On the Derivatives of the Delta Function

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Abstract

In this writing the conventional law concerning the derivatives of the delta function and those representing the delta function have been considered to bring out certain discrepancies. The errors and their source have been discussed

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Introduction

The conventional derivative laws in relation to the delta function and their examples have been analyzed to bring out certain conflicting features. These conventional laws become questionable but not the concept of the delta function by itself in so far as this writing is concerned.

Inconsistencies with Derivatives of the Delta Function

We consider the fundamental result[1][2] on derivatives of the delta function as given below

$$\int_{-\infty}^{+\infty} f(x)\delta^n(x)\,dx = -\int_{-\infty}^{+\infty} f'(x)\delta^{n-1}(x)\,dx \quad (1)$$

The above holds for any arbitrary function as well as for any subinterval on $(-\infty, +\infty)$ and we have the following result[3]

$$f(x)\delta'(x) = -f'(x)\delta(x) \quad (2)$$

$$\delta'(x) = -\frac{f'(x)}{f(x)}\delta(x) \quad (3)$$

$$\int_{-\infty}^{+\infty} \delta'(x)dx = -\int_{-\infty}^{+\infty} \frac{f'(x)}{f(x)}\delta(x)$$

$$[\delta(x)]_{-\infty}{}^{+\infty} = -\frac{f'(0)}{f(0)}$$

$$0 = -\frac{f'(0)}{f(0)} \Rightarrow f'(0) = 0 \quad (4)$$

Since $f(x)$ Is an arbitrary function, well behaved in relation to continuity and differentiability of course,, equation (4) becomes questionable.

Equation (2) is differentiated with respect to x:

$$f'(x)\delta'(x) + f(x)\delta''(x) = -f'(x)\delta'(x) - f''(x)\delta(x)$$

$$2f'(x)\delta'(x) + f(x)\delta''(x) + f''(x)\delta(x) = 0$$

Applying (3) on the last equation we have,

$$-2f'(x)\left[\frac{f'(x)}{f(x)}\delta(x)\right] + f(x)\delta''(x) + f''(x)\delta(x) = 0$$

$$\delta(x)\left[f''(x) - 2\frac{[f'(x)]^2}{f(x)}\right] + f(x)\delta''(x) = 0 \quad (5)$$

$$\delta''(x) = -\frac{1}{f(x)}\left[f''(x) - 2\frac{[f'(x)]^2}{f(x)}\right]\delta(x) \quad (6)$$

$\delta''(x)$ depends on the nature of the test function $f(x)$ which is not an acceptable idea.

Integrating (4) with respect to x we obtain,

$$\int_{-\infty}^{+\infty} \delta(x)\left[f''(x) - 2\frac{[f'(x)]^2}{f(x)}\right]dx + \int_{-\infty}^{+\infty} f(x)\delta''(x)dx = 0 \ (7)$$

$$\left[f''(0) - 2\frac{[f'(0)]^2}{f(0)}\right] + \int_{-\epsilon}^{+\epsilon} f(x)\delta''(x)dx = 0$$

Since for $x \neq 0$,$\delta(x) = 0$ we have $\delta'(x) = 0$ and $\delta''(x) = 0$ [for $x \neq 0$]. Moreover from (5) $\delta''(x)$ is a peaked function like $\delta(x)$: $f(x)$ is expected to vary much slowly than $\delta''(x)$ on an infinitesimally small interval $-\epsilon < x < +\varepsilon$. Therefore

$$\left[f''(0) - 2\frac{[f'(0)]^2}{f(0)}\right] + f(0)\int_{-\epsilon}^{+\epsilon} \delta''(x)dx = 0$$

$$\int_{-\epsilon}^{+\epsilon} \delta''(x)dx = -\frac{1}{f(0)}\left[f''(0) - 2\frac{[f'(0)]^2}{f(0)}\right] \ (8)$$

$$[\delta'(x)]_{-\epsilon}{}^{+\epsilon} = -\frac{1}{f(0)}\left[f''(0) - 2\frac{[f'(0)]^2}{f(0)}\right]$$

$$-\frac{1}{f(0)}\left[f''(0) - 2\frac{[f'(0)]^2}{f(0)}\right] = 0$$

$$f''(0) = 2\frac{[f'(0)]^2}{f(0)} \quad (9)$$

The above formula[represented by (9)]is not acceptable

We consider the following result[4] :

$$x\delta'(x) = -\delta(x) \quad (10)$$

$$x^n\delta^n(x) = -n!\,(-1)^n\delta(x)$$

$$x^2\delta'^{(x)} = -x\delta(x)$$

$$\int_{-\infty}^{\infty} x^2\delta'\,(x) = -\int_{-\infty}^{+\infty} x\delta(x)\,dx$$

$$\Rightarrow \int_{-\epsilon}^{\epsilon} x^2\delta'\,(x)dx = -\int_{-\epsilon}^{+\epsilon} x\delta(x)\,dx = 0$$

$$\Rightarrow \int_{-\epsilon}^{\epsilon} x^2\delta'\,(x)dx = 0 \;(11)$$

The above is true of any arbitrary interval$(-\epsilon, \epsilon)$. Therefore $x^2\delta'(x)$ should be an odd function. Since $x^2$is an even function $\delta'(x)$ should be odd. That implies $\delta(x)$should be even.

Indeed by integration

$$\int \delta'(x)\,dx = f_{even}(x)$$

$$\delta(x) = f_{even}(x)$$

Since a constant is an even function it may be included in $f_{even}(x)$

[In general any arbitrary function may be expressed as the sum of an even and an odd function. If the even part is not a constant the derivative will be the sum of an even and an odd function.]

Now we consider

$$x^3\delta'(x) = -x^2\delta(x)$$

$$\int_{-\infty}^{\infty} x^3 \delta'(x) = -\int_{-\infty}^{+\infty} x^2 \delta(x)\, dx$$

$$\Rightarrow \int_{-\epsilon}^{\epsilon} x^3 \delta'(x) dx = -\int_{-\epsilon}^{+\epsilon} x^2 \delta(x)\, dx = 0$$

$$\int_{-\epsilon}^{\epsilon} x^3 \delta'(x) dx = 0 \ (12)$$

The above is true of any $(-\epsilon, \epsilon)$. Therefore δ'(x) should be an even function. With $\delta'(x)$ we have

$$\int \delta'(x)\, dx = f_{odd}(x) + C$$

$$\delta(x) = f_{odd}(x) + C$$

[In general any arbitrary function may be expressed as the sum of an even and an odd function. If the even part is not a constant the derivative will be the sum of an even and an odd function.]

and $\delta(x)$, consequently, an odd function at most with an additive constant as opposed to what we saw earlier: $\delta(x)$= $f_{even}(x)$,

From (10) cannot arrive at (2) by power series technique:$(10) \nRightarrow (2)$. Consequently

Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots .... + a_2 x^2 + a_1 x + a_0$$

$$f'(x) = a_n n x^{n-1} + (n-1) a_{n-1} x^{n-2} + (n-2) a_{n-2} x^{n-3} + \cdots .... + 2a_2 x + a_1$$

From the above expansions it is evident that $f(x)\delta'(x) = -f'(x)\delta(x) \nRightarrow x\delta'(x) = -\delta(x)$ and

$x\delta'(x) = -\delta(x) \nRightarrow f(x)\delta'(x) = -f'(x)\delta(x)$ though $f'(x) = 1$ if $f(x) = x$

The reason ,as we shall see soon is , that for each function f(x) we require a separate sequence of functions representing the delta function: we have to consider distributions: mapping from functions to real numbers in the form of a linear functional . Even that does not help as we shall see. The delta function, as we know and the idea is a highlighted one in literature , is not a function in the usual sense of being a function. We have ignored this fact while arriving at the contradiction.

Next we consider the standard formula[5]

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx} dx \quad (13)$$

Differentiating with respect to x, we have,

$$\Rightarrow \delta'(x) = ix\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx} dk = ik\delta(x)$$

$$\delta'(x) = ix\delta(x) \quad (14)$$

The result given by (14) stands opposed to the standard result given by (10)

$$x\delta'(x) = -\delta(x) \quad (15)$$

The Delta Function in Formal Theory and with Applications

We consider the formal definition of the delta function[6] as a distribution, The Dirac delta function is a linear functional that maps every function to its value at zero. ... In many applications, the Dirac delta is regarded as a kind of limit (a weak limit) of a sequence of functions having a tall spike at the origin (in theory of distributions, this is a true limit).

$$\langle \delta, \varphi \rangle = \varphi(0) \ (16)$$

We have a mapping from a function to a real number[functional]

$\delta: \varphi \to \varphi(0)$

By way of example the mapping may be achieved as

$$\int_{-\infty}^{+\infty} \varphi(\mathrm{x})\delta(\mathrm{x})\mathrm{dx} = \varphi(0) \quad (17)$$

The above example is relevant in applications like physics . As an example[7] we may refer to the derivation of Helmholtz theorem where the following is considered

$$\nabla \mathrm{F} = -\nabla^2 \mathrm{U} = -\frac{1}{4\pi}\int \mathrm{D}(\vec{\mathrm{r}}')\nabla^2 \frac{1}{|\vec{\mathrm{r}}' - \vec{\mathrm{r}}|}\mathrm{dV}' = \int \mathrm{D}(\vec{\mathrm{r}}')\delta(\vec{\mathrm{r}}' - \vec{\mathrm{r}})\mathrm{dV}' = \mathrm{D}(\vec{\mathrm{r}}) \quad (18)$$

But we have seen the serious errors with

$$\int_{-\infty}^{+\infty} \varphi(\mathrm{x})\delta(\mathrm{x})\mathrm{dx} = \varphi(0)$$

when it comes to the derivatives

Further Investigation[for locating the source of error]

A distribution is a mapping from a asset of functions to real numbers. To that end we consider a sequence of functions $G_n(x)$ such that

1. $G_n(x) \neq 0$for $-\epsilon_1(n) < x < \epsilon_2(n)$ else $G_n(x) = 0; \epsilon_1(n) > 0, \epsilon_2(n) > 0$
2. . $lim_{n\to\infty}\epsilon_i(n) = 0; i = 1,2; lim_{n\to\infty}G_n(0) = \infty$;and $lim_{n\to\infty}\int_{-\infty}^{+\infty} G_n(x) = 1$

Next we consider for very large 'n'[n tending to infinity] $lim_{n\to\infty}\int_{-\infty}^{+\infty} f(x)G_n(x)\,dx$

$$\int_{-\infty}^{+\infty} f(x)G_n(x)\,dx = \int_{-\infty}^{-\epsilon_1(n))} f(x)G_n(x)dx + \int_{-\epsilon_1(n)}^{+\epsilon_2(n)} f(x)G_n(x)dx + \int_{\epsilon_2(n)}^{+\infty} f(x)G_n(x)dx$$

Since $\int_{-\infty}^{-\epsilon_1(n)} f(x)G_n(x)dx = 0$and $\int_{\epsilon_2(n)}^{+\infty} f(x)G_n(x)dx = 0$ we have

$$\int_{-\infty}^{+\infty} f(x)G_n(x)\,dx = \int_{-\epsilon_1(n)}^{+\epsilon_2(n)} f(x)G_n(x)\,dx \quad (19)$$

If $f(x)$ changes much slowly with respect to $G_n(x)$ on the interval $-\epsilon_1(n) < x < \epsilon_2(n)$

$$lim_{n\to\infty}\int_{-\epsilon_1(n)}^{+\epsilon_2(n)} f(x)G_n(x)\,dx = f(0);$$

$$lim_{n\to\infty}\int_{-\infty}^{+\infty} f(x)G_n(x)\,dx = f(0)\ (16)$$

Equation (16) relates to the defining criterion for the delta function

For the nth function

$$\int_{-\infty}^{+\infty} f(x){G_n}'(x)\,dx = \int_{-\epsilon(n)}^{+\epsilon(n)} f(x){G_n}'(x)\,dx = [f(x)G_n(x)]_{-\epsilon(n)}{}^{+\epsilon(n)} - \int_{-\epsilon(n)}^{+\epsilon(n)} f'(x)G_n(x)\,dx \quad (20)$$

Now for n tending to infinity $G_n(x) \to \infty$ and at the same time we may consider $f(x) \ll G_n(x)$.We may replace $f(x)$ by a straight line[tangent] on $(-\varepsilon, \varepsilon)$. $f(x)G_n$ will not be exactly symmetrical unless f(x) isparallrel to the x axis on the interval$(-\varepsilon, \varepsilon)$.

We may rewrite equation (17) as

$$[f(x)G_n'(x)]_{Avg}(\varepsilon_2 + \varepsilon_1) = f(\varepsilon_2)G_n(\varepsilon_2) - f(-\varepsilon_1)G_n(-\varepsilon_1) - f'(0)\ (21)$$

The first term on the right side of (17) or (18) may be written as

$$f(\varepsilon_2)G_n(\varepsilon_2) - f(-\varepsilon_1)G_n(-\varepsilon_1) = f(\varepsilon_2)G_n(\varepsilon_2) - f(\varepsilon_2)G_n(-\varepsilon_1) + f(\varepsilon_2)G_n(-\varepsilon_1) - f(-\varepsilon_1)G_n(-\varepsilon_1)$$

$$= f(\varepsilon_2)[G_n(\varepsilon_2) - G_n(-\varepsilon_1)] + [f(\varepsilon_2) - f(-\varepsilon_1)]G_n(-\varepsilon_1)$$

We recall the fundamental law for derivatives of the delta function:

$$\int_{-\infty}^{+\infty} f(x){G_n}'(x)\,dx = -\int_{-\epsilon(n)}^{+\epsilon(n)} f'(x)G_n(x)\,dx$$

When viewed in respect of (20) we have,

$$\left|[f(x)G_n(x)]_{-\epsilon(n)}{}^{+\epsilon(n)}\right| \ll \left|\int_{-\epsilon(n)}^{+\epsilon(n)} f'(x)G_n(x)\,dx\right| \quad (22)$$

$$\Rightarrow [f(x)G_n(x)]_{-\epsilon(n)}{}^{+\epsilon(n)} \ll f'(0)$$

$$|f(\varepsilon_2)G_n(\varepsilon_2) - f(-\varepsilon_1)G_n(-\varepsilon_1)| \ll |f'(0)|$$

$$\left|\frac{f(\varepsilon_2)G_n(\varepsilon_2) - f(-\varepsilon_1)G_n(-\varepsilon_1)}{\varepsilon_2 - (-\varepsilon_1)}\right| \left(\varepsilon_2 - (-\varepsilon_1)\right) \ll |f'(0)|$$

$$\left|\frac{f(\varepsilon_2)[G_n(\varepsilon_2) - G_n(-\varepsilon_1)] + [f(\varepsilon_2) - f(-\varepsilon_1)]G_n(-\varepsilon_1)}{\varepsilon_2 - (-\varepsilon_1)}\right| (\varepsilon_2 + \varepsilon_1) \ll |f'(0)|; (\varepsilon_2 + \varepsilon_1) > 0$$

$$\left[f(\varepsilon_2)\frac{[G_n(\varepsilon_2) - G_n(-\varepsilon_1)]}{\varepsilon_2 - (-\varepsilon_1)} + G_n(-\varepsilon_1)\frac{[f(\varepsilon_2) - f(-\varepsilon_1)]G_n(-\varepsilon_1)}{\varepsilon_2 - (-\varepsilon_1)}\right] (\varepsilon_2 + \varepsilon_1) \ll |f'(0)|$$

On taking limits on either side: $n \to \infty, Max\{\varepsilon_1, \varepsilon_2\} \to 0$

$$|f(0)G_n'(0) + G_n(0)f'(0)|(\varepsilon_2 + \varepsilon_1) \ll |f'(0)|$$

$$\left|\frac{f(0)}{f'(0)}G_n'(0) + G_n(0)\right| (\varepsilon_2 + \varepsilon_1) \ll 1$$

$$\left|\frac{f(0)}{f'(0)}G_n'(0)(\varepsilon_2 + \varepsilon_1) + G_n(0)(\varepsilon_2 + \varepsilon_1)\right| \ll 1$$

Since $lim_{n\to\infty} G_n(x)(\varepsilon_2 + \varepsilon_1) = 1; -\varepsilon_1 < x < \varepsilon_2$

$$\left|\frac{f(0)}{f'(0)}G_n'(0)(\varepsilon_2 + \varepsilon_1) + 1\right| \ll 1 \quad (23)$$

The last relation will fail if $\left|\frac{f(0)}{f'(0)} G_n'(0)\right| > 0$ [keeping in mind that $(\varepsilon_2 + \varepsilon_1) > 0$ and also with $n \to \infty, G_n(0) \to \infty, G_n'(0) \to \infty$. Therefore equation (20) represents an absurd relation.

If we have a look at the fundamental equation for the derivative of the delta function we may impose

Again imposing $f(x)G_n'(x) = -f'(x)G_n(x)$for very large n]that is by using equation (2)

$$\frac{G_n{}'}{G_n} = -\frac{f'(x)}{f(x)}$$

$$\int \frac{G_n{}'}{G_n} dx = -\int \frac{f'(x)}{f(x)} dx$$

$$ln(G_n/C) = -lnf(x)$$

$$G_n = -Cf(x) \ (24)$$

Equation (21) indicates that each function f(x) requires its own distinct sequence$\{G_n\}$ representing the delta function. Even that will not suffice[example:$\delta(x)$ is both even and odd as we saw earlier in the section “**Inconsistencies with Derivatives of the Delta Function]**

As $n \to \infty, C_n \to \infty$

In the vicinity of zero f(x) will be a straight line of positive ,negative or zero gradient: we may replace f(x) by the tangent at x=0

in the vicinity of the x axis we may approximate $f(x) = ax + b$

In any case

$$G_n = -Cf(x) = -C_n(ax + b) \ (25)$$

or $G_n \approx -C_n(ax + b)$

We apply equation (22) on the relation: $f(x)G_n'(x) = -f'(x)G_n(x)$

$$-(ax + b)C_n = aC_n(ax + b)$$

Considering C_n arbitrarily large but finite

$$(1 + a)(ax + b)C_n = 0(26)$$

$C_n = 0$or $a = -1$ or $ax + b = 0 \Rightarrow b \approx 0$ since $x \approx 0$ on $(-\epsilon, \epsilon)$

The derivative law in relation to the as given by (10) is at stake .But the delta function as represented by equation (16) , viewed independent of the conventional derivative law as given by (1), stands unchallenged .

[We may consider $\int_{-\infty}^{+\infty} G_n(x) = 1$for all n in place of and $lim_{n\to\infty} \int_{-\infty}^{+\infty} G_n(x) = 1$. Theconclusionswe have arrived at in this article remain unaffected].

The derivatives of the delta function in so far as the conventional law is concerned its allied derivations come into question.

Conclusions

As claimed the analysis of the Delta function brings out unacceptable features in relation to the conventional law in regarding its derivatives. The concept of the delta function viewed independent of these conventional derivative law stands unchallenged.

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