Chapter

SOME RECENT ASPECTS OF DEVELOPMENTS OF CHERN-SIMON GAUGE FIELD THEORIES

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Abstract

In this chapter, we present the basic elements of Chern-Simon theory and then we review some recent aspects of developments in Chern-Simon gauge field theory as a topological quantum field theory on a three-manifold.

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1. INTRODUCTION

We focus on some aspects of developments in Chern-Simon gauge field theory since there are good resources to explore for the interested reader (see [3], [22], [23], [29], [43], [49] and [50]). Indeed, this is a good example to illustrate how the ideas from quantum field theory can be used to study topology. This theory provides a field theoretic framework for the study of knots and links in a given three manifold (see [5], [40], [41], [57]). It was A. S. Schwarz who first conjectured in [57] that Jones polynomials (see [37] and [38]) may be related to Chern-Simon theory. Edward Witten in [63] demonstrated this 20 years ago. Moreover, he came up with a general field theoretic framework to study knots and links. Since then, enormous effort has gone into developing an exact explicit non-perturbative solution of this field theory. The interplay between quantum field theory and knot theory has given some rich results in both directions. Many of the open problems in knot theory have found answers in the process of investigation.

Topological quantum field theories are independent of the background metric and the involved operators are also metric independent. Degrees of freedom are also topological. Wilson loops operators are the topological operators of the Chern-Simons gauge field theory. Their vacuum expectation values are the topological invariants for knots and links which do not depend on the exact shape, location or form of the knots and links but reflect only their topological properties. The power of this framework is so deep that it allows us to study these invariants not only on simple manifolds such as 3-sphere but also on any arbitrary 3-manifold. The knots and links invariants obtained from these field theories are also intimately related to the integrable vertex models in two dimensions (see [60]).

Indeed a quantum group approach to these invariants has been developed by A. N. Kirillow and N. Reshetikhin. Also there have been some developments in these directions in algebraic topology. Chern-Simons theory has also played a fundamental role in quantum gravity. For example 3-dimensional gravity with a negative cosmological constant, itself a topological field theory, can be described by two copies of $SU(2)$ Chern-Simons theory.

In 4-dimensional gravity, Chern-Simons theories also find applications. For example, the boundary degree of freedom of a black hole in 4-dimensions can be described by an $SU(2)$ Chern-Simons field theory. This made it possible to calculate exactly. The quantum entropy of a non-rotating black hole. This agrees with the Bekenstein-Hawking formula for large areas and goes indeed beyond the semi-classical results.

2. Basic Elements of Chern-Simon Theory

In this section we give a summery of how the Chern-Simons forms are constructed. Then we explain these forms in the four dimensional version since it has something to do with knot theory and quantum gravity. This section is heavily based on [7], [20], [48] and [51].

Action principles are very important in physics and the motivation initiated by Fermat's discovery which says that the taken path of the ray of light between two given points is the path that can be traversed in the least time. In Newtonian Mechanics, the general solution to the Newton's differential equations of second order has two constants which implies that if we want to know where the particle is going on we need to know about the initial position and momentum. Therefore in this case it is profitable to focus on initial and final positions to define the action by taking integral over the path of a quantity called the Lagrangian, denoted by L, which is the difference between kinetic energy (dented by T) and potential energy (denoted by V), that's $L = T - V$. This type of integral is called the action, denoted by S which is a functional since it takes a function and spit out a number, i.e., it gives a number to each path. By minimizing the variance of this action we find the path of least time taken by the particle. By applying the quantity of the Lagrangian, we can better observe the symmetries and some other properties of the equations. Indeed the Lagrangian read off all the dynamic behaviour of the equations. Moreover, if L is given then we can obtain the equations of motions from the so-called Euler-Lagrange equations. In effect the principle of least action generates the Euler-Lagrange equations.

Atiyah [[5], p. 2] pointed out"the prototype of all gauge theories is electromagnetism". For example one can obtain the Yang-Mills equation by a comparison of Maxwell's equations or specifically as a special case of Maxwell's equations (see for more details in [[7], chapters 3 and 4]). It is well-known that Yang-Mills Lagrangian which is a n-form is given as follows:

$$\mathcal{L}_{\mathcal{YM}}= \tfrac{1}{2}tr(F \wedge \star F)$$

where F is the curvature of D and D is a connection on the vector bundle E over a n-dimensional semi-Riemanian oriented manifold M. The star in front of F is the Hodge star operator. Indeed F is a Lie-algebra valued 2-form and in three dimensions the dual of F, that is, $\star F$ is a 1-form (see [5]). By taking the integral of the Yang-Mills Lagrangian over M, we obtain the Yang-Mills action, that is

$$S_{YM}(A) = \tfrac{1}{2}\int_M tr(F \wedge \star F))$$

where A is the vector potential. The involved Hodge star operator indicates that the Yang-Mills equation depends on the metric, i.e. the fixed background structure, defined on spacetime. If we take A to be self-dual, then $\star F = F$. By utilizing the Bianchi identity, $d_A F = 0$ and therefore the Yang-Mills equation is $d_A \star F = 0$. Indeed, the action can be defined to be the integral of the Lagrangian, nth Chern form, i.e., $tr(F \wedge ... \wedge F) = tr(F^n)$ which is a $2n$-form , denoted by $\Omega^{2n}(F)$, over a 2n-dimensional manifold M:

$$S(A) = \int_M tr(F^n))$$

In this situation, the Lagrangian produce only trivial equations. It means that $\delta S = 0$ for all A, where δ is an infinitesimal variation:

$$\begin{aligned}
\delta S(A) &= \delta \int_M tr(F^n)) && (1)\\
&= n \int_M tr(\delta F \wedge F^{n-1}) && (2)\\
&= n \int_M tr(d_D \delta A \wedge F^{n-1}) && (3)\\
&= n \int_M tr(\delta A \wedge d_D F^{n-1}) && (4)\\
&= n \int_M tr(\delta A \wedge 0) && (5)\\
&= 0 && (6)
\end{aligned}$$

But $\delta S = 0$ also can be considered as the independency of $S(A)$ from A, only by being dependent on the vector bundle $\pi : E \to M$, where M is an oriented manifold and F is the curvature of any connection on E. Chern forms are all invariant under gauge transformation and also closed, according to Chern-Weil theorem. This means that $d\Omega^{2n}(F) = tr(d_D F \wedge ... \wedge F + ...F \wedge ... \wedge d_D F) = 0$ by utilizing the fact that $tr(d_D F) = dtr(F)$, the graded cyclic property of the trace and the Bianchi identity such that $d_D F = dF + [A, F] = 0$. It follows that the nth Chern form defines a cohomology class in $H^{2n}(M)$. Let A and A' be two vector potentials with curvatures F and F' respectively. Consider the variation $\delta A = A' - A$ and $A_s = A + s\delta A$, where for $s = 0$ and $s = 1$ we have A and A', respectively. In the following, it is shown that the difference of Chern forms is exact:

$$\begin{aligned}
\Omega^{2n}(F') - \Omega^{2n}(F) &= tr(F'^n) - tr(F^n) && (7)\\
&= \int_0^1 \frac{d}{ds} tr(F_s^n) ds && (8)\\
&= n \int_0^1 dtr(\delta A \wedge F_s^{n-1}) ds && (9)\\
&= nd \int_0^1 tr(\delta A \wedge F_s^{n-1}) ds && (10)
\end{aligned}$$

Chern-Weil theorem says that if F and F' are curvature two-forms corresponding to different connections D and D', then $\Omega^{2n}(F') - \Omega^{2n}(F)$ is exact.

The nth Chern class $c_n(E)$ can be defined as a Chern class of the vector bundle E over M considering it as a cohomology class of $\Omega^{2n}(F)$, where F is the curvature of any connection on E. Observe that the Chern form depends on the connection A but its cohomology class does not. It should be noted that from a topological viewpoint what concerns the definition of the Chern classes the result of their integrals over any compact orientable 2n-dimensional manifold is an integer by proper normalizing. For example if E is a complex vector bundle and M is compact and orientable, then the integral of the nth Chern form over M is an integer called the nth Chern number.

Now we want to obtain Chern-Simons 3-form. We showed above that when one changes the vector potential then the Chern form changes by an exact and if we continue in the same line then its proof offers an explicit 3-form whose exteriror derivative is $tr(F \wedge F)$. Assume that $A_s = sA$ and let $F_s = ds \wedge A + sdA + s^2 A \wedge A$, where F_s is the curvature of A_s. Thus

$$tr(F \wedge F) = \int_0^1 \frac{d}{ds} tr(F_s \wedge F_s) ds \tag{11}$$

$$= 2d \int_0^1 tr(A \wedge (sdA + s^2 A \wedge A)) ds \tag{12}$$

$$= d\, tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) ds. \tag{13}$$

Hence $tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$ is the well-known Chern-Simons 3-form. The action of Chern-Simons theory, denoted by S_{CS}, is proportional to the integral of the Chern-Simons 3-form, that is $S_{CS} = k \int tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$. Utilizing path integral quantization approach, we get Chern-Simon action which is

$$S_{SC}(A) = \tfrac{k}{4\pi} \textstyle\int_M tr(A \wedge dA + (\frac{2}{3}) A \wedge A \wedge A),$$

where the positive integer k is called the Chern-Simon level ([for more details see [48]]). Fundamental data are:

1. A compact and smooth 3-manifold M;

2. A semi-simple and compact gauge group G;

3. An integer parameter k, called the coupling constant.

The action of the theory is the integral of the Chern-Simons form associated to a gauge connection A corresponding to a gauge group G. As it can be seen, the action is metric independent, which means that it is a topological quantum field theory. Therefore the Chern-Simon theory is a 2+1 dimensional gauge theory which is independent of a metric or a background structure. This theory is invariant under small gauge transformations and diffeomorphisms but it is not invariant under large gauge transformation (see [7], [35] and [48]). In the case of non-Abelian Chern-Simon theory, the Chern-Simon form is not invariant under large gauge transformations and the Wilson loop is not invariant under large gauge transformation. This implies, as Gambini and Pullin pointed out (see[[24], p. 259]), that we do not have a function but a loop. It should be noted that this is not the case in the Abelian situation which we have small gauge transformation. But in both cases, it is not known how this problem is related to framing ambiguity.

3. Some Aspects of Developments

Observables in the theory lead to vacuum expectations values (briefly vevs) which correspond to topological invariant. This means that observables should satisfy two properties:

1. They must be metric independent;

2. They must be gauge invariant.

Wilson loops satisfy both. They correspond to the holonomy of the gauge connection A along a loop. Indeed the products of these operators are the natural candidates to obtain topological invariants after computing their vevs. In his paper [63], Witten showed that the vevs of the products of wilson loops correspond to the Jones polynomials when one considers $SU(2)$ as gauge group. He also showed that if one uses $SU(N)$ and that if Wilson loops carries the fundamental representations then we get the Homfly polynomials as the resulting invariant. Witten in his paper [64] describes how Chern-Simon gauge theory in three dimensions can arise as a string theory with a world-sheet model involving a topological sigma model related to Floer/Gromov theory. The perturbation theory of this string theory coincides with chern-Simon perturbation theory.

It should be noticed that if one considers other groups and representations, then one gets different set of knot and link invariants. For example if we take $SO(N)$ as gauge group then we get Kauffman polynomial; or if we take $SU(2)$ as gauge group then we get the Akutsu-Wadati polynomials. All these means that Chern-Simons gauge theory can be investigated for arbitrary groups and arbitrary representations. All these also made it possible to approach to the polynomial invariants via quantum groups and also has been established a categorical view point to study knots, links and graphs (see [39]). A particular invariant of 3-manifolds is the partition function of Chern-Simons gauge theory. It is hard to obtain this from a field theory point of view. But it has been defined by using triangulation of 3-manifold (A mathematical viewpoint). This invariant is now called Witten-Reshetekhin-Turaev invariant. Moreover, on a triangulated 3-manifold. One can obtain this invariant from Chern-Simons gauge theory by using lattice gauge theory methods (this has been done by R. Gambini). It has been done several studies of Chern-Simons gauge theory from the non-perturbative aspects for the last 25 years. The quantization of the theory has been investigated from the operator theory aspect. Also a powerful method for general computation of knot and graph invariants was constructed by Kaul and others(see [41]).

The connection between Chern-Simon gauge theory and rational conformal field theory has been used to construct knot and link invariants from any conformal field theory. Chern-Simons gauge theory showed to be important with connection to canonical quantum gravity (C. Rovelli, L. Smolin, B. Bruegmann, R. Gambini, J. Griego and J. Pullin)). Chern-Simons gauge theory has also connection to the Gromov-Witten theory of non-compact Calabi-Yau threefolds (see [[1], [17], [18], [25], [26]]). Chern-Simons gauge field theory has also been studied from the perturbative aspect by pioneer works of Gaudagnini, Martellini and Mintchev (see [27]) and Bar-Natan (see [8]). Subsequent works in this direction led to the theory of Vassiliev invariants which are group factors associated to chord diagrams or weight systems from the point of view of Chern-Simon theory (see [13]).

Witten E. in [65] investigated the analytic continuation of three-dimensional Chern-Simons gauge theory by not considering integer values of the usual coupling parameter k in order to examine questions concerning the volume conjecture, or analytic continuation of three-dimensional quantum gravity (to the extent that it can be described by gauge theory) from Lorentzian to Euclidean signature. As he pointed out, such analytic continuation can be carried out by generalizing the usual integration cycle of the Feynman path integral. Among others, he exhibits that the space of possible integration cycles for Chern-Simons theory can be interpreted as the "physical Hilbert space" of a twisted version of $N = 4$ super Yang-Mills theory in four dimensions.

In the paper [42] from 1993, M. Kontsevich answered to the following open question affirmatively:

Could weight systems on chord diagrams be integrated to obtain invariants for non-singular knots?

He proved that a weight system on chord diagrams determines a unique Vassiliev invariant on non-singular knots. This provides an explicit expression for the Vassiliev invariant for non-singular knots. This is known as the Kontsevich integral. The origin of the Kontsevich factor in non-covariant gauges is still an open question. This factor is introduced in both the light-cone and the temporal gauges but it is not understood.

A. Iqbal and Amir-Kian Kashani-Poor [33] have offered an application of Chern-Simon theory and showed that a generalization of the topological closed string partition function whose field theory limit is the generalization of the instanton partition function, proposed by Nekrasov, can be determined easily from the Chern-Simons theory. Lawrence and Rozansky [44] have shown that the partition function via the exact solution of Chern-Simons theory has a very simple structure and it can be expressed as a sum of local contributions from the flat connections on a Seifert manifold M. In 2005, Beasely and Witten [9] have considered Chern-Simon Gauge theory on a Seifert manifold M and based on the observation exhibited by Lawrence and Rozansky, showed that how this observation is a natural consequence of the technique of non-abelian localization applied to the Chern-Simons path integral. They also explained how the partition function of Chern-Simons theory on M admits a topological interpretation in terms of the equivariant cohomology of the moduli space of flat connections on M. Beasely in [10] extended this result by utilizing it to the expectation values of Wilson loop operators which wrap the circle fibers of M over a Riemannian surface Σ. In [11], Beasely showed that how certain Wilson loop observables in Chern-Simons gauge theory on a Seifert three-manifold M can be given an analogous symplectic description. Indeed he discusses the symplectic geometry of Chern-Simon theory and in order to make it possible to hsave a symplectic interpretation for the Chern-Simons path integral, he introduces a contact structure on M.

In 2007, Gukov and Murakami verified some aspects of the conjecture that the asymptotic behaviour of the colored Jones polynomial is equal to the perturbative expansion of the Chern-Simons gauge theory with complex gauge group $SL(2, C)$ on the hyperbolic knot complement (see [31]). Oda has shown "why" the Chern-Simon state exists in Yang-Mills theory and general relativity in 4-dimensions. Indeed he showed that Chern-Simons state provides us a window of catching a glimpse of a relationship between general relativity (Yang-Mills theory) and topological quantum field theory. Cheren-Simon states exists only when the cosmological constant is non-vanishing. This of course gives some indications to how Chern-Simons state can be useful in understanding the real world.

Open question: Is the Lorentzian Chern-Simon state normalizable under an appropriate inner product? (see [52])

Gopakumar and Vafa proposed a new duality, that is, the large N limit of $SU(N)$ Chern-Simons theory on 3-sphere is exactly the same as an $N = 2$ topological closed string on the 2-sphere blow up of the conifold geometry (see [26]). The detailed understanding of Wilson loop in their correspondence remains a fruitful area of research mathematically. Nothing has been don in this respect. The correspondence can be generalized to Chern-Simons theory on lens spaces. This is not completely studied. H. Ooguri and C. Vafa indicated new evidence for the above conjecture (see [53]). They extended this conjecture to the observables of Wilson loop which are Wilson loop operators. They considered the computation of the expectation value of the Wilson loop for simple knot. They proposed the following open question:

Question: Can this be generalized to arbitrary knots?

Recently Gukov and Witten considered analysis of Chern-Simons gauge theory from a new viewpoint of quantization and not those views from conformal field theory, algebraic geometry and deformation quantization. They constructed the space of physical states of Chern-Simons theory with compact gauge group G on an oriented 2-manifold without boundary C. Then they defined M to be the moduli space of homomorphisms from fundamental group of C into G of a given topological type. M is the classical phase space of the Chern-Simons theory and has a natural symplectic form. They showed that quantization of this symplectic manifold finds applications in Chern-Simon theory. They pointed out that Chern-Simons gauge theory of a non-compact gauge group is not well-understood. So to generalize this view to this type of gauge group is an open area of research! A difficult one! (see [30]).
S. Gukov, M. Marino and P. Putrov [32] investigated resurgence properties of partition function of $SU(2)$ Chern-Simons theory (WRT invariant) on closed three-manifolds. They exhibited that in various examples Borel transforms of asymptotic expansions posses expected analytic properties.

Chern-Simons theory is also related to the open string field theory of topological A-model (see [64]). A. Klemm considered Chern-Simon theory and topological theory on non-compact Calabi-Yau manifolds. $SU(N)$ Chern-Simon theory on 3-sphere is equivalent to open topological string theory on the cotangent bundle of 3-sphere. Large duality relates this open topological string theory to closed topological string . He proposed that generalization of this topological realization of the gauge theory string theory duality of 't Hooft and Maldacena leads to a solution of topological string theory on non-compact Calabi-Yau toric manifolds. T. Dimofte, S. Gukov, J. Lenells and D. Zagier (see [19]) developed several methods that allow us to compute all-loop partition functions in perturbative Chern-Simons theory with complex gauge group G, sometimes in multiple ways. In the background of a non-abelian irreducible flat connection, perturbative G invariants turn out to be interesting topological invariants, which are very different from finite type Vassiliev invariants obtained in a theory with compact gauge group G. They investigated several aspects of these invariants and offered an example where they computed them explicitly to high loop order. They introduced a new theory arithmetic topological quantum field theory and conjectured that $SL(2, C)$ Chern-Simons theory is an example of such a theory!

Konstantin Wernli [61] in his lecture notes not only gives an excellent presentation of the classical Chern-Simon theory and perturbative quantization but also tries to show that the perturbative quantization of Chern-Simons theory is a better mathematical understanding of the Feynman path integral [21] in the domain of quantum field theory.

Schwarz in his lecture in Max-Planck (august 2009) [58] (see also [59]) talked about BV-formulation of Chern-Simons theory by considering the action functional $S = \frac{1}{2} < A, dA > + \frac{1}{3} < A,$[A, A]> and showed that every Z-graded L_∞ algebra is quasi-isomorphic to differential graded Lie algebra which implies that the Chern-Simon is universal. He also showed that multi-dimensional Chern-Simons implies algebra of differential forms on an odd-dimensional manifold M. Qiu and Zabzine studied a toy model which was an odd analogue of Chern-Simons theory. It is worth-noticing that their toy model reproduces the same weight function as in Chern-Simons and Rozansky-Witten theory. They offered some explicit computation of two point functions and exhibited that its purterbation theory is identical to Chern-Simons theory. (see [55]).

Recently, B. Lian, C. Vafa, F. Vafa and Shou-Cheng Zhang [47] showed that for 3-dimensional time-reversal invariant superconductors, a generalized Berry gauge field behaves as a fluctuating field of a Chern-Simons gauge theory.

Lehum et al. in their paper [45] investigated some properties of the effective superpotential in the three-dimensional superspace. Among others, they describe the classical action of the $N = 2$ Chern-Simons-matter theory written in the $N = 1$ superspace. They also calculate the complete leading log effective superpotential of the $N = 2$ Chern-Simons-matter theory in terms of a $N = 1$ background superfield. Lehum et al. in [46] presented the calculation of the two-point functions and the eective potential for the mass-deformed N = 3 Chern-Simons-matter theory in the context of the supereld formalism. Indeed the supersymmetric Chern-Simons theory has been the focus of researchers in the field after the advent of the paper [2] in which the $N = 6$ superconformal Chern-Simons-matter theory was introduced and investigated within the context of the AdS/CFT correspondence.

Santos et al. [56] analyzed the physical basis of the molecular biochirality, which is responsible by a Parity Violation Energy Dierence (for short, $PVED$) in some organic molecules such as amino acids, which occur as levogyrous enantiomers in Nature. They studied the role which is played by the $4 - D$ Chern-Simons theory in the origin of $PVED$. As the authors pointed out, $PVED$ arises from Chern-Simons Theory, Loop Quantum Gravity (for short, LQG) and modied Leitner-Okubo gravitational potentials.

It is well-known that large N quantum eld theories with matrix elds are without solutions in general. But the case is different with the Chern-Simons matter theories in an 't Hooft limit. As Jensen and Patil [36] pointed out these theories offer a theoretical lamppost, under which we can reliably compute many observables, like the S-matrix of massive phases, operator product expansion coefficients at fixed points, finite temperature response functions, and so on. In their paper, they have studied $U(N)_k$ Chern-Simons theory coupled to fundamental fermions and scalars in a large N 't Hooft limit. Moreover, they computed not only the thermal free energy at high temperature, but also the two- and three-point functions of simple gauge-invariant operators. They also showed that the outcome of their results support various dualities between Chern-Simons-matter theories with $N = 0, 1$, and 2 supersymmetry.

Lee and Park [54] investigated Chern-Simons theory with real places. In their paper, they generalized the arithmetic Chern-Simons theory over totally imaginary number fields to arbitrary number fields (with real places) by utilizing cohomology with compact support to deal with real places. They were also able to offer some non-trivial examples confined to non-abelian gauge group with coefficient $Z/2Z$ and the abelian cyclic gauge group with coefficient Z/nZ.

Costello and Yamazaki [15] also studied, among others, two-dimensional integrable field theories in the context of the four-dimensional Chern-Simons-type gauge theory. It is worth noticing that the integrable field theories are realized as effective theories for the four-dimensional theory coupled with two-dimensional surface defects as the author pointed out. They also computed their Lagrangians and the Lax operators satisfying the zero-curvature condition.

Costello and Li in [16] exhibited the coupling of holomorphic Chern-Simons theory at large N with Kodaira-Spencer gravity. Also they present a new anomaly cancellation mechanism at all loops in perturbation theory for open-closed topological B-model.

Halder and Minwalla [34] studied matter Chern-Simons theories in the large N limit. More specifically, they studied the large N $2+1$ dimensional fermions in the fundamental representation of a $SU(N)_k$ Chern-Simons gauge group in the presence of a uniform background magnetic field for the $U(1)$ global symmetry of this theory. They also investigated the simplest and best studied matter Chern-Simons theories in their paper, that is, the regular fermion theory and critical boson theory, i.e. the theory of a single multiplet of fundamental Wilson Fisher bosons interacting with a $U(N_B)$ Chern-Simons gauge field.

In [12], M. Blau et al. checked out the path integral for the partition function of Chern-Simons gauge theory with a compact gauge group on a general Seifert 3-manifold. As they pointed out, this approach extends previous results and relies on abelianisation, a background field method and local application of the Kawasaki Index theorem.

William and Williams [62] considered Chern-Simons theory as a deformation of a 3-dimensional BF theory that is partially holomorphic and partially topological. Indeed, they constructed a one-loop exact, and finite, quantization of the mixed topological-holomorphic theory on product 3-manifolds of the form $\Sigma \times M$, where Σ is a Riemann surface and M is a smooth 1-manifold. Further, they set forth a novel gauge that led naturally to a one-loop exact quantization of this BF theory and Chern-Simons theory. As they pointed out, this approach reveals several important features of Chern-Simons theory, of which the bulk-boundary correspondence of Chern-Simons theory with chiral WZW theory.

Andrianopoli et al. in [4] focused on the $AdS3$ N-extended Chern-Simons supergravity in the sense of Achucarro-Townsend and investigated its gauge symmetries. Moreover, they developed these gauge symmetries to a $BRST$ symmetry. Then they achieved its quantization by choosing suitable gaugefixings. The outcome led to quantum theories with different features. They presented the reproduction of the Ansatz by Alvarez, Valenzuela and Zanelli for the graphene fermion by a special choice of the gauge-fixing.

Bade and Beasely [6] discuss the situation in which gauge invariance in Chern-Simons-matter theories in three dimensions may require the Chern-Simons level k to be half-integral. This implies that the parity is violated. In this relation they investigate as a result the analytic aspects of this factorization for non-abelian gauge groups and general matter representations by utilizing the known formulas for the partition function. They also discuss in an appendix, the analytic continuation of torus knot observables in the $SU(2)$ Chern-Simons-matter theory.

Capozziello et al. in their paper [14], present a new model which simulates the motion of free electrons in graphene by the evolution of strings on manifold. In this relation, they present the construction of superconductors by graphene utilizing their model. In this respect, by breaking the gravitational-analogue symmetry of graphene sheets, they exhibit that the existence Chern-Simons bridge of two separated child sheets give rise to a Chern-Simons Wormhole.

There are many other recent research papers on Chern-Simons gauge field theories which we could not bring here because of the lack of space. We focused only on some of them in order to some extent report on the development of these theories in different aspects.

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