Exact Solution of Navier-Stokes Equations

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ABSTRACT

In Navier-Stokes equations, we discover the exact solution by Newton potential and time-function.

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1. Introduction

We discover the exact solution in Navier-Stokes equation by Newton potential and time function.

According NASA's Navier-Stokes Equations(3-dimensional-unsteady),

Coordinate: (x, y, z), Time: t, Pressure: p, Heat Flux: q

Density: ρ, Stress: τ, Reynolds Number: R_e,

Velocity Components: (u, v, w) Total Energy: E_t Plandtl Number: P_r

Continuity: $$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad (1)$$

X-Momentum: $$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z}$$

$$= -\frac{\partial p}{\partial x} + \frac{1}{R_e}[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}] \quad (2)$$

Y-Momentum: $$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(\rho vw)}{\partial z}$$

$$= -\frac{\partial p}{\partial y} + \frac{1}{R_e}[\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}] \quad (3)$$

Z-Momentum: $$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho wv)}{\partial y} + \frac{\partial(\rho w^2)}{\partial z}$$

$$= -\frac{\partial p}{\partial z} + \frac{1}{R_e}[\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}] \quad (4)$$

Energy: $$\frac{\partial E_t}{\partial t} + \frac{\partial(E_t u)}{\partial x} + \frac{\partial(E_t v)}{\partial y} + \frac{\partial(E_t w)}{\partial z} = -\frac{\partial(pu)}{\partial x} - \frac{\partial(pv)}{\partial y} - \frac{\partial(pw)}{\partial z}$$

$$-\frac{1}{R_e P_r}[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}]$$

$$+\frac{1}{R_e}[\frac{\partial}{\partial x}(u\tau_{xx} + v\tau_{xy} + w\tau_{xz}) + \frac{\partial}{\partial y}(u\tau_{xy} + v\tau_{yy} + w\tau_{yz}) + \frac{\partial}{\partial z}(u\tau_{xz} + v\tau_{yz} + w\tau_{zz})]$$

(5)

2. Exact Solution in 3-Dimentional Navier-Stokes Equation (Include time)

For we solve equations, we use Newton potential and time function. If we think the solution of Laplace equation,

$$u = \frac{C_1}{r^3} x f(t) \quad , \quad v = \frac{C_1}{r^3} y f(t) \ , \ w = \frac{C_1}{r^3} z f(t) \ , \quad r = \sqrt{x^2 + y^2 + z^2} \ , \quad \rho = \rho_0$$

(6)

In this case, we solve Eq(1).

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = f(t)\rho_0[-\frac{3C_1}{r^5}(x^2 + y^2 + z^2) + \frac{3C_1}{r^3}] = 0 \qquad (7)$$

Second point, in Eq(2)

$$\tau_{xx} = \frac{C_2}{r^6} x^2 [f(t)]^2, \quad \tau_{xy} = \frac{C_2}{r^6} xy[f(t)]^2, \qquad \tau_{xz} = \frac{C_2}{r^6} xz[f(t)]^2 \quad ,$$

$$,\dots, \tau_{ij} = \frac{C_2}{r^6} x^i x^j [f(t)]^2 \qquad (8)$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z}$$

$$= \rho_0 \dot{f}(t) \frac{C_1}{r^3} x + [f(t)]^2 \rho_0 [-\frac{6C_1^2}{r^8}(x^3 + xy^2 + xz^2) + \frac{C_1^2}{r^6}(2x + x + x)]$$

$$= \rho_0 \dot{f}(t) \frac{C_1}{r^3} x - [f(t)]^2 \rho_0 \frac{2C_1^2}{r^6} x \qquad (9)$$

In this time, if $f(t)$ is,

$$\frac{1}{[f(t)]^2} \frac{d}{dt}[f(t)] = 1 \ \rightarrow \ f(t) = \frac{1}{C' - t}, \ C' \text{ is constant} \qquad (10)$$

Therefore,

$$= \rho_0 \frac{1}{(C' - t)^2} \frac{C_1}{r^3} x - \frac{1}{(C' - t)^2} \rho_0 \frac{2C_1^2}{r^6} x$$

$$= -\frac{\partial p}{\partial x} + \frac{1}{R_e}[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}]$$

$$= -\frac{\partial p}{\partial x} + \frac{1}{R_e}[-\frac{6C_2}{r^8}(x^3 + xy^2 + xz^2) + \frac{C_2}{r^6}(2x + x + x)] \frac{1}{(C' - t)^2}$$

$$= -\frac{\partial p}{\partial x} - \frac{1}{R_e} \frac{2C_2}{r^6} x \qquad (11)$$

Hence, in Eq(2),in Eq(3) and in Eq(4),

$$\frac{\partial p}{\partial x} = [\frac{2x}{r^6}(\rho_0 C_1^2 - \frac{C_2}{R_e}) - \rho_0 \frac{C_1}{r^3} x] \frac{1}{(C' - t)^2} = [\frac{2C_3}{r^6} x - \rho_0 \frac{C_1}{r^3} x] \frac{1}{(C' - t)^2} \ ,$$

$$\frac{\partial p}{\partial y}=[\frac{2y}{r^6}(\rho_0 C_1^{\ 2}-\frac{C_2}{R_e})-\rho_0\frac{C_1}{r^3}y]\frac{1}{(C'-t)^2}=[\frac{2C_3}{r^6}y-\rho_0\frac{C_1}{r^3}y]\frac{1}{(C'-t)^2} \quad ,$$

$$\frac{\partial p}{\partial z}=[\frac{2z}{r^6}(\rho_0 C_1^{\ 2}-\frac{C_2}{R_e})-\rho_0\frac{C_1}{r^3}z]\frac{1}{(C'-t)^2}=[\frac{2C_3}{r^6}z-\rho_0\frac{C_1}{r^3}z]\frac{1}{(C'-t)^2}$$

$$C_3=\rho_0 C_1^{\ 2}-\frac{C_2}{R_e} \tag{12}$$

Therefore,

$$p=[-\frac{1}{2}\frac{C_3}{r^4}+\rho_0\frac{C_1}{r}]\frac{1}{(C'-t)^2} \quad , \quad C_3=\rho_0 C_1^{\ 2}-\frac{C_2}{R_e} \tag{13}$$

In Eq(5), if E_t, q_i is,

$$E_t=E_0,\ q_x=\frac{C_4}{r^4}\frac{x}{(C'-t)^3},q_y=\frac{C_4}{r^4}y\frac{1}{(C'-t)^3},q_z=\frac{C_4}{r^4}z\frac{1}{(C'-t)^3} \tag{14}$$

$$\frac{\partial E_t}{\partial t}+\frac{\partial(E_t u)}{\partial x}+\frac{\partial(E_t v)}{\partial y}+\frac{\partial(E_t w)}{\partial z}=0 \tag{15}$$

As,

$$(pu,pv,pw)=[-\frac{1}{2}\frac{C_1C_3}{r^7}+\rho_0\frac{C_1^{\ 2}}{r^4}](x,y,z)\frac{1}{(C'-t)^3} \tag{16}$$

Hence,

$$-\frac{\partial(pu)}{\partial x}-\frac{\partial(pv)}{\partial y}-\frac{\partial(pw)}{\partial z}$$

$$=[\{-7\frac{C_3C_1}{2r^9}(x^2+y^2+z^2)+3\frac{C_3C_1}{2r^7}\}+\{4\rho_0\frac{C_1^{\ 2}}{r^6}(x^2+y^2+z^2)-3\rho_0\frac{C_1^{\ 2}}{r^4}\}]\frac{1}{(C'-t)^3}$$

$$=[-\frac{2C_1C_3}{r^7}+\rho_0\frac{C_1^{\ 2}}{r^4}]\frac{1}{(C'-t)^3} \tag{17}$$

So,

$$-\frac{1}{R_eP_r}[\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}]$$

$$=-\frac{1}{R_eP_r}[\{-\frac{4C_4}{r^6}(x^2+y^2+z^2)+\frac{3C_4}{r^4}\}\frac{1}{(C'-t)^3}]=\frac{1}{R_eP_r}\frac{C_4}{r^4}\frac{1}{(C'-t)^3} \tag{18}$$

In this time,

$$\tau_{ij}=\frac{C_2}{r^6}x^ix^j[f(t)]^2=\frac{C_2}{r^6}x^ix^j\frac{1}{(C'-t)^2},$$

$$(\tau_{xj}u, \tau_{yj}v, \tau_{zj}w) = \frac{C_1C_2}{r^9}(x^2, y^2, z^2)x^j \frac{1}{(C'-t)^3}$$

$$\frac{1}{R_e}[\frac{\partial}{\partial x}(u\tau_{xx} + v\tau_{xy} + w\tau_{xz}) + \frac{\partial}{\partial y}(u\tau_{xy} + v\tau_{yy} + w\tau_{yz}) + \frac{\partial}{\partial z}(u\tau_{xz} + v\tau_{yz} + w\tau_{zz})]$$

$$= \frac{1}{R_e}[-9\frac{C_1C_2}{r^{11}}x^2(x^2 + y^2 + z^2) + \frac{C_1C_2}{r^9}(3x^2 + y^2 + z^2)$$

$$-9\frac{C_1C_2}{r^{11}}y^2(x^2 + y^2 + z^2) + \frac{C_1C_2}{r^9}(x^2 + 3y^2 + z^2)$$

$$-9\frac{C_1C_2}{r^{11}}z^2(x^2 + y^2 + z^2) + \frac{C_1C_2}{r^9}(x^2 + y^2 + 3z^2)]\frac{1}{(C'-t)^3}$$

$$= \frac{1}{R_e}[-9\frac{C_1C_2}{r^7} + 5\frac{C_1C_2}{r^7}]\frac{1}{(C'-t)^3} = -\frac{1}{R_e}\frac{4C_1C_2}{r^7}\frac{1}{(C'-t)^3} \tag{19}$$

Hence, Eq(5) is

$$0 = [(-2\frac{C_1C_3}{r^7} + \rho_0\frac{C_1^2}{r^4}) + \frac{1}{R_eP_r}\frac{C_4}{r^4} - \frac{1}{R_e}\frac{4C_1C_2}{r^7}]\frac{1}{(C'-t)^3} \quad ,$$

$$C_3 = -\frac{2C_2}{R_e} \qquad C_4 = -\rho_0 C_1^2 R_e P_r$$

$$C_3 = \rho_0 C_1^2 - \frac{C_2}{R_e} = -\frac{2C_2}{R_e} \quad , \quad C_2 = -\rho_0 R_e C_1^2 \tag{20}$$

Therefore, P is

$$p = [-\frac{1}{2}\frac{C_3}{r^4} + \rho_0\frac{C_1}{r}]\frac{1}{(C'-t)^2} = [\frac{C_2}{R_e}\frac{1}{r^4} + \rho_0\frac{C_1}{r}]\frac{1}{(C'-t)^2} = [-\rho_0\frac{C_1^2}{r^4} + \rho_0\frac{C_1}{r}]\frac{1}{(C'-t)^2} \tag{21}$$

$$C_4 = -\rho_0 C_1^2 R_e P_r, \; q_x = \frac{C_4}{r^4}x\frac{1}{(C'-t)^3} = -\rho_0\frac{C_1^2 R_e P_r}{r^4}\frac{x}{(C'-t)^3}$$

$$q_y = \frac{C_4}{r^4}y\frac{1}{(C'-t)^3} = -\rho_0\frac{C_1^2 R_e P_r}{r^4}y\frac{1}{(C'-t)^3}$$

$$q_z = \frac{C_4}{r^4}z\frac{1}{(C'-t)^3} = -\rho_0\frac{C_1^2 R_e P_r}{r^4}z\frac{1}{(C'-t)^3} \tag{22}$$

3. Conclusion

Therefore, the exact solution of Navier-Stokes equations (3-dimensional-unsteady) is

Pressure: $p = [-\rho_0 \frac{C_1^2}{r^4} + \rho_0 \frac{C_1}{r}] \frac{1}{(C'-t)^2}$

Heat Flux:

$$q_x = -\rho_0 \frac{C_1^2 R_e P_r}{r^4} \frac{x}{(C'-t)^3}, q_y = -\rho_0 \frac{C_1^2 R_e P_r}{r^4} \frac{y}{(C'-t)^3}, q_z = -\rho_0 \frac{C_1^2 R_e P_r}{r^4} \frac{z}{(C'-t)^3}$$

Density: $\rho = \rho_0$,

Stress: $\tau_{ij} = \frac{C_2}{r^6} x^i x^j \frac{1}{(C'-t)^2} = -\rho_0 \frac{R_e}{r^6} C_1^2 x^i x^j \frac{1}{(C'-t)^2}$

Reynolds Number: R_e, Plandtl Number: P_r

Velocity Components: $(u, v, w) = \frac{C_1}{r^3} (x, y, z) \frac{1}{(C'-t)}$

Total Energy: $E_t = E_0$

References

[1]Three-dimensional unsteady form of the Navier-Stokes equations: Glenn Research Center,NASA