From Taylor Series

Anamitra Palit

palit.anamitra@gmail.com

Cell:+919163892336

Abstract

The writing intends to bring out certain inconsistent aspects relating to the Taylor expansion. The Taylor series is not an identity that leads to a host of problems.

Introduction

The Taylor series is well known for its application in mathematics and in physics. The article brings out some anomalous features about the Taylor expansion

Various Inconsistencies

Case 1.

We consider

$$f(x+2h)=f\big((x+h)+h\big)\ (1)$$

Expanding about $(x+h)$

$$f(x+2h)=f(x+h)+\frac{h}{1!}f'(x+h)+\frac{h^2}{2!}f''(x+h)+\frac{h^3}{3!}f'''(x+h)+\cdots\ldots.(2)$$

Expanding about $x=x$

$$f(x+2h)=f(x)+\frac{2h}{1!}f'(x+h)+\frac{4h^2}{2!}f''(x+h)+\frac{8h^3}{3!}f'''(x+h)+\cdots\ldots.(3)$$

From (2) and (3)

$$f(x+h)+\frac{h}{1!}f'(x+h)+\frac{h^2}{2!}f''(x+h)+\frac{h^3}{3!}f'''(x+h)+\cdots\ldots.$$
$$=f(x)+\frac{2h}{1!}f'(x)+\frac{4h^2}{2!}f''(x)+\frac{8h^3}{3!}f'''(x)+\cdots\ldots$$

$$f(x+h)-f(x)+h[f'(x+h)-2f'(x)]+\frac{1}{2!}h^2[f''(x+h)-4f''(x)]$$
$$+\frac{1}{3!}h^3[f'''(x+h)-8f'''(x)]+\cdots..=0\ (4)$$

$$\frac{f(x+h)-f(x)}{h}\frac{1}{h}+\frac{[f'(x+h)-2f'(x)]}{h}+\frac{1}{2!}[f''(x+h)-4f''(x)]+\frac{1}{3!}h[f'''(x+h)-8f'''(x)] + \cdots .. = 0$$

$$\frac{f(x+h)-f(x)}{h}\frac{1}{h}+\frac{[f'(x+h)-f'(x)]}{h}-\frac{f'(x)}{h}+\frac{1}{2!}[f''(x+h)-4f''(x)] + \frac{1}{3!}h[f'''(x+h)-8f'''(x)]+h[\dots..]=0 \ (5)$$

Equation (5) is considered for $h \neq 0$. Even when we go for $h \to 0$, h does not become equal to zero. It is in the neighborhood of zero without becoming equal to zero]

$$\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]\frac{1}{h}+\frac{[f'(x+h)-f'(x)]}{h}+\frac{1}{2!}[f''(x+h)-4f''(x)] + \frac{1}{3!}h[f'''(x+h)-8f'''(x)]+h[\dots..]=0$$

$$Lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]\frac{1}{h}+Lim_{h\to 0}\frac{[f'(x+h)-f'(x)]}{h} + \frac{1}{2!}Lim_{h\to 0}[f''(x+h)-4f''(x)]+\frac{1}{3!}Lim_{h\to 0}h[f'''(x+h)-8f'''(x)]+h[\dots..] = 0 \ \ (6)$$

We are considering a function for which

$$Lim_{h\to 0}h[f'''(x+h)-8f'''(x)]+h[\dots..]=0$$

Then

$$Lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]\frac{1}{h}+f''(x)-\frac{3}{2}f''(x)=0$$

$$Lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]\frac{1}{h}=\frac{1}{2}f''(x)$$

$$Lim_{h\to 0}\frac{\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]}{h}=\frac{1}{2}f''(x) \ (7)$$

We apply L' Hospital's rule[1`] to obtain

$$Lim_{h\to 0}\frac{\frac{d}{dh}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]}{1}=\frac{1}{2}f''(x)$$

$$Lim_{h\to 0}\frac{\frac{d}{dh}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]}{1}=\frac{1}{2}f''(x)$$

$$Lim_{h\to 0}\left[-\frac{1}{h^2}\big(f(x+h)-f(x)\big)+\frac{1}{h}\big(f'(x+h)-f'(x)\big)\right]=\frac{1}{2}f''(x)$$

$$\left[-Lim_{h\to 0}\frac{1}{h^2}\big(f(x+h)-f(x)\big)+Lim_{h\to 0}\frac{1}{h}\big(f'(x+h)-f'(x)\big)\right]=\frac{1}{2}f''(x)(8)$$

$$-\infty+f''(x)=\frac{1}{2}f''(x)$$

$$-\frac{1}{2}f''(x)=-\infty$$

As claimed we have brought out an aspect of inconsistency with Taylor Series.

Case 2.Let us have another situation for our analysis. We write the Taylor series

$$f(x_0+h)=f(x_0)+\frac{h}{1!}f'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+\cdots\ldots(9).$$

The increment h may be sufficiently large subject to the fact that the series has to converge.

$$\frac{\partial f(x_0+h)}{\partial h}=f'(x_0)+hf''(x_0)+\frac{h^2}{2!}f'''(x_0)+\cdots(10).$$

$$lim_{h\to 0}\frac{\partial f(x_0+h)}{\partial h}=f'(x_0)\ \ (11)$$

$$lim_{h\to 0}F_h(x_0+h)=f'(x_0)$$

The limit $f'(x_0)$ is independent of h.This is an example of uniform convergence . We may analyze as follows:

$$\frac{\partial f(x)}{\partial h}=\frac{\partial f(x_0+h)}{\partial h}$$

is evaluated for different values of h: $\left[\frac{\partial f(x)}{\partial h}\right]_{h_1},\left[\frac{\partial f(x)}{\partial h}\right]_{h_2},\left[\frac{\partial f(x)}{\partial h}\right]_{h_3}\ldots.$

The limit $f'(x_0)$ is independent of x

Therefore we can interchange the derivative and the limit[2].

$$\frac{\partial}{\partial h}lim_{h\to 0}\frac{\partial f(x_0+h)}{\partial h}=0$$

$$lim_{h\to 0}\frac{\partial}{\partial h}\left[\frac{\partial f(x_0+h)}{\partial h}\right]=0\ \ (12)$$

$$lim_{h\to 0}\frac{\partial}{\partial h}\left[\frac{\partial f(x_0+h)}{\partial h}\right]=0$$

$$lim_{h\to 0}\frac{\partial^2 f(x_0+h)}{\partial h^2}=0$$

$$lim_{h\to 0}\frac{\partial}{\partial h}\left[\frac{\partial f(x_0+h)}{\partial h}\right]=0$$

$$lim_{h\to 0}\frac{\partial}{\partial h}\left[\frac{\partial f(x_0+h)}{\partial (x_0+h)}\frac{\partial (x_0+h)}{\partial h}\right]=0$$

$$lim_{h\to 0}\frac{\partial}{\partial h}\left[\frac{\partial f(x_0+h)}{\partial (x_0+h)}\right]=0$$

$$lim_{h\to 0}\frac{\partial}{\partial (x_0+h)}\left[\frac{\partial f(x_0+h)}{\partial (x_0+h)}\right]\frac{\partial (x_0+h)}{\partial h}=0$$

$$lim_{h\to 0}\frac{\partial}{\partial x}\left[\frac{\partial f(x)}{\partial x}\right]=0$$

where $x=x_0+h$

We now have,

$$lim_{h\to 0}\frac{\partial^2 f(x)}{\partial x^2}=0\ (13)$$

$$\left[\frac{\partial^2 f(x)}{\partial x^2}\right]_{x=x_0}=0$$

But $x=x_0$ could be any arbitrary point.

By differentiating (10) we obtain the expected result

$$\frac{\partial^2 f(x_0+h)}{\partial h^2}=f''(x_0)(14)$$

which contradicts the earlier result given by (13)unless $f''(x_0)=0$

Direct Calculations

We write the Taylor series

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots \ldots (15).$$

$$\frac{\partial}{\partial x}f(x+h) = f'(x) + \frac{h}{1!}f''(x) + \frac{h^2}{2!}f'''(x) + \frac{h^3}{3!}f''''(x) + \cdots \ldots (16).$$

$$\frac{\partial}{\partial h}f(x+h) = f'^{(x)} + \frac{h}{1!}f''(x) + \frac{h^2}{2!}f'''(x) + \frac{h^3}{3!}f''''(x) + \cdots (17)$$

From (10) and (11) we have,

$$\frac{\partial}{\partial x}f(x+h) = \frac{\partial}{\partial h}f(x+h) \; (18)$$

Differentiating (10) with respect to $x + h$[holding x as constant]

$$\frac{\partial}{\partial x}f(x+h) = \frac{d}{d(x+h)}f(x+h) \; (19)$$

$$\left[\frac{\partial}{\partial x}f(y)\right]_{y=x} = \left[\frac{\partial}{\partial h}f(y)\right]_{y=y} \quad (20)$$

$\frac{\partial}{\partial x}f(y) = \frac{df(x)}{dx}$ is a constant on $(x, x+h)$. This notion may be considered to show that$\frac{df(x)}{dx}$ is constant everywhere.[we take $(x, x+h), (x+h, x+2h), (x+2h, x+3h)$ ....and consider the proof given over and over again]

$$\frac{\partial}{\partial x}f(x) = const \Rightarrow \frac{\partial^2}{\partial x^2}f(x) = 0$$

which we got earlier

Now [treating f as a function of x and h we may write

$$df(x+h) = \frac{\partial}{\partial x}f(x+h)dx + \frac{\partial}{\partial h}f(x+h)dh \; (21)$$

Again

$$df(x+h) = \frac{\partial}{\partial(x+h)}f(x+h)d(x+h) \quad (22)$$

$$\Rightarrow df(x+h) = \frac{\partial}{\partial(x+h)}f(x+h)dx + \frac{\partial}{\partial(x+h)}f(x+h)dh \quad (23)$$

From (21) and (22) we have,

$$\left[\frac{\partial}{\partial(x+h)}f(x+h) - \frac{\partial}{\partial x}f(x+h)\right]dx + \left[\frac{\partial}{\partial(x+h)}f(x+h) - \frac{\partial}{\partial h}f(x+h)\right]dh = 0$$

$$\frac{\partial}{\partial(x+h)}f(x+h)=\frac{\partial}{\partial x}f(x+h)=\frac{\partial}{\partial h}f(x+h)\ \ (24)$$

We clearly see that the function $\frac{df}{dx}$is a constant function that is $\frac{d^2f}{dx^2}=0$

Further Considerations

We recall (9)

$$f(x_0+h)=f(x_0)+\frac{h}{1!}f'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+\cdots\ldots(9)$$

We differentiate the above with respect to $x=x_0+h';h'<h$

$$\left[\frac{df(x_0+h)}{dh}\right]_{h=h'}=f'(x_0)+h'f''(x_0)+\frac{h'^2}{2!}f'''(x_0)+\cdots..=f'(x_0+h')\ (25)$$

$$\frac{df(x_0+h)}{d(x_0+h)}=\frac{df(x_0+h)}{dh}\frac{dh}{d(x_0+h)}=\frac{df(x_0+h)}{dh}$$

$$\frac{df(x_0+h)}{d(x_0+h)}=\frac{df(x_0+h)}{dh}\ \ (26)$$

We obtain an indication of constancy of $\frac{df(x_0+h)}{dh}$ from (26) and keeping in mind equation (18) we have

$$\frac{\partial}{\partial x}f(x+h)=\frac{\partial}{\partial h}f(x+h)=\frac{df(x+h)}{d(x+h)}$$

Next we consider a truncated Taylor series which has been approximated with 'n' terms. Now we have an equation and not an identity and there are discrete solutions for h. Since we have taken an approximation to the Taylor series it is least likely the corresponding roots will cause a divergence of the infinite series in the Taylor expansion. It would be better to take a truncation which is not an approximation but the infinite Taylor series is convergent for it. These solutions for 'h' will not satisfy the entire Taylor series with an infinite number of terms. Suppose one solution of 'h' from approximated equation[equation with finite number of terms] satisfied the infinite Taylor series, we will have (9) as well as a truncated (9)[approximated up to 'n' terms. The situation has been delineated below

We no0wcconsider the Maclaurin expansion for e^x

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots..$$

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\epsilon_n(x)\ \ (26)$$

$\epsilon_n(x)$: Remainder after the nth term count starting from zero: n=0,1,2…….

$$\epsilon_0(x) = e^x - 1$$

Differentiating (26) with respect to 'x' for a fixed 'n'we obtain

$$\frac{de^x}{dx} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{d\epsilon_n(x)}{dx} \quad (27)$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{d\epsilon_n(x)}{dx} \quad (28)$$

$$\frac{d\epsilon_n(x)}{dx} - \epsilon_n(x) = \frac{x^n}{n!} (29)$$

The nth terms should vanish for n tending to infinity. Also by direct limit evaluation we may show that

$$lim_{n\to\infty}\frac{x^n}{n!} = 0$$

With n tending to infinity

$$\frac{d\epsilon_\infty(x)}{dx} - \epsilon_\infty(x) = 0$$

$$ln\epsilon_\infty(x) = \mathrm{x} + C' \quad (30)$$

If C'=0

$$\epsilon_\infty(x) = e^x$$

If $C' \neq 0, C' = lnC$

$$\epsilon_\infty(x) = Ce^x$$

$$\epsilon_\infty(x) = Ce^x$$

Again

$$C = 0 \Rightarrow C' = -\infty$$

That means we used $-\infty$ as the constant of integration in equation (30)

Conclusions

As claimed we have arrived at some inconsistent aspects of the Taylor expansion

References

1. Wikipedia,L' Hospital'sRule, Link:

https://en.wikipedia.org/wiki/L%27H%C3%B4pital%27s_rule

Accessed on 11th March, 2020

2. Wikipedia, Uniform Convergence, Applications, Differentiability

https://en.wikipedia.org/wiki/Uniform_convergence#To_differentiability