On the Derivatives of the Delta Function

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Abstract

In this writing the conventional law concerning the derivatives of the delta function and those representing the delta function have been considered to bring out certain discrepancies. The errors and their source have been discussed

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Introduction

The conventional derivative laws in relation to the delta function and their examples have been analyzed to bring out certain conflicting features.

Inconsistencies with Derivatives of the Delta Function

We consider the fundamental result[1][2] on derivatives of the delta function as given below

$$\int_{-\infty}^{+\infty} f(x)\delta^{n}(x)\,dx = -\int_{-\infty}^{+\infty} f'(x)\delta^{n-1}(x)\,dx \quad (1)$$

The above holds for any arbitrary function and we have the following result[3]

$$f(x)\delta'(x) = -f'(x)\delta(x) \quad (2)$$

But we have considered the same delta function for all f(x). This notion will be proved erroneous in the article later.

It would be considered more reasonable to consider (1) valid for any subinterval on $(-\infty, +\infty)$ and then conclude(2). If the subinterval does not contain zero each side of (1) would be zero else each side would be the same non zero quantity.

$$\int_{-\epsilon}^{+\epsilon} f(x)\delta^{n}(x)\,dx = -\int_{-\epsilon}^{+\epsilon} f'(x)\delta^{n-1}(x)\,dx$$

$$\delta'(x) = -\frac{f'(x)}{f(x)}\delta(x) \quad (3)$$

$$\int_{-\infty}^{+\infty} \delta'(x)dx = -\int_{-\infty}^{+\infty} \frac{f'(x)}{f(x)}\delta(x)$$

$$[\delta(x)]_{-\infty}{}^{+\infty} = -\frac{f'(0)}{f(0)}$$

$$0 = -\frac{f'(0)}{f(0)} \Rightarrow f'(0) = 0 \quad (4)$$

Since $f(x)$ Is an arbitrary function, well behaved in relation to continuity and differentiability of course,, equation (4) becomes questionable.

Equation (2) is differentiated with respect to x:

$$f'(x)\delta'(x) + f(x)\delta''(x) = -f'(x)\delta'(x) - f''(x)\delta(x)$$

$$2f'(x)\delta'(x) + f(x)\delta''(x) + f''(x)\delta(x) = 0$$

Applying (3) on the last equation we have,

$$-2f'(x)\left[\frac{f'(x)}{f(x)}\delta(x)\right] + f(x)\delta''(x) + f''(x)\delta(x) = 0$$

$$\delta(x)\left[f''(x) - 2\frac{[f'(x)]^2}{f(x)}\right] + f(x)\delta''(x) = 0 \quad (5)$$

$$\delta''(x) = -\frac{1}{f(x)}\left[f''(x) - 2\frac{[f'(x)]^2}{f(x)}\right]\delta(x) \quad (6)$$

$\delta''(x)$ depends on the nature of the test function $f(x)$ which is not an acceptable idea.

Integrating (4) with respect to x we obtain,

$$\int_{-\infty}^{+\infty} \delta(x)\left[f''(x) - 2\frac{[f'(x)]^2}{f(x)}\right]dx + \int_{-\infty}^{+\infty} f(x)\delta''(x)dx = 0 \ (7)$$

$$\left[f''(0) - 2\frac{[f'(0)]^2}{f(0)}\right] + \int_{-\epsilon}^{+\epsilon} f(x)\delta''(x)dx = 0$$

Since for $x \neq 0$,$\delta(x) = 0$ we have $\delta'(x) = 0$ and $\delta''(x) = 0$ [for $x \neq 0$]. Moreover from (5) $\delta''(x)$ is a peaked function like $\delta(x)$: $f(x)$ is expected to vary much slowly than $\delta''(x)$ on an infinitesimally small interval $-\epsilon < x < +\varepsilon$. Therefore

$$\left[f''(0) - 2\frac{[f'(0)]^2}{f(0)}\right] + f(0)\int_{-\epsilon}^{+\epsilon} \delta''(x)dx = 0$$

$$\int_{-\epsilon}^{+\epsilon} \delta''(x)dx = -\frac{1}{f(0)}\left[f''(0) - 2\frac{[f'(0)]^2}{f(0)}\right] \ (8)$$

$$[\delta'(x)]_{-\epsilon}{}^{+\epsilon} = -\frac{1}{f(0)}\left[f''(0) - 2\frac{[f'(0)]^2}{f(0)}\right]$$

$$-\frac{1}{f(0)}\left[f''(0) - 2\frac{[f'(0)]^2}{f(0)}\right] = 0$$

$$f''(0) = 2\frac{[f'(0)]^2}{f(0)} \ \ (9)$$

The above formula[represented by (9)]is not acceptable

We consider the following result[4] :

$$x\delta'(x) = -\delta(x) \ \ (10)$$

$$x^n\delta^n(x) = -n!\,(-1)^n\delta(x)$$

$$x^2\delta'^{(x)} = -x\delta(x)$$

$$\int_{-\infty}^{\infty} x^2\delta'\,(x) = -\int_{-\infty}^{+\infty} x\delta(x)\,dx$$

$$\Rightarrow \int_{-\epsilon}^{\epsilon} x^2\delta'\,(x)dx = -\int_{-\epsilon}^{+\epsilon} x\delta(x)\,dx = 0$$

$$\Rightarrow \int_{-\epsilon}^{\epsilon} x^2\delta'\,(x)dx = 0 \ (11)$$

The above is true of any arbitrary interval$(-\epsilon, \epsilon)$. Therefore $x^2\delta'(x)$ should be an odd function. Since $x^2$is an even function $\delta'(x)$ should be odd. That implies $\delta(x)$should be even.

Indeed by integration

$$\int \delta'(x)\,dx = f_{even}(x)$$

$$\delta(x) = f_{even}(x)$$

Since a constant is an even function it may be included in $f_{even}(x)$

[In general any arbitrary function may be expressed as the sum of an even and an odd function. If the even part is not a constant the derivative will be the sum of an even and an odd function.]

Now we consider

$$x^3\delta'(x) = -x^2\delta(x)$$

$$\int_{-\infty}^{\infty} x^3\delta'(x) = -\int_{-\infty}^{+\infty} x^2\delta(x)\,dx$$

$$\Rightarrow \int_{-\epsilon}^{\epsilon} x^3\delta'(x)dx = -\int_{-\epsilon}^{+\epsilon} x^2\delta(x)\,dx = 0$$

$$\int_{-\epsilon}^{\epsilon} x^3\delta'(x)dx = 0\ (12)$$

The above is true of any $(-\epsilon, \epsilon)$. Therefore δ'(x) should be an even function. With $\delta'(x)$ we have

$$\int \delta'(x)\,dx = f_{odd}(x) + C$$

$$\delta(x) = f_{odd}(x) + C$$

[In general any arbitrary function may be expressed as the sum of an even and an odd function. If the even part is not a constant the derivative will be the sum of an even and an odd function.]

and $\delta(x)$, consequently, an odd function at most with an additive constant as opposed to what we saw earlier: $\delta(x)$= $f_{even}(x)$,

From (10) cannot arrive at (2) by power series technique:$(10) \nRightarrow (2)$. Consequently

Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots \ldots . + a_2x^2 + a_1x + a_0$$

$$f'(x) = a_n n x^{n-1} + (n-1)a_{n-1}x^{n-2} + (n-2)a_{n-2}x^{n-3} + \cdots \ldots . + 2a_2x + a_1$$

From the above expansions it is evident that $f(x)\delta'(x) = -f'(x)\delta(x) \nRightarrow x\delta'(x) = -\delta(x)$ and

$x\delta'(x) = -\delta(x) \nRightarrow f(x)\delta'(x) = -f'(x)\delta(x)$ though $f'(x) = 1$ if $f(x) = x$

The reason ,as we shall see soon is , that for each function f(x) we require a separate sequence of functions representing the delta function: we have to consider distributions: mapping from functions to real numbers in the form of a linear functional . Even that does not help as we shall see. The delta function, as we know and the idea is a highlighted one in literature , is not a function in the usual sense of being a function. We have ignored this fact while arriving at the contradiction.

Next we consider the standard formula[5]

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx} dx \quad (13)$$

Differentiating with respect to x, we have,

$$\Rightarrow \delta'(x) = ix\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx} dk = ik\delta(x)$$

$$\delta'(x) = ix\delta(x) \quad (14)$$

The result given by (14) stands opposed to the standard result given by (10)

$$x\delta'(x) = -\delta(x) \quad (15)$$

The Delta Function in Formal Theory and with Applications

We consider the formal definition of the delta function[6] as a distribution, The Dirac delta function is a linear functional that maps every function to its value at zero. ... In many applications, the Dirac delta is regarded as a kind of limit (a weak limit) of a sequence of functions having a tall spike at the origin (in theory of distributions, this is a true limit).

$$\langle \delta, \varphi \rangle = \varphi(0) \ (16)$$

We have a mapping from a function to a real number[functional]

$\delta: \varphi \rightarrow \varphi(0)$

By way of example the mapping may be achieved as

$$\int_{-\infty}^{+\infty} \varphi(x)\delta(x)dx = \varphi(0) \quad (17)$$

The above example is relevant in applications like physics . As an example[7] we may refer to the derivation of Helmholtz theorem where the following is considered

$$\nabla F = -\nabla^2 U = -\frac{1}{4\pi}\int D(\vec{r}')\nabla^2 \frac{1}{|\vec{r}' - \vec{r}|} dV' = \int D(\vec{r}')\delta(\vec{r}' - \vec{r})dV' = D(\vec{r}) \quad (18)$$

But we have seen the serious errors with

$$\int_{-\infty}^{+\infty} \varphi(x)\delta(x)dx = \varphi(0)$$

when it comes to the derivatives

Further Investigation[for locating the source of error]

A distribution is a mapping from a asset of functions to real numbers. To that end we consider a sequence of functions $G_n(x)$ such that

1. $G_n(x) \neq 0$for $-\epsilon_1(n) < x < \epsilon_2(n)$ else $G_n(x) = 0; \epsilon_1(n) > 0, \epsilon_2(n) > 0$

2. . $lim_{n\to\infty}\epsilon_i(n) = 0; i = 1,2; lim_{n\to\infty}G_n(0) = \infty$;and $lim_{n\to\infty}\int_{-\infty}^{+\infty} G_n(x) = 1$

Next we consider for very large 'n'[n tending to infinity] $lim_{n\to\infty}\int_{-\infty}^{+\infty} f(x)G_n(x)\,dx$

$$\int_{-\infty}^{+\infty} f(x)G_n(x)\,dx = \int_{-\infty}^{-\epsilon_1(n))} f(x)G_n(x)dx + \int_{-\epsilon_1(n)}^{+\epsilon_2(n)} f(x)G_n(x)dx + \int_{\epsilon_2(n)}^{+\infty} f(x)G_n(x)dx$$

Since $\int_{-\infty}^{-\epsilon_1(n)} f(x)G_n(x)dx = 0$and $\int_{\epsilon_2(n)}^{+\infty} f(x)G_n(x)dx = 0$ we have

$$\int_{-\infty}^{+\infty} f(x)G_n(x)\,dx = \int_{-\epsilon_1(n)}^{+\epsilon_2(n)} f(x)G_n(x)\,dx \quad (19)$$

If $f(x)$ changes much slowly with respect to $G_n(x)$ on the interval $-\epsilon_1(n) < x < \epsilon_2(n)$

$$lim_{n\to\infty}\int_{-\epsilon_1(n)}^{+\epsilon_2(n)} f(x)G_n(x)\,dx = f(0);$$

$$lim_{n\to\infty}\int_{-\infty}^{+\infty} f(x)G_n(x)\,dx = f(0) \ (16)$$

Equation (16) relates to the defining criterion for the delta function

For the nth function

$$\int_{-\infty}^{+\infty} f(x)G_n'(x)\,dx = \int_{-\epsilon(n)}^{+\epsilon(n)} f(x)G_n'(x)\,dx = [f(x)G_n(x)]_{-\epsilon(n)}^{+\epsilon(n)} - \int_{-\epsilon(n)}^{+\epsilon(n)} f'(x)G_n(x)\,dx \quad (20)$$

$$\int_{-\infty}^{+\infty} f(x){G_n}'(x)\,dx = \int_{-\epsilon(n)}^{+\epsilon(n)} f(x){G_n}'(x)\,dx = [f(x)G_n(x)]_{-\epsilon(n)}{}^{+\epsilon(n)} - f'(0) \quad (20')$$

Case1

We assume

$$lim_{n\to\infty}[f(x)G_n(x)]_{-\epsilon(n)}{}^{+\epsilon(n)} = 0$$

together with (20),we have the fundamental laws given by (1)

We have from (20')

$$\int_{-\infty}^{+\infty} f(x){G_n}'(x)\,dx = -\int_{-\epsilon(n)}^{+\epsilon(n)} f'(x)G_n(x)\,dx = -f'(0) = finite$$

for sufficiently large 'n'

$$\Rightarrow lim_{n\to\infty}\int_{-\infty}^{+x;|x|\epsilon} f(x){G_n}'(x)\,dx = lim_{n\to\infty}a_n(x) = f'(0)finite$$

Differentiating with respect to x: $f(x)G_n'(x) = {a_n}'(x)finite$ for arbitrary x

${a_n}'(x)$ cannot remain finite as n tends to infinity if${G_n}'$ tends to infinity

The fundamental law is not working unless G_n is finite with n tending to infinity

Case 2

We assume for sufficiently large 'n'

$$[f(x)G_n(x)]_{-\epsilon(n)}{}^{+\epsilon(n)} = a_n = finite\ and\ non\ zero$$

$f(x)$ and $G_n(x)$ are continuous on $(-\epsilon,\epsilon)$

$$\left|\frac{f(\varepsilon_2)G_n(\varepsilon_2) - f(-\varepsilon_1)G_n(-\varepsilon_1)}{\varepsilon_2 - (-\varepsilon_1)}\right|\big(\varepsilon_2 - (-\varepsilon_1)\big) = finite\ non\ zero$$

$$\left|\frac{f(\varepsilon_2(n))G_n\big(\varepsilon_2(n)\big) - f(-\varepsilon_1(n))G_n\big(-\varepsilon_1(n)\big)}{\varepsilon_2 - (-\varepsilon_1)}\right|\Big(\varepsilon_2(n) - \big(-\varepsilon_1(n)\big)\Big) = a_n$$

$$\left|\frac{f(\varepsilon_2)[G_n(\varepsilon_2)-G_n(-\varepsilon_1)]+[f(\varepsilon_2)-f(-\varepsilon_1)]G_n(-\varepsilon_1)}{\varepsilon_2-(-\varepsilon_1)}\right|(\varepsilon_2+\varepsilon_1)=a_n$$

$$\left[f(\varepsilon_2)\frac{[G_n(\varepsilon_2)-G_n(-\varepsilon_1)]}{\varepsilon_2-(-\varepsilon_1)}+G_n(-\varepsilon_1)\frac{[f(\varepsilon_2)-f(-\varepsilon_1)]G_n(-\varepsilon_1)}{\varepsilon_2-(-\varepsilon_1)}\right](\varepsilon_2+\varepsilon_1)=a_n$$

$lim_{n\to\infty}\left[f(\varepsilon_2)\frac{[G_n(\varepsilon_2)-G_n(-\varepsilon_1)]}{\varepsilon_2-(-\varepsilon_1)}+G_n(-\varepsilon_1)\frac{[f(\varepsilon_2)-f(-\varepsilon_1)]G_n(-\varepsilon_1)}{\varepsilon_2-(-\varepsilon_1)}\right](\varepsilon_2+\varepsilon_1)=finite\ non\ zero=$ a'[bounded]

$$|f(0)G_n'(0)+G_n(0)f'(0)|(\varepsilon_2+\varepsilon_1)=a'$$

$$lim_{n\to\infty}\left|\int_{-\infty}^{+\infty}f(x)G_n'(x)\,dx+\int_{-\infty}^{+\infty}G_n(x)f'(x)dx\right|=a'$$

$$lim_{n\to\infty}\left|\int_{-\infty}^{+\infty}f(x)G_n'(x)\,dx+f'(0)\right|=a'$$

$$lim_{n\to\infty}\int_{-\infty}^{+x:|x|<\epsilon}f(x)G_n'(x)\,dx=a''=finite$$

$$lim_{n\to\infty}\int_{-\infty}^{x:|x|<\epsilon}f(x)G_n'(x)dx=h(x)=finite\quad(28)$$

Differentiating both sides with respect to $x$we have: $f(x)G_n'(x)=h'(x)$,finite$\Longrightarrow$ $G_n'(x)$:finite for all ‘n’ and for all x for non zero f(x)and hence $G_n(x)$finite for all ‘n’ which is not true.

Case 3

We assume

$$[f(x)G_n(x)]_{-\epsilon(n)}{}^{+\epsilon(n)}=\infty$$

and also that both $f(x)$ and $G_n(x)$ are continuous on $(-\epsilon,\epsilon)$

From (20’)

$$\int_{-\infty}^{+\infty}f(x)G_n'(x)\,dx$$

becomes divergent

$$\left|\int_{-\infty}^{+\infty}f(x)G_n'(x)\,dx+\int_{-\infty}^{+\infty}G_n(x)f'(x)dx\right|=\infty$$

$$\left|\int_{-\infty}^{+\infty} f(x)G_n'(x)\,dx + f'(0)\right| = \infty$$

$$\left|\int_{-\infty}^{+\infty} f(x)G_n'(x)\,dx\right| = \infty$$

$$\left|\int_{-\epsilon}^{+\epsilon} f(x)G_n'(x)\,dx\right| = \infty$$

$$\left|\int_{-\infty}^{+\infty} f(x)G_n'(x)\,dx\right| = \infty$$

The above integral is of divergent nature.

The derivative law in relation to the as given by (10) is at stake in that and we do not have the fundamental theorem for the first order derivative of the delta function. The delta function itself is at stake.

[We may consider $\int_{-\infty}^{+\infty} G_n(x) = 1$for all n in place of and $lim_{n\to\infty} \int_{-\infty}^{+\infty} G_n(x) = 1$. The conclusions we have arrived at in this article remain unaffected].

Conclusions

As claimed the analysis of the Delta function brings out unacceptable features in relation to the conventional law in regarding its derivatives.

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