum_k j_{free,k}, \sum_k j_{free,k} > + \sum_k < j_{free,k}, S^2 j_{free,k} > \\ + 2 \sum_{1 \leq k < l \leq N} Re < j_{free,k}, S^2 j_{free,l} > . \end{aligned}$$

By the proposition above, $\sum_k < j_{free,k}, S^2 j_{free,k} >= 0$, so we get that the square E^2 of the total energy of the interacting N-part system equals the square E_{free}^2 of the total energy of the N non-interacting systems plus a sum V of mixed, real-valued terms $Re < j_{free,k}, S^2 j_{free,l} >$, $(k < l)$, so for $\left|E_{free}\right| \gg |V|$:

$$|E| = +\left|E_{free}\right| \sqrt{1 + \frac{V}{E_{free}}} \approx \left|E_{free}\right| + \frac{V}{2\left|E_{free}\right|} =$$
$$= \left|E_{free}\right| + \frac{1}{\left|E_{free}\right|} \sum_{1 \leq k < l \leq N} Re < j_{free,k}, S^2 j_{free,l} > .$$

For $N = 2$ and restricting to real-valued j_1, j_2, we have

$$\frac{1}{\left|E_{free}\right|} Re < j_{free,1}, S^2 j_{free,2} >$$
$$= \frac{1}{\left|E_{free}\right|} \int j_1(x) \cdot (S^2 j_2)(x) d^4x = \int j_1(x) \Phi(x) d^4x$$

with $\Phi(x) := \frac{1}{\left|E_{free}\right|} S^2 j_{free,2}(x)$, which then is the vector field of gravitational interaction, of which the first component converges to the classical gravitational field as $c \to \infty$, and the other components converge to zero. (Doing the same for $N > 2$, would lead to a vector field Φ which depends on N space-time quadruples $x_1, \ldots, x_N \in \mathbb{R}^4$, however.)

In all, it was shown that the gravitational field can be derived as an approximation from the Lorentz-invariant quadratic field equation 4.1, which by its simple, yet rich $U(4)$-symmetry allows to include all gauge fields of the standard model.

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