

Remarks on Infinitesimal Amount of Riemann Zeta Zeros

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Abstract

This remarks proves, that Riemann zeta function has infinitesimal amount of zeros.

Riemann zeta function expressed as follow

$$\zeta(r, \theta) = \sum_{n=1}^{\infty} \frac{1}{n^{r \cos \theta + ir \sin \theta}} \quad (1)$$

According Abel's summation formula [1]

$$\zeta(r, \theta) = \sum_{n=1}^{\infty} \frac{1}{n^{r \cos \theta + ir \sin \theta}} = \lim_{x \rightarrow \infty} \frac{[x]}{x^{r \cos \theta + ir \sin \theta}} - \int_{u=1}^{\infty} \frac{[u] du}{u^{r \cos \theta + ir \sin \theta + 1}} \quad (2)$$

For $r > 1, \cos \theta > 0, \sin \theta > 0$ the first term in (2) goes to zero and we can rewrite (2) without losing of generalisation as follow

$$\zeta(r, \theta) = \gamma_{r \cos \theta + ir \sin \theta} - \int_{u=1}^{\infty} \frac{du}{u^{r \cos \theta + ir \sin \theta}} \quad (3)$$

where $\gamma_{r \cos \theta + ir \sin \theta}$ is constant for each r and θ . After integration we obtains

$$\zeta(r, \theta) = \gamma_{r \cos \theta + ir \sin \theta} - \frac{u^{-r \cos \theta - ri \sin \theta + 1}}{-r \cos \theta - ri \sin \theta + 1} \Big|_{u=1}^{u=\infty} \quad (4)$$

Let's restrict r and θ to $r \cos \theta - 1 > 0, r \sin \theta > 0$. So,

$$\zeta(r, \theta) = \gamma_{r \cos \theta + ir \sin \theta} + \frac{1}{-r \cos \theta - ri \sin \theta + 1} \quad (5)$$

or

$$\zeta(r, \theta) = \sqrt{(\Re \zeta(r, \theta))^2 + (\Im \zeta(r, \theta))^2} \exp \left(i \arctan \frac{\Im \zeta(r, \theta)}{\Re \zeta(r, \theta)} + i2\pi n \right) \quad (6)$$

where

$$\Re \zeta(r, \theta) = \Re \gamma_{r \cos \theta + ir \sin \theta} + \frac{-r \cos \theta + 1}{(1 - r \cos \theta)^2 + r^2 \sin^2 \theta} \quad (7)$$

$$\Im \zeta(r, \theta) = \Im \gamma_{r \cos \theta + ir \sin \theta} + \frac{r \sin \theta}{(1 - r \cos \theta)^2 + r^2 \sin^2 \theta} \quad (8)$$

Obviously, for $r \gg 1$ we can find infinitesimal amount of Riemann zeta zeros by solving equation as follow

$$\frac{r^{\Im} \gamma_{r \cos \theta + ir \sin \theta} + \sin \theta}{r^{\Re} \gamma_{r \cos \theta + ir \sin \theta} - \cos \theta} - 2\pi n = 0, \forall n \in (-\infty, \infty) \quad (9)$$

with solution of $r = \frac{2\pi n \cos \theta + \sin \theta}{2\pi n \Re \gamma_{r \cos \theta + ir \sin \theta} - \Im \gamma_{r \cos \theta + ir \sin \theta}}$. It implies that Riemann zeta function has infinitesimal amount of zeros.

References

- [1] Apostol, Tom (1976), Introduction to Analytic Number Theory, Undergraduate Texts in Mathematics, Springer-Verlag.