Dynamics of boundary layer separating from surface to the outer flow

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Solving the Navier-Stokes equations for 3D boundary layer of the incompressible flow of Newtonian fluids is the most unsolved problem in fluid mechanics. A lot of authors have been executing their researches to obtain the analytical and semi-analytic solutions for the boundary layer approximation of Navier-Stokes equations, even for example for 2D case of *compressible* gas flow. But there is an essential deficiency of 3D solutions for the boundary layer indeed.

In current research, an elegant ansatz is developed to obtain 3D solutions for the boundary layer approximation of Navier-Stokes equations of incompressible fluids (in the vicinity of the point of separating of boundary layer from surface to the outer ideal flow). The governing equation for such the process is proved to be the Poisson equation for each the component of velocity field of boundary layer flow, which could nevertheless be reduced to the Laplace equation in case of the uniform outer flow.

Keywords: Navier-Stokes equations, non-stationary flow, boundary layer.

1. Introduction, the system of equations.

Navier-Stokes equations for the boundary layer of incompressible flow of Newtonian fluids [1], along with the equations for the outer ideal flow with respect to such the boundary layer, could be presented in the Cartesian coordinates as below [2-6] (*under the proper initial conditions, including no-slip condition at the wall*):

$$a)\ \nabla \cdot \vec{u} \ = \ 0\,, \quad b)\ \nabla \cdot \vec{v} \ = \ 0\,, \tag{1.1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{\nabla p}{\rho} + \nu \cdot \nabla^2 \vec{u} + \vec{F}\,, \tag{1.2}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla (p)}{\rho} + \vec{F}\,, \tag{1.3}$$

- where

ρ is the fluid density;

p is the pressure (which should be the same for both the boundary layer, and the outer ideal flow limiting the influence of the boundary layer),

u is the flow velocity of the boundary layer, a vector field;

v is the velocity of the outer flow (outer with respect to the boundary layer), outer flow is supposed to be presented by the ideal flow of Newtonian fluids, a vector field;

ν is the kinematic viscosity,

and ***F*** = -∇ ϕ represents external force (*per unit of mass in a volume*) acting on the fluid, with potential ϕ (which should be the same for both the boundary layer, and the outer ideal flow limiting the influence of the boundary layer).

As for the boundary conditions for the outer flow, we will consider only the Cauchy problem in the whole space (the domain in which the flow occurs).

Using the identity $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = (1/2)\nabla(\boldsymbol{u}^2) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})$ for (1.2) along with identity $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = (1/2)\nabla(\boldsymbol{v}^2) - \boldsymbol{v}\times(\nabla\times\boldsymbol{v})$ for (1.3), we could present equations (1.2)-(1.3) as below [7-8]:

$$\frac{\partial\vec{u}}{\partial t} = \vec{u}\times(\nabla\times\vec{u}) + \left(\nu\cdot\nabla^2\vec{u} - \frac{1}{2}\nabla(\vec{u}^2) - \frac{\nabla p}{\rho} - \nabla\phi\right), \qquad (1.4)$$

$$\frac{\partial\vec{v}}{\partial t} = \vec{v}\times(\nabla\times\vec{v}) - \left(\frac{1}{2}\nabla(\vec{v}^2) + \frac{\nabla p}{\rho} + \nabla\phi\right) \qquad (1.5)$$

Let us consider the point of separating of the boundary layer from the surface (to the outer ideal flow), see Fig.1.

Let us recall that boundary layer is known to be separating from surface (to the outer flow) only at the moment when kinetic energy of the boundary layer is significantly decreased up to the negligible magnitude of the square of velocity field ($\boldsymbol{u}^2$) [1].

It means that solutions of the boundary layer equations (1.4) could be transformed and mutually combined with the outer flow (1.5) if only the condition below is valid for such a solutions, in the vicinity of the point of separating of boundary layer from surface (to the outer ideal flow):

$$\Delta\vec{u} + \left(\frac{1}{2\nu}\right)\nabla(\vec{v}^2) = 0 \qquad (1.6)$$

- where $\Delta\ \boldsymbol{u} = \nabla^2\boldsymbol{u}$ is the designation of vector Laplacian in Cartesian coordinate system.

2. The spatial part of solution at the point of separating of the boundary layer.

Equation (1.6) is known to be the Poisson equation; in case of the uniform outer flow ***v*** (∇ (***v***^{2}) = ***0***), it is reduced to the Laplace equation for each component of flow velocity ***u*** at the point of separating of the boundary layer from surface (to the outer flow):

$$\Delta\vec{u} \; = \; 0 \tag{2.1}$$

- which is proved to have the proper exact solution as below ($r \neq 0$, δ = const):

$$\vec{u}(r,t) \; = \; -\frac{\delta}{4\pi}\frac{\vec{u}(t)}{r}, \quad r = \sqrt{x^2 + y^2 + z^2} \tag{2.2}$$

Let us present the plot of solution (2.2) of Eq. (2.1) at Fig.2 (for the chosen meanings of time-parameter $t = t_0$ and $z = z_0$), in the vicinity of the point of separating of the boundary layer from surface (to the outer ideal flow):

Under the appropriate dynamical influence from the outer flow, such the emerging bubble of the boundary layer (at the point of separating of the boundary layer) should be then distorted according to the direction of the velocity of the outer flow ***v*** (Figs. 5-7).
So, 2D-profile of the emerging bubble of the boundary layer should be presented as at Fig.1: it is well-known fact [1] that velocity field changes its direction in the vicinity of the point where boundary layer is separating from the surface to the outer ideal flow.

3. Discussion.

Navier-Stokes equations for the boundary layer of incompressible flow of Newtonian fluids [1] are known still to be the most desired for solving but unsolved problem in fluid mechanics.

The main motivation of current research is the understanding the dynamics of the boundary layer separating from the surface (to the outer flow). The ultimate condition for this proccess is that the boundary layer solutions (1.4) should be transformed to the flows of ideal fluid and then it should be mutually combined with the solutions for the outer ideal flow (1.5). It let us obtain equation (1.6) for the spatial part of solution which corresponds to the case of boundary layer separating from the surface.

We should especially note that the main assumption which is afterwards successfully converted to the key governing equation (1.6) is that the kinetic energy of the boundary layer should be significantly decreasing up to the negligible magnitude at this point [1]. Indeed, there is a lot of examples of such the solutions in fluid mechanics (see bubbles of separating of the boundary layer at Figs.3-4).

Having been dynamically loaded from the outer flow, the aforementioned bubbles of the boundary layer (at the point of separating of the boundary layer) should be then distorted according to the direction of the velocity of the outer flow $\boldsymbol{v}$ (see Figs.5-7).

Conclusion.

Solving the Navier-Stokes equations for 3D boundary layer of the incompressible flow of Newtonian fluids is the most unsolved problem in fluid mechanics. A lot of authors have been executing their researches to obtain the analytical and semi-analytic solutions for the boundary layer approximation of Navier-Stokes equations [12-13], even for 2D case of *compressible* gas flow [14-15]. But there is an essential deficiency of 3D solutions for the boundary layer indeed.

According to previously suggested ansatz [10], we have developed such the elegant approach in the current research to obtain 3D solutions for the boundary layer approximation of Navier-Stokes equations of incompressible flow of Newtonian fluids, in the vicinity of the point of separating of the boundary layer from surface to the outer ideal flow.

The uniqueness of the presented solutions is not considered. In this respect we confine ourselves to mention the paper [16], in which all the difficulties concerning the uniqueness in unbounded domain are remarked.

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References:

[1]. Theodore von Kármán (1963), *Aerodynamics*. McGraw-Hill.

[2]. Batchelor G.K. (1999), *An introduction to fluid dynamics*, Cambridge Mathematical Library, Cambridge University Press, Cambridge.

[3]. Constantin P. and Foias C. (1988), *Navier–Stokes equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL.

[4]. Fefferman C.L. (2000), *Existence and smoothness of the Navier–Stokes equations*, Clay Mathematics Institute Publications.

[5]. Leray J. (1934), *Sur le mouvement d'un fluide visqueux emplissant l'espace*, Acta Math.63 (1934), pp.193–248.

[6]. R. Temam (1997), *Infinite-dimensional dynamical systems in mechanics and physics*, Applied Mathematical Sciences, vol. 68, Springer-Verlag, New York.

[7]. Ershkov S.V. (2015). *On Existence of General Solution of the Navier-Stokes Equations for 3D Non-Stationary Incompressible Flow*, International Journal of Fluid Mechanics Research, 06/2015, 42(3), pp. 206-213. Begell House.

[8]. Ershkov S.V. (2016). *A procedure for the construction of non-stationary Riccati-type flows for incompressible 3D Navier-Stokes equations,* Rendiconti del Circolo Matematico di Palermo (2016), Volume 65, Issue 1, pp.73-85

[9]. Kamke E. (1971), *Hand-book for Ordinary Differential Eq*. Moscow: Science.

[10]. Ershkov S.V. (2015). *Non-stationary Riccati-type flows for incompressible 3D Navier-Stokes equations*, Computers and Mathematics with Applications, vol. 71, no. 7, pp. 1392–1404.

[11]. Polyanin, Andrei D. (2002), *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Boca Raton (FL): Chapman & Hall/CRC Press. See also: https://en.wikipedia.org/wiki/Poisson%27s_equation

[12]. Landau, L.D.; Lifshitz, E.M. (1987), *Fluid mechanics*, *Course of Theoretical Physics 6* (2nd revised ed.), Pergamon Press, ISBN 0-08-033932-8.

[13]. Drazin, P.G. and Riley N. (2006), *The Navier-Stokes Equations: A Classification of Flows and Exact Solutions*, Cambridge, Cambridge University Press.

[14]. Schlichting H. (1979). *Boundary-Layer Theory* (7 ed.). New York (USA): McGraw-Hill.

[15]. Blasius H. (1908). "*Grenzschichten in Flüssigkeiten mit kleiner Reibung*". Z. Math. Phys. 56: 1–37. (English translation).

[16]. Galdi G.P., Rionero S. (1979), *The weight function approach to uniqueness of viscous flows in unbounded domains*, Archive for Rational Mechanics and Analysis, Volume 69, Issue 1, pp 37-52.

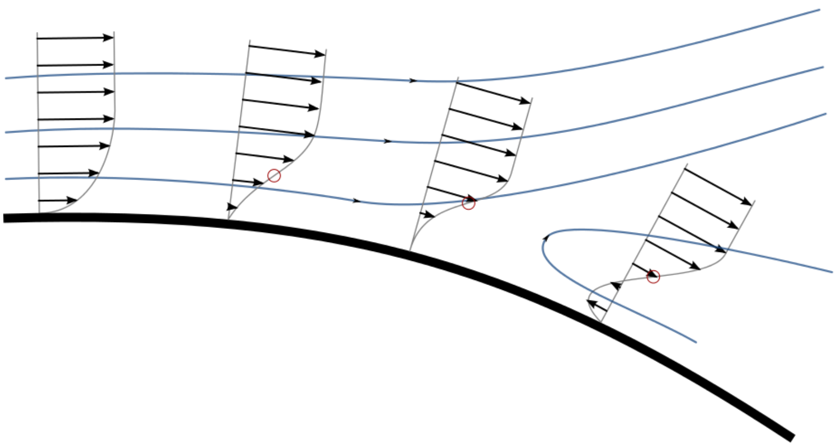


Fig.1. Separation proccess in the boundary layer (schematically imagined).

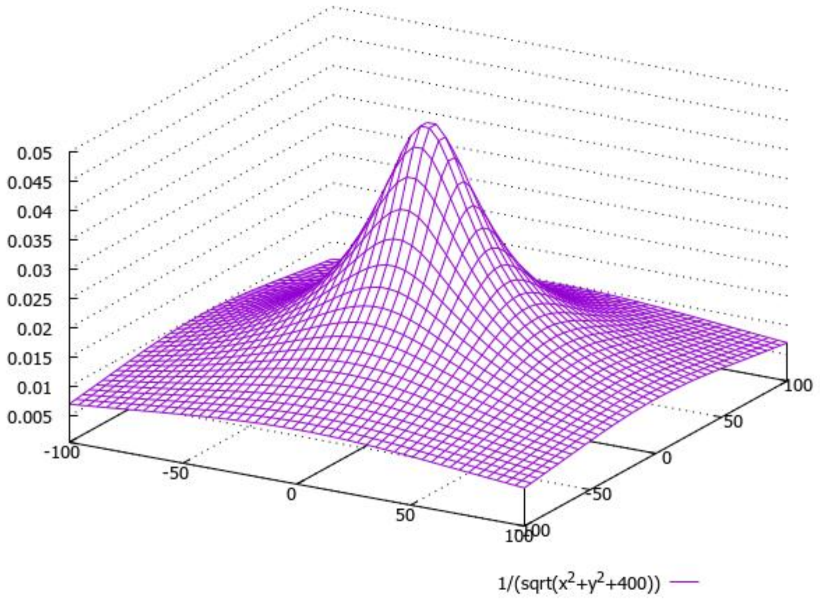


Fig. 2. Solution (2.2) (z = 20), at the point of separating of boundary layer from surface.

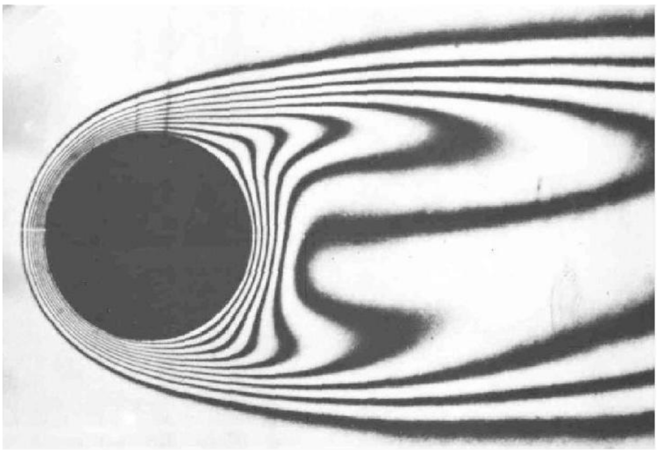


Fig.3. Separating of bubbles in the boundary layer (see 2 points on the sphere).

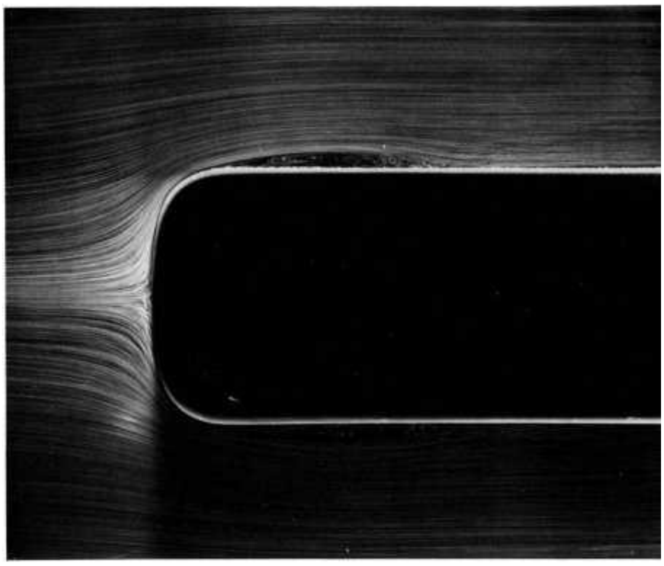
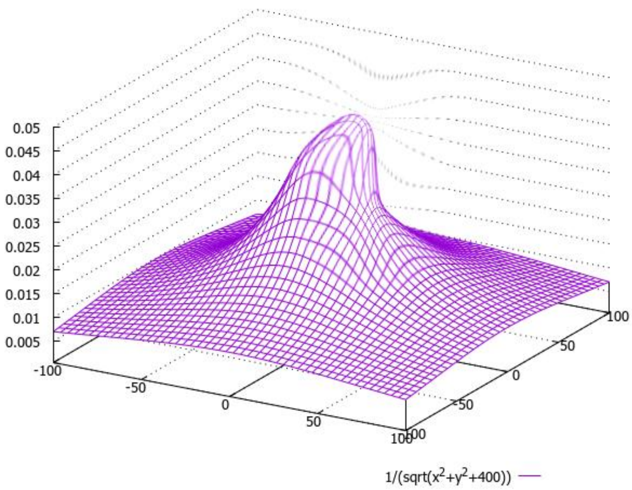
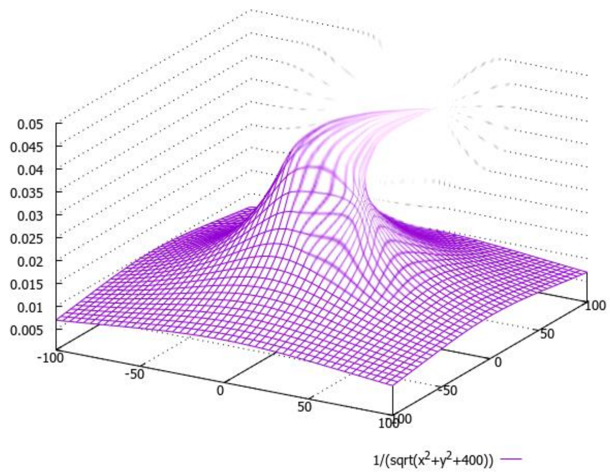


Fig.4. Separating of the bubbles in the boundary layer
(at the top of the cylinder with the rounded edges).



Figs. 5. Solution (2.2) (z = 20), under the appropriate dynamical influence
from the outer flow $\boldsymbol{v}$ (at the point of separating of boundary layer).



Figs. 6. Solution (2.2) (z = 20), under the appropriate dynamical influence from the outer flow $\boldsymbol{v}$ (at the point of separating of boundary layer).

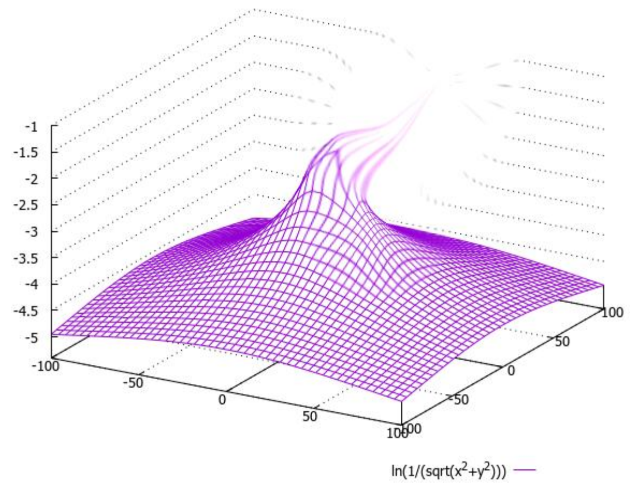


Fig. 7. Solution (2.2) (z = 20), under the appropriate dynamical influence from the outer flow $\boldsymbol{v}$ (at the point of separating of boundary layer).