ON DIRAC NEGATIVE MASS AND MAGNETIC MONOPOLE

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Abstract: In this work first we discuss the possibility of existence of Dirac negative mass and magnetic monopole by formulating Dirac and Maxwell equations from a symmetrical system of linear first order partial differential equations. Then, by establishing a complete symmetry between space and time, in particular their dual dynamics, we show that the existence of negative mass is the result of dynamical symmetry between space and time, and magnetic monopole is a manifestation of the temporal topological structure of an elementary particle classified by the homotopy group of closed surfaces. We also show that the quantum relationship between the electric charge and the magnetic charge obtained by Dirac can be derived by imposing a topological relationship between the Gaussian curvatures of the temporal and spatial manifolds.

In 1928, Dirac developed a relativistic wave equation to describe the quantum dynamics of an elementary charged particle. The equation, however, not only can be used to describe a spin half particle, such as an electron, but also predicts the existence of an elementary charged particle with negative mass. When the positron was discovered, it was then assumed that a negative charged electron that has a negative mass manifests as a positive charged particle that has a positive mass of equal magnitude [1]. Equivalently, this assumption states that a single Dirac wave equation can be used to describe the dynamics of two different elementary particles whose signs of mass and charge are exchanged. Or Dirac equation in fact describes the dynamics of a single quantum matter field that simply has two different components one of which has a positive mass and one negative mass, similar to the case of Maxwell field equations of electromagnetism that describe the dynamics of both the electric field with an electric charge q_e and the magnetic field with a vanishing magnetic charge $q_m = 0$. Interestingly, the missing magnetic charge in Maxwell field equations led Dirac, in 1931, to develop a different theory that shows that the symmetry between electricity and magnetism implies the existence of a fundamental magnetic monopole q_m that is connected to the fundamental charge q_e by the relation $\hbar c/q_e q_m = 2$ [2]. As stated in his works, Dirac uncovered unknown physical entities mainly from his belief that in order to advance in scientific investigation we would need to modify and generalise the axiomatic foundations at the base of mathematics rather than simply develop logically from an established scheme. However, besides the epistemological problems that arise from the paradigm shift in the axiomatic foundations of mathematics, the more important facts that we are facing in physics are related to the question of why the predicted physical entities of negative mass and magnetic charge have never been observed. In this work we will discuss this fundamental question and show that the unobservability of negative mass and magnetic charge may be due to the fact that both physical entities are in fact associated with the temporal dynamics of elementary particles [3]. We have shown in our previous works that both Dirac equation and Maxwell field equations, which should contain in them negative mass and magnetic charge in order to be more symmetrical, can be derived from a general system of linear first order partial differential equations [4,5]. An explicit form of the system can be written as follows [6,7]

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{r}\frac{\partial\psi_i}{\partial x_j} = k_1\sum_{l=1}^{n} b_l^r\,\psi_l + k_2 c^r, \qquad r = 1,2,\dots,n \tag{1}$$

The system of equations given in Equation (1) can be rewritten in a matrix form as

$$\left(\sum_{i=1}^{n} A_i\frac{\partial}{\partial x_i}\right)\psi = k_1\sigma\psi + k_2 J \tag{2}$$

where $\psi = (\psi_1,\psi_2,\dots,\psi_n)^T$, $\partial\psi/\partial x_i = (\partial\psi_1/\partial x_i\,,\partial\psi_2/\partial x_i\,,\dots,\partial\psi_n/\partial x_i)^T$, A_i, σ and J are matrices representing the quantities a_{ij}^k, b_l^r and c^r, and k_1 and k_2 are undetermined constants. Now, if we apply the operator $\sum_{i=1}^{n} A_i\frac{\partial}{\partial x_i}$ on the left on both sides of Equation (2) then we obtain

$$\left(\sum_{i=1}^{n} A_i\frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{n} A_j\frac{\partial}{\partial x_j}\right)\psi = \left(\sum_{i=1}^{n} A_i\frac{\partial}{\partial x_i}\right)(k_1\sigma\psi + k_2 J) \tag{3}$$

If we assume further that the coefficients a_{ij}^k and b_l^r are constants and $A_i\sigma = \sigma A_i$, then Equation (3) can be rewritten in the following form

$$\left(\sum_{i=1}^{n} {A_i}^2\frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n}\sum_{j>i}^{n}\left(A_iA_j + A_jA_i\right)\frac{\partial^2}{\partial x_i\partial x_j}\right)\psi = k_1^2\sigma^2\psi + k_1k_2\sigma J + k_2\sum_{i=1}^{n} A_i\frac{\partial J}{\partial x_i} \tag{4}$$

In order for the above systems of partial differential equations to be used to describe physical phenomena, the matrices A_i must be determined. We have shown that, as in the case of Dirac and Maxwell field equations, the matrices A_i must take a form so that Equation (4) reduces to the following equation

$$\left(\sum_{i=1}^{n} {A_i}^2\frac{\partial^2}{\partial x_i^2}\right)\psi = k_1^2\sigma^2\psi + k_1k_2\sigma J + k_2\sum_{i=1}^{n} A_i\frac{\partial J}{\partial x_i} \tag{5}$$

For the classical electromagnetic field, with the notation $\psi = \left(E_x,E_y,E_z,B_x,B_y,B_z\right)^T = (\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6\,)^T$, and $\epsilon\mu = 1$, the most symmetric form of Maxwell field equations of the electromagnetic field that are derived from Faraday's law and Ampere's law can be written as

$$\frac{\partial \psi_1}{\partial t} + \mu j_1 = \frac{\partial \psi_6}{\partial y} - \frac{\partial \psi_5}{\partial z} \tag{6}$$

$$\frac{\partial \psi_2}{\partial t} + \mu j_2 = \frac{\partial \psi_4}{\partial z} - \frac{\partial \psi_6}{\partial x} \tag{7}$$

$$\frac{\partial \psi_3}{\partial t} + \mu j_3 = \frac{\partial \psi_5}{\partial x} - \frac{\partial \psi_4}{\partial y} \tag{8}$$

$$\frac{\partial \psi_4}{\partial t} + j_4 = \frac{\partial \psi_2}{\partial z} - \frac{\partial \psi_3}{\partial y} \tag{9}$$

$$\frac{\partial \psi_5}{\partial t} + j_5 = \frac{\partial \psi_3}{\partial x} - \frac{\partial \psi_1}{\partial z} \tag{10}$$

$$\frac{\partial \psi_6}{\partial t} + j_6 = \frac{\partial \psi_1}{\partial y} - \frac{\partial \psi_2}{\partial x} \tag{12}$$

where $J = (j_1, j_2, j_3, j_4, j_5, j_6)^T$ is the electromagnetic current in which the electric current is $j_e = (j_1, j_2, j_3)$ and the magnetic current is $j_m = (j_4, j_5, j_6)$. The system of equations given in Equations (6-12) can be written the following matrix form

$$\left(A_1 \frac{\partial}{\partial t} + A_2 \frac{\partial}{\partial x} + A_3 \frac{\partial}{\partial y} + A_4 \frac{\partial}{\partial z}\right) \psi = A_5 J \tag{13}$$

with the matrices A_i are given as

$$A_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad A_5 = \begin{pmatrix} \mu & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

Furthermore, if an additional condition that imposes on the function ψ that requires that it also satisfies the wave equation given by Equation (5) then Gauss's laws will be recovered. On the other hand, Dirac equation can be derived from Equation (4) by simply imposing the following conditions on the matrices A_i

$$A_i^2 = \pm 1 \tag{15}$$

$$A_i A_j + A_j A_i = 0 \quad \text{for} \quad i \neq j \tag{16}$$

For the case of $n = 4$, the matrices A_i can be shown to take the form

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{17}$$

With $k_1 = m$, $\sigma = 1$ and $k_2 = 0$, the system of linear first order partial differential equations given in Equation (2) reduces to Dirac equation

$$-\frac{\partial \psi_1}{\partial t} - im\psi_1 = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_4 + \frac{\partial \psi_3}{\partial z} \tag{18}$$

$$-\frac{\partial \psi_2}{\partial t} - im\psi_2 = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_3 - \frac{\partial \psi_4}{\partial z} \tag{19}$$

$$\frac{\partial \psi_3}{\partial t} - im\psi_3 = \left(-\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_2 - \frac{\partial \psi_1}{\partial z} \tag{20}$$

$$\frac{\partial \psi_4}{\partial t} - im\psi_4 = \left(-\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_1 + \frac{\partial \psi_2}{\partial z} \tag{21}$$

Except for the fact that Dirac equation is expressed in complex mathematics and Maxwell field equations are real, the two formulations look remarkably similar. With the form of the field equations given in Equations (18-21), we may interpret that the change of the field (ψ_1, ψ_2) with respect to time generates the field (ψ_3, ψ_4), similar to the case of Maxwell field equations given in Equations (6-8), the change of the electric field generates the magnetic field. With this observation it may be suggested that, like the Maxwell electromagnetic field which is composed of two essentially different physical fields, the Dirac field of massive particles may also be viewed as being composed of two different physical fields, namely the field (ψ_1, ψ_2), which plays the role of the electric field in Maxwell field equations, and the field (ψ_1, ψ_2), which plays the role of the magnetic field. Even though Dirac equation and Maxwell field equations when derived from a general system of linear first order partial differential equations predict the existence of negative mass and magnetic monopole, with the assumption of a complete symmetry with respect to the physical entries of the system, in addition to the prediction of their existence from current formulations of physical theories such as grand unified theory and superstring theory [8,9,10], they have never been observed. In the following we will try to address this observational problem by showing that these physical entities may belong to a temporal dynamics rather than the conventional spatial dynamics that has been adopted in the current formulations of physical theories.

As shown in our previous work on temporal dynamics that we can generalise to formulate a 3-dimensional temporal dynamics that involves the second rate of change of time with respect to distance [3]. Mathematically, space-time can be assumed to be a six-dimensional metrical continuum, which is a union of a 3-dimensional spatial manifold and a 3-dimensional temporal manifold. The spatial manifold is a simply connected Euclidean space R^3 and the temporal manifold is also a simply connected Euclidean manifold R^3. The points of this space-time are expressed as $(x^1, x^2, x^3, x^4, x^5, x^6)$, where (x^4, x^5, x^6) representing (t^1, t^2, t^3), and the square of the infinitesimal space-time length is of a quadratic form $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. In this work, however, we will consider space-time as two separate Euclidean manifolds which are connected dynamically. In this case, the quadratic forms for the infinitesimal spatial arc length and the temporal arc length are reduced respectively to the forms $ds^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2$ and $d\tau^2 = (dt_1)^2 + (dt_2)^2 + (dt_3)^2$. In Newtonian physics, the dynamics of a particle is a description of the rate of change of its position in space with respect to time according to Newton's laws of motion, where time is assumed to flow at a constant rate and is considered to be a 1-dimensional continuum. We can generalise this formulation by considering the dynamics of a particle as a description of the mutual rates of change of the position and the time of a particle with respect to one another, where not only space but time is also considered to be a 3-dimensional manifold. This generalisation will yield new insights that can be used to explain physical phenomena. Consider a particle of inertial mass m that occupies a position in space. In a coordinate system S, the position of the particle at the time τ is determined by the position vector $\mathbf{r}(\tau) = x_1(\tau)\mathbf{i} + x_2(\tau)\mathbf{j} + x_3(\tau)\mathbf{k}$. We have assumed the Newtonian time is the temporal arc length τ. As in classical physics, the classical dynamics of the particle is governed by Newton's laws of motion. We will term Newton's laws as spatial laws. The spatial second law is

$$m\frac{d^2\mathbf{r}}{d\tau^2} = \mathbf{F} \tag{22}$$

These spatial laws determine the dynamics of a particle in space with the assumption that time is 1-dimensional, universal and flowing at a constant rate. Similar to the case of 1-dimensional time, we can establish a dynamics for a 3-dimensional temporal manifold by considering space as an independent variable. However, due to the symmetry between space and time we may use the following argument to formulate. As in classical dynamics, in order for a particle to change its position it needs a flow of time. So, similarly, we assume that in order for the particle to change its time it would need an expansion of space. We consider the motion of a particle in space as its local spatial expansion. This assumption then allows us to define the rate of change of time with respect to space. From this mutual symmetry between space and time, a temporal dynamics, which is identical to Newtonian dynamics, can be assumed. Consider a particle of a temporal mass D that occupies a time in the 3-dimensional temporal manifold. In the coordinate system S, the time of the particle at the position specified by the spatial vector $\mathbf{r}$ is determined by the temporal vector $\mathbf{t}(s) = t_1(s)\mathbf{i} + t_2(s)\mathbf{j} + t_3(s)\mathbf{k}$, where s is the spatial arc length in the 3-dimensional spatial manifold and $ds^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2$. We assume the temporal dynamics of the particle is governed by dynamical laws which are similar to Newton's laws of motion in space. In the following we will term these laws as temporal laws. The temporal second law is

$$D\frac{d^2\mathbf{t}}{ds^2} = \mathbf{F} \tag{23}$$

With the view that time is a 3-dimensional manifold, it follows that time flow is a complex description with regards to a physical process. Time is not simply specified as past, present and future, but also dependent on its direction of flow. Only when the direction of flow of time can be specified then the state and the dynamics of a particle can be determined completely. For example, if time is a 3-dimensional continuum whose topology is Euclidean R^3 then the time of a particle with a temporal distance of unit length from the origin of a reference system is a temporal sphere of unit radius. In fact, the 3-dimensional temporal manifold can be reduced to 1-dimensional continuum by considering the 3-dimensional temporal manifold as a compactified manifold of the form $\mathrm{R}\times S^2$, where S^2 is a 2-dimensional compact manifold whose size is much smaller than any length. However, in the following we will only consider forces that act along a radial spatial direction, such as the force of gravity and Coulomb force, therefore even though we can assume time as a 3-dimensional continuum whose topology is Euclidean R^3, we will also only consider the dynamics of a particle along its radial time. In this case time is effectively a 1-dimensional continuum. Therefore, otherwise stated, we will assume $ds = dr$ and $d\tau = dt$. Now consider the case when a force that produces the same dynamics to an elementary particle if we apply Equations (22) and (23) to it separately. Suppose the temporal dynamics of the particle and its spatial dynamics are influenced by the same force $\mathbf{F}$ that gives rise to the same physical process, then we have

$$m\frac{d^2\mathbf{r}}{dt^2} = D\frac{d^2\mathbf{t}}{dr^2} \tag{24}$$

Since *m* and *D* are constant, Equation (24) can be re-written in magnitude form as

$$m\frac{d^2r}{dt^2} = D\frac{d^2t}{dr^2} \tag{25}$$

Equation (25) can be shown to take the form

$$\frac{d^2r}{dt^2}\left(\frac{D}{m} + \left(\frac{dr}{dt}\right)^3\right) = 0. \tag{26}$$

From this equation we obtain the following equations

$$\frac{d^2r}{dt^2} = 0 \tag{27}$$

and

$$\frac{dr}{dt} = \sqrt[3]{-D/m}\ . \tag{28}$$

Now we consider the case in with $dr/dt > 0$. In this case, if we assume $m > 0$ then $D < 0$. In particular, if the temporal rate of flow equals the spatial rate then we have $D = -m$. From this result we then may assume that the negative mass that appears in Dirac equation is associated with the temporal dynamics of an elementary particle, in addition to its spatial dynamics which is associated with the positive mass. The positive and negative masses are manifestations of the two field components of matter wave.

We now show that the relationship between the electric charge and the magnetic charge $\hbar c/q_e q_m = 2$ obtained by Dirac can be derived by imposing a tolopogical relationship between the Gaussian curvatures of temporal and spatial manifolds. Even though the following results are similar to those obtained for the spatial Euclidean continuum, for clarity, we will give an abbreviated version by first defining a temporal Gaussian curvature in the temporal Euclidean continuum R^3 and then deriving a quantised magnetic charge from Feynman integral method [11]. As in spatial dimensions, we consider a temporal surface defined by the relation $t^3 = f(t^1, t^2)$. Then, as shown in differential geometry, the temporal Gaussian curvature denoted by K_T can be determined by $f(t^1, t^2)$ and given as $K_T = (f_{11}f_{22} - (f_{12})^2)/(1 + f_1^2 + f_2^2)^2$, where $f_\mu = \partial f/\partial t^\mu$ and $f_{\mu\nu} = \partial^2 f/\partial t^\mu \partial t^\nu$. Let P_T be a 3-dimensional physical quantity which will be identified with the surface density of a magnetic substance, such as magnetic charge of an elementary particle. We therefore assume that an elementary particle is assigned not only with an electric charge q_e but also a magnetic charge q_m. We further assume that the quantity P_T is proportional to the temporal Gaussian curvature K_T. Now, as in the case with spatial dimensions, if we consider a surface action integral of the form $S = \int P_T dA_T = \int (q_m/2\pi)\, K_T dA_T$, then we have

$$S = \frac{q_m}{2\pi} \int \frac{f_{11}f_{22} - (f_{12})^2}{(1 + f_1^2 + f_2^2)^{3/2}} dt^1 dt^2 \tag{29}$$

According to the calculus of variations, similar to the case of path integral, in order to extremise the action integral $S = \int L(f, f_\mu, f_{\mu\nu}, x^\mu) dt^1 dt^2$, the functional $L(f, f_\mu, f_{\mu\nu}, t^\mu)$ must satisfy the differential equations

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial t^\mu}\frac{\partial L}{\partial f_\mu} + \frac{\partial^2}{\partial t^\mu \partial t^\nu}\frac{\partial L}{\partial f_{\mu\nu}} = 0 \tag{30}$$

However, it is straightforward to verify that with the functional of the form given by the relation $L = (q_m/2\pi)\,(f_{11}f_{22} - (f_{12})^2)/(1 + f_1^2 + f_2^2)^{3/2}$ the differential equations given by Equation (30) are satisfied by any surface. Hence, we can generalise Feynman's postulate to formulate a quantum theory in which the transition amplitude between states of a quantum mechanical system is a sum over random surfaces, provided the functional P_T in the action integral $S = \int P_T dA_T$ is taken to be proportional to the temporal Gaussian curvature K_T of a temporal surface. Consider a closed surface and assume that we have many such different surfaces which are described by the higher dimensional homotopy groups. As in the case of the fundamental homotopy group of paths, we choose from among the homotopy class a representative spherical surface, in which case we can write

$$\oint P_T dA_T = \frac{q_m}{4\pi} \oint d\Omega, \tag{31}$$

where $d\Omega$ is an element of solid angle. Since $\oint d\Omega$ depends on the homotopy class of the spheres that it represents, we have $\oint d\Omega = 4\pi n$, where n is the topological winding number of the homotopy group. From this result we obtain a generalised Bohr quantum condition

$$\oint P_T dA_T = n_T q_m \tag{32}$$

The action integral $(q_m/4\pi)\oint K_T dA_T$ is similar to Gauss's law in electrodynamics. In this case the constant q_m can be identified with the magnetic charge of a particle. In particular, the magnetic charge q_m represents the topological structure of a physical system must exist in multiples of q_m. Hence, the magnetic charge of a physical system, such as an elementary particle, may depend on the topological structure of the system and is classified by the homotopy group of closed surfaces. This result may shed some light on why magnetic charge is quantised. We are now in the position to show that it is possible to obtain the relationship between the electric charge q_e and the magnetic charge q_m derived by Dirac by considering a spatiotemporal Gaussian curvature K which is defined as a product of the temporal Gaussian curvature K_T and the spatial Gaussian curvature K_S as follows

$$K = K_T \times K_S \tag{33}$$

The spatiotemporal submanifold that gives rise to this form of curvature is homeomorphic to $S^2 \times S^2$. If K_T and K_S are independent from each other then we can write

$$\oint K\, dA = \oint K_T \times K_S\, dA_T dA_S = \oint K_T\, dA_T \times \oint K_S\, dA_S \tag{34}$$

If we assume further that $\oint K\, dA = k$, where k is an undetermined constant, then using the results $\oint K_S\, dA_S = \oint (q_e/2\pi) K_S dA_S$ and $\oint K_T\, dA_T = \oint (q_m/2\pi) K_T dA_T$, we obtain a general relationship between the electric charge q_e and the magnetic charge q_m

$$\frac{k}{q_e q_m} = n_S n_T \tag{35}$$

In particular, if $n_S = 1$, $n_T = 2$ and $k = \hbar c$, or $n_S = 2$, $n_T = 1$ and $k = \hbar c$, then we recover the relationship obtained by Dirac, $\hbar c / q_e q_m = 2$.

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