FORMULATION OF MAXWELL FIELD EQUATIONS FROM A GENERAL SYSTEM OF LINEAR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract: In this work we show that, as in the case of Dirac equation, it is possible to formulate Maxwell field equations of electromagnetism from a general system of linear first order partial differential equations. A prominent feature that emerges from formulating Dirac and Maxwell field equations from a general system of linear first order partial differential equations is that the field equations must be formed in such a way that the functions that represent the system also obey a wave equation. This type of mathematical duality may be related to the wave-particle duality in quantum physics.

In our previous work on Dirac equation, we showed that it is possible to derive Dirac equation from a general system of linear first order partial differential equations [1]. In fact we showed that, depending on the space dimension of a physical system, not only Dirac equation but other systems of first order linear partial differential equations, such as the Cauchy-Riemann equations, could also be derived in the same manner from a general system of linear first order partial differential equations. To extend our discussion, in this work we show that Maxwell field equations of electromagnetism can also be formulated from a general system of linear first order partial differential equations. A general system of linear first order partial differential equation can be written as follows [2]

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{r}\frac{\partial \psi_i}{\partial x_j} = k_1\sum_{l=1}^{n} b_l^r\,\psi_l + k_2 c^r, \qquad r = 1,2,\dots,n \tag{1}$$

The system of equations given in Equation (1) can be rewritten in a matrix form as

$$\left(\sum_{i=1}^{n} A_i\frac{\partial}{\partial x_i}\right)\psi = k_1\sigma\psi + k_2 J \tag{2}$$

where $\psi = (\psi_1,\psi_2,\dots,\psi_n)^T$, $\partial\psi/\partial x_i = (\partial\psi_1/\partial x_i\,,\partial\psi_2/\partial x_i\,,\dots,\partial\psi_n/\partial x_i)^T$, $A_i$, $\sigma$ and $J$ are matrices representing the quantities $a_{ij}^k$, $b_l^r$ and $c^r$, and $k_1$ and $k_2$ are undetermined constants. Now, if we apply the operator $\sum_{i=1}^{n} A_i\frac{\partial}{\partial x_i}$ on the left on both sides of Equation (2) then we have

$$\left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j}\right)\psi = \left(\sum_{i=1}^{n} A_i \frac{\partial}{\partial x_i}\right)(k_1\sigma\psi + k_2 J) \tag{3}$$

If we assume further that the coefficients $a_{ij}^k$ and $b_l^r$ are constants and $A_i\sigma = \sigma A_i$, then Equation (3) can be rewritten in the following form

$$\left(\sum_{i=1}^{n} {A_i}^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n}\sum_{j>i}^{n}\left(A_iA_j + A_jA_i\right)\frac{\partial^2}{\partial x_i \partial x_j}\right)\psi = k_1^2\sigma^2\psi + k_1k_2\sigma J + k_2\sum_{i=1}^{n} A_i \frac{\partial J}{\partial x_i} \tag{4}$$

In order for the above systems of partial differential equations to be used to describe physical phenomena, the matrices $A_i$ must be determined. It is observed that, as in the case of Dirac equation and as shown in this work for the case of Maxwell field equations, the matrices $A_i$ must take a form so that Equation (4) reduces to the following equation

$$\left(\sum_{i=1}^{k} {A_i}^2 \frac{\partial^2}{\partial x_i^2} - \sum_{i=k+1}^{n} {A_i}^2 \frac{\partial^2}{\partial x_i^2}\right)\psi = k_1^2\sigma^2\psi + k_1k_2\sigma J + k_2\sum_{i=1}^{n} A_i \frac{\partial J}{\partial x_i} \tag{5}$$

The simplest way for Equation (4) to be reduced to Equation (5) is to impose the following conditions on the matrices $A_i$

$$A_i^2 = \pm 1 \tag{6}$$

$$A_iA_j + A_jA_i = 0 \quad \text{for} \quad i \neq j \tag{7}$$

For the case of $n = 4$, the matrices $A_i$ in general can be shown to take the form

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{8}$$

The matrices given in Equation (8) are Dirac matrices and the system of linear first order partial differential equations given in Equation (1) reduces to Dirac equation [3]

$$i\frac{\partial\psi_1}{\partial t} + i\frac{\partial\psi_4}{\partial x} + \frac{\partial\psi_4}{\partial y} + i\frac{\partial\psi_3}{\partial z} = m\psi_1 \tag{9}$$

$$i\frac{\partial\psi_2}{\partial t} + i\frac{\partial\psi_3}{\partial x} - \frac{\partial\psi_3}{\partial y} - i\frac{\partial\psi_4}{\partial z} = m\psi_2 \tag{10}$$

$$-i\frac{\partial\psi_3}{\partial t} - i\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_2}{\partial y} - i\frac{\partial\psi_1}{\partial z} = m\psi_3 \tag{11}$$

$$-i\frac{\partial\psi_4}{\partial t}-i\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_1}{\partial y}+i\frac{\partial\psi_2}{\partial z}=m\psi_4 \tag{12}$$

However, it is noted that the conditions given in Equations (6) and (7) are only a particular set of conditions that are satisfied by the matrices $A_i$ in order to reduce Equation (4) to Equation (5). For example, a complicated set of conditions that give rise to the same Equation (5) if we set $\left(\sum_{i=1}^{n}\sum_{j>i}^{n}\left(A_iA_j+A_jA_i\right)\frac{\partial^2}{\partial x_i\partial x_j}\right)\psi=0$ together with $A_i^2=\pm 1$. An even more complicated set of conditions that would also give rise to the same Equation (5) could be devised by setting more complicated relationships between all quantities that form Equation (4). In this work we will show that this is in fact the case for Maxwell field equations of electromagnetism. In the case of Dirac equation, the forms of the matrices $A_i$ and their relationships can be suggested from the relativistic relationship between the energy and momentum of a particle. On the other hand, Maxwell field equations were derived from various physical laws in which the physical quantities have different types of relativistic relationships between them. Fortunately, since we already know Maxwell field equations therefore we simply obtain the matrices $A_i$ from the field equations and derive their relationships. As shown below, without the knowledge of the mathematical formulation of the field equations, it is almost impossible to suggest the forms for the matrices $A_i$ to formulate Maxwell field equations from the system of linear first order partial differential equations given in Equation (1). In classical electrodynamics, Maxwell field equations of the electromagnetic field are written as

$$\mathbf{\nabla}.\mathbf{E}=\frac{\rho_e}{\epsilon} \tag{13}$$

$$\mathbf{\nabla}.\mathbf{B}=0 \tag{14}$$

$$\mathbf{\nabla}\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0 \tag{15}$$

$$\mathbf{\nabla}\times\mathbf{B}-\epsilon\mu\frac{\partial\mathbf{E}}{\partial t}=\mu\mathbf{j}_e \tag{16}$$

where the charge density $\rho_e$ and the current density $\mathbf{j}_e$ satisfy the conservation law

$$\mathbf{\nabla}.\mathbf{j}_e+\frac{\partial\rho_e}{\partial t}=0 \tag{17}$$

In terms of the components $\mathbf{E}=\left(E_x,E_y,E_z\right)$, $\mathbf{B}=\left(B_x,B_y,B_z\right)$ and $\mathbf{j}_e=\left(j_x,j_y,j_z\right)$ Maxwell field equations can be rewritten in an expanded vector form as follows

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=\frac{\rho_e}{\epsilon} \tag{18}$$

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0 \tag{19}$$

$$\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\mathbf{k}+\frac{\partial B_x}{\partial t}\mathbf{i}+\frac{\partial B_y}{\partial t}\mathbf{j}+\frac{\partial B_z}{\partial t}\mathbf{k}=0 \tag{20}$$

$$\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)\mathbf{k}-\epsilon\mu\frac{\partial E_x}{\partial t}\mathbf{i}-\epsilon\mu\frac{\partial E_y}{\partial t}\mathbf{j}-\epsilon\mu\frac{\partial E_z}{\partial t}\mathbf{k}$$
$$=\mu j_x\mathbf{i}+\mu j_y\mathbf{j}+\mu j_z\mathbf{k} \tag{21}$$

It is observed from the system of equations given in Equations (18-21) that there are eight equations that involve only six unknowns therefore Maxwell equations seem to be over-determined. However, it has been shown that physical systems that satisfy Faraday's law and Ampere's law also satisfy Gauss laws therefore in the following we will consider for the case Maxwell equations are given by Equations (20) and (21) only. Actually, as will be shown later, Gauss's laws are additional conditions that are required to reduce the second order partial differential equation given in Equation (4) to the wave equation given in Equation (5). In components, Equations (20) and (21) are written out as a system of linear first order partial differential equations as

$$\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}-\epsilon\mu\frac{\partial E_x}{\partial t}=\mu j_x \tag{22}$$

$$\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}-\epsilon\mu\frac{\partial E_y}{\partial t}=\mu j_y \tag{23}$$

$$\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}-\epsilon\mu\frac{\partial E_z}{\partial t}=\mu j_z \tag{24}$$

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{\partial B_x}{\partial t}=0 \tag{25}$$

$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}+\frac{\partial B_y}{\partial t}=0 \tag{26}$$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{\partial B_z}{\partial t}=0 \tag{27}$$

In the following we will introduce the following new notation

$$\psi=\left(E_x,E_y,E_z,B_x,B_y,B_z\right)^T=(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6\,)^T \tag{28}$$

For simplicity we also set $\epsilon\mu=1$. In terms of the new notation, the system of equations given in Equations (22-27) becomes

$$-\frac{\partial \psi_1}{\partial t}+\frac{\partial \psi_6}{\partial y}-\frac{\partial \psi_5}{\partial z}=\mu j_1 \tag{29}$$

$$-\frac{\partial \psi_2}{\partial t}+\frac{\partial \psi_4}{\partial z}-\frac{\partial \psi_6}{\partial x}=\mu j_2 \tag{30}$$

$$-\frac{\partial \psi_3}{\partial t}+\frac{\partial \psi_5}{\partial x}-\frac{\partial \psi_4}{\partial y}=\mu j_3 \tag{31}$$

$$\frac{\partial \psi_4}{\partial t}+\frac{\partial \psi_3}{\partial y}-\frac{\partial \psi_2}{\partial z}=0 \tag{32}$$

$$\frac{\partial \psi_5}{\partial t}+\frac{\partial \psi_1}{\partial z}-\frac{\partial \psi_3}{\partial x}=0 \tag{33}$$

$$\frac{\partial \psi_6}{\partial t}+\frac{\partial \psi_2}{\partial x}-\frac{\partial \psi_1}{\partial y}=0 \tag{34}$$

The system of equations given in Equations (29-34) can be written the following matrix form

$$\left(A_0\frac{\partial}{\partial t}+A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}\right)\psi=A_4 J \tag{35}$$

where $\psi=(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6)^T,\ J=(j_1,j_2,j_3,0,0,0)^T$ and the matrices $A_i$ are

$$A_0=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\quad A_1=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&-1&0&0&0\\0&1&0&0&0&0\end{pmatrix}\quad A_2=\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&-1&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\-1&0&0&0&0&0\end{pmatrix}$$

$$A_3=\begin{pmatrix}0&0&0&0&-1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&-1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\quad A_4=\begin{pmatrix}\mu&0&0&0&0&0\\0&\mu&0&0&0&0\\0&0&\mu&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix} \tag{36}$$

It is seen that when Maxwell field equations are written in the form given by Equation (35) then, like Dirac equation, they can be seen as a particular case of the more general system of linear first order partial differential equations given in Equation (2) with the matrices $A_i$ are specified in the forms given in Equations (36). However, unlike Dirac matrices, the commutation relations between these matrices are more complicated even though they also seem to follow a particular pattern. From Equation (36) we obtain

$$A_0^2=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\quad A_1^2=\begin{pmatrix}0&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\end{pmatrix}\quad A_2^2=\begin{pmatrix}-1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-1\end{pmatrix}$$

$$A_3^2=\begin{pmatrix}-1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&0\end{pmatrix}\quad A_4^2=\begin{pmatrix}\mu^2&0&0&0&0&0\\0&\mu^2&0&0&0&0\\0&0&\mu^2&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}$$

$$A_1A_2+A_2A_1=\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{pmatrix}\quad A_1A_3+A_3A_1=\begin{pmatrix}0&0&1&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&1&0&0\end{pmatrix}\quad A_2A_3+A_3A_2=\begin{pmatrix}0&0&0&0&0&0\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{pmatrix}$$

$$A_0 A_i + A_i A_0 = 0 \quad \text{for} \quad i = 1, 2, 3 \tag{37}$$

Now, if we apply the differential operator $(A_0\, \partial/\partial t + A_1\, \partial/\partial x + A_2\, \partial/\partial y + A_3\, \partial/\partial z)$ to Equation (35) then we arrive at

$$\left(\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}\frac{\partial^2}{\partial t^2} + \begin{pmatrix} 0&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&-1 \end{pmatrix}\frac{\partial^2}{\partial x^2} + \begin{pmatrix} -1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&-1 \end{pmatrix}\frac{\partial^2}{\partial y^2} \right.$$
$$+ \begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&0 \end{pmatrix}\frac{\partial^2}{\partial z^2} + \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0 \end{pmatrix}\frac{\partial^2}{\partial x \partial y} + \begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}\frac{\partial^2}{\partial x \partial z}$$
$$\left. + \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{pmatrix}\frac{\partial^2}{\partial y \partial z}\right)\psi = -\left(\begin{pmatrix} \mu&0&0&0&0&0\\ 0&\mu&0&0&0&0\\ 0&0&\mu&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}\frac{\partial}{\partial t}\right)J \tag{38}$$

From the equation given in Equation (38), we obtain the following equations for the electric field $\mathbf{E} = (E_x, E_y, E_z) = (\psi_1, \psi_2, \psi_3)$

$$\frac{\partial^2 \psi_1}{\partial t^2} - \frac{\partial^2 \psi_1}{\partial y^2} - \frac{\partial^2 \psi_1}{\partial z^2} + \frac{\partial}{\partial x}\left(\frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right) = -\mu \frac{\partial j_1}{\partial t} \tag{39}$$

$$\frac{\partial^2 \psi_2}{\partial t^2} - \frac{\partial^2 \psi_2}{\partial x^2} - \frac{\partial^2 \psi_2}{\partial z^2} + \frac{\partial}{\partial y}\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_3}{\partial z}\right) = -\mu \frac{\partial j_2}{\partial t} \tag{40}$$

$$\frac{\partial^2 \psi_3}{\partial t^2} - \frac{\partial^2 \psi_3}{\partial x^2} - \frac{\partial^2 \psi_3}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y}\right) = -\mu \frac{\partial j_3}{\partial t} \tag{41}$$

Using Gauss's law for the electric field

$$\boldsymbol{\nabla}.\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z} = \frac{\rho_e}{\epsilon} \tag{42}$$

we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{\rho_e}{\epsilon} - \frac{\partial \psi_1}{\partial x}\right) = -\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\rho_e}{\epsilon}\right) \tag{43}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_3}{\partial z}\right) = \frac{\partial}{\partial y}\left(\frac{\rho_e}{\epsilon} - \frac{\partial \psi_2}{\partial y}\right) = -\frac{\partial^2 \psi_2}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\rho_e}{\epsilon}\right) \tag{44}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y}\right) = \frac{\partial}{\partial z}\left(\frac{\rho_e}{\epsilon} - \frac{\partial \psi_3}{\partial z}\right) = -\frac{\partial^2 \psi_3}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{\rho_e}{\epsilon}\right) \tag{45}$$

In vector form, the wave equation for the electric field can be written as

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = \nabla\left(\frac{\rho_e}{\epsilon}\right) - \mu \frac{\partial \mathbf{J}_e}{\partial t} \tag{46}$$

where $\mathbf{J}_e = (j_1, j_2, j_3)$. Similarly for the magnetic field $\mathbf{B} = (B_x, B_y, B_z) = (\psi_4, \psi_5, \psi_6)$ we obtain the following equations

$$\frac{\partial^2 \psi_4}{\partial t^2} - \frac{\partial^2 \psi_4}{\partial y^2} - \frac{\partial^2 \psi_4}{\partial z^2} + \frac{\partial}{\partial x}\left(\frac{\partial \psi_5}{\partial y} + \frac{\partial \psi_6}{\partial z}\right) = 0 \tag{47}$$

$$\frac{\partial^2 \psi_5}{\partial t^2} - \frac{\partial^2 \psi_5}{\partial x^2} - \frac{\partial^2 \psi_5}{\partial z^2} + \frac{\partial}{\partial y}\left(\frac{\partial \psi_4}{\partial x} + \frac{\partial \psi_6}{\partial z}\right) = 0 \tag{48}$$

$$\frac{\partial^2 \psi_6}{\partial t^2} - \frac{\partial^2 \psi_6}{\partial x^2} - \frac{\partial^2 \psi_6}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{\partial \psi_4}{\partial x} + \frac{\partial \psi_5}{\partial y}\right) = 0 \tag{49}$$

$$\boldsymbol{\nabla}.\mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \frac{\partial \psi_4}{\partial x} + \frac{\partial \psi_5}{\partial y} + \frac{\partial \psi_6}{\partial z} = 0 \tag{50}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi_5}{\partial y} + \frac{\partial \psi_6}{\partial z}\right) = -\frac{\partial^2 \psi_4}{\partial x^2} \tag{51}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \psi_4}{\partial x} + \frac{\partial \psi_6}{\partial z}\right) = -\frac{\partial^2 \psi_5}{\partial y^2} \tag{52}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial \psi_4}{\partial x} + \frac{\partial \psi_5}{\partial y}\right) = -\frac{\partial^2 \psi_6}{\partial z^2} \tag{53}$$

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0 \tag{54}$$

An interesting question that arises from the above considerations is whether the electric field and the magnetic field, if in general they obey the same system of linear first order partial differential equations given in Equation (1), can have other relationships so that they also give rise to the wave equations given in Equations (46) and (54). In this case the simplest forms the matrices $A_i$ would take are the same as Dirac matrices but in six dimensions

$${A_0}^2 = 1 \tag{55}$$

$${A_i}^2 = -1 \quad \text{for} \quad i = 1, 2, 3 \tag{56}$$

$$A_i A_j + A_j A_i = 0 \quad \text{for} \quad i = 0, 1, 2, 3 \tag{57}$$

Therefore we need to look for new six by six matrices so that the wave equations given in Equations (46) and (54) can be obtained. If we define the operators $A_i$ as follows

$$A_0 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \end{pmatrix} \tag{58}$$

then it can be verified that the matrices given in Equations (58) satisfy the following relations

$$A_1A_2 + A_2A_1 = 0 \tag{59}$$

$$A_1A_3 + A_3A_1 = 0 \tag{60}$$

$$A_2A_3 + A_3A_2 = 0 \tag{61}$$

$$A_0A_1 + A_1A_0 = 0 \tag{62}$$

$$A_0A_2 + A_2A_0 = 0 \tag{63}$$

$$A_0A_3 + A_3A_0 \neq 0 \tag{64}$$

Even though the set of matrices given in Equation (58) satisfy the conditions imposed by the conditions given in Equations (55-56), due to the commutation relation given by Equation (64), they do not completely satisfy the conditions imposed by Equation (57). Now, if we apply the differential operator $(A_0\,\partial/\partial t + A_1\,\partial/\partial x + A_2\,\partial/\partial y + A_3\,\partial/\partial z)$ to Equation (35) then we obtain

$$\left( \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial t^2} - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial x^2} - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial y^2} - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial z^2} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2i & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \frac{\partial^2}{\partial t \partial z} \right) \psi = 0 \tag{65}$$

where we have assumed $A_4 = 0$ for simplicity. In this case a wave equation for the electric field and magnetic field is obtained if the unified wavefunction of the electromagnetic field to take the form $\psi = (\psi_1, \psi_2, 0, 0, \psi_5, \psi_6)^T$. This is an electromagnetic field that is polarised so that the electric field is in the x-y plane and the magnetic field is in the y-z plane. Finally, it is also worth mentioning here that if the field equations are time-independent then all components of the wavefunction $\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6)^T$ satisfy Laplace equation

$$\left( \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial x^2} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial y^2} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial z^2} \right) \psi = 0 \tag{65}$$

However, since solutions to Laplace equation in a three-dimensional space $\nabla^2\psi_\mu = 0$ can be written in the form

$$\psi_\mu(x,y,z) = \frac{k_\mu}{\sqrt{x^2+y^2+z^2}}, \quad \mu = 1,2,3,4,5,6 \tag{66}$$

where $k_\mu$ are undetermined constants, therefore it could be suggested that the Coulomb electrostatic field is a static solution to Maxwell field equations. The field in this case has a dual character, which can manifest itself either as an electric field or a potential and this duality may be related to the so-called Aharonov-Bohm effect [5].

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