Inferences from the Taylor Series

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Abstract

The writing intends to bring out certain inconsistent aspects relating to the Taylor expansion. The Taylor series does not hold for the entire real axis that leads to a host of problems.

Introduction

The Taylor series is well known for its application in mathematics and in physics. The article brings out some anomalous features about the Taylor expansion

Various Inconsistencies

Case 1.

We consider

$$f(x+2h) = f\big((x+h)+h\big) \quad (1)$$

Expanding about $(x+h)$

$$f(x+2h) = f(x+h) + \frac{h}{1!}f'(x+h) + \frac{h^2}{2!}f''(x+h) + \frac{h^3}{3!}f'''(x+h) + \cdots \ldots . (2)$$

Expanding about $x = x$

$$f(x+2h) = f(x) + \frac{2h}{1!}f'(x+h) + \frac{4h^2}{2!}f''(x+h) + \frac{8h^3}{3!}f'''(x+h) + \cdots \ldots . (3)$$

From (2) and (3)

$$f(x+h) + \frac{h}{1!}f'(x+h) + \frac{h^2}{2!}f''(x+h) + \frac{h^3}{3!}f'''(x+h) + \cdots \ldots .$$
$$= f(x) + \frac{2h}{1!}f'(x) + \frac{4h^2}{2!}f''(x) + \frac{8h^3}{3!}f'''(x) + \cdots \ldots$$

$$f(x+h)-f(x)+h[f'(x+h)-2f'(x)]+\frac{1}{2!}h^2[f''(x+h)-4f''(x)] \\ +\frac{1}{3!}h^3[f'''(x+h)-8f'''(x)]+\cdots..=0\ (4)$$

$$\frac{f(x+h)-f(x)}{h}\frac{1}{h}+\frac{[f'(x+h)-2f'(x)]}{h}+\frac{1}{2!}[f''(x+h)-4f''(x)]+\frac{1}{3!}h[f'''(x+h)-8f'''(x)] \\ +\cdots..=0$$

$$\frac{f(x+h)-f(x)}{h}\frac{1}{h}+\frac{[f'(x+h)-f'(x)]}{h}-\frac{f'(x)}{h}+\frac{1}{2!}[f''(x+h)-4f''(x)] \\ +\frac{1}{3!}h[f'''(x+h)-8f'''(x)]+h[\ldots..]=0\ (5)$$

Equation (5) is considered for $h \neq 0$. Even when we go for $h \to 0, h$ does not become equal to zero. It is in the neighborhood of zero without becoming equal to zero]

$$\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]\frac{1}{h}+\frac{[f'(x+h)-f'(x)]}{h}+\frac{1}{2!}[f''(x+h)-4f''(x)] \\ +\frac{1}{3!}h[f'''(x+h)-8f'''(x)]+h[\ldots..]=0$$

$$Lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]\frac{1}{h}+Lim_{h\to 0}\frac{[f'(x+h)-f'(x)]}{h} \\ +\frac{1}{2!}Lim_{h\to 0}[f''(x+h)-4f''(x)]+\frac{1}{3!}Lim_{h\to 0}h[f'''(x+h)-8f'''(x)]+h[\ldots..] \\ =0\ \ (6)$$

We are considering a function for which

$$Lim_{h\to 0}h[f'''(x+h)-8f'''(x)]+h[\ldots..]=0$$

Then

$$Lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]\frac{1}{h}+f''(x)-\frac{3}{2}f''(x)=0$$

$$Lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]\frac{1}{h}=\frac{1}{2}f''(x)$$

$$Lim_{h\to 0}\frac{\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]}{h}=\frac{1}{2}f''(x)\ (7)$$

We apply L' Hospital's rule[1`] to obtain

$$Lim_{h\to 0}\frac{\frac{d}{dh}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]}{1}=\frac{1}{2}f''(x)$$

$$Lim_{h\to 0}\frac{\frac{d}{dh}\left[\frac{f(x+h)-f(x)}{h}-f'(x)\right]}{1}=\frac{1}{2}f''(x)$$

$$Lim_{h\to 0}\left[-\frac{1}{h^2}\big(f(x+h)-f(x)\big)+\frac{1}{h}\big(f'(x+h)-f'(x)\big)\right]=\frac{1}{2}f''(x)$$

$$\left[-Lim_{h\to 0}\frac{1}{h^2}\big(f(x+h)-f(x)\big)+Lim_{h\to 0}\frac{1}{h}\big(f'(x+h)-f'(x)\big)\right]=\frac{1}{2}f''(x)(8)$$

$$-\infty+f''(x)=\frac{1}{2}f''(x)$$

$$-\frac{1}{2}f''(x)=-\infty$$

As claimed we have brought out an aspect of inconsistency with Taylor Series.

Case 2.Let us have another situation for our analysis. We write the Taylor series

$$f(x_0+h)=f(x_0)+\frac{h}{1!}f'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+\cdots\ldots(9).$$

The increment h may be sufficiently large subject to the fact that the series has to converge.

$$\frac{\partial f(x_0+h)}{\partial h}=f'(x_0)+hf''(x_0)+\frac{h^2}{2!}f'''(x_0)+\cdots(10).$$

$$lim_{h\to 0}\frac{\partial f(x_0+h)}{\partial h}=f'(x_0)\ \ (11)$$

$$lim_{h\to 0}F_h(x_0+h)=f'(x_0)$$

The limit $f'(x_0)$ is independent of h.This is an example of uniform convergence . We may analyze as follows:

$$\frac{\partial f(x)}{\partial h}=\frac{\partial f(x_0+h)}{\partial h}$$

is evaluated for different values of h: $\left[\frac{\partial f(x)}{\partial h}\right]_{h_1},\left[\frac{\partial f(x)}{\partial h}\right]_{h_2},\left[\frac{\partial f(x)}{\partial h}\right]_{h_3}\ldots.$

The limit $f'(x_0)$ is independent of x

Therefore we can interchange the derivative and the limit[2].

$$\frac{\partial}{\partial h} lim_{h\to 0} \frac{\partial f(x_0+h)}{\partial h} = 0$$

$$lim_{h\to 0} \frac{\partial}{\partial h}\left[\frac{\partial f(x_0+h)}{\partial h}\right] = 0 \ (12)$$

$$lim_{h\to 0} \frac{\partial}{\partial h}\left[\frac{\partial f(x_0+h)}{\partial h}\right] = 0$$

$$lim_{h\to 0} \frac{\partial^2 f(x_0+h)}{\partial h^2} = 0$$

$$lim_{h\to 0} \frac{\partial}{\partial h}\left[\frac{\partial f(x_0+h)}{\partial h}\right] = 0$$

$$lim_{h\to 0} \frac{\partial}{\partial h}\left[\frac{\partial f(x_0+h)}{\partial (x_0+h)}\frac{\partial (x_0+h)}{\partial h}\right] = 0$$

$$lim_{h\to 0} \frac{\partial}{\partial h}\left[\frac{\partial f(x_0+h)}{\partial (x_0+h)}\right] = 0$$

$$lim_{h\to 0} \frac{\partial}{\partial (x_0+h)}\left[\frac{\partial f(x_0+h)}{\partial (x_0+h)}\right]\frac{\partial (x_0+h)}{\partial h} = 0$$

$$lim_{h\to 0} \frac{\partial}{\partial x}\left[\frac{\partial f(x)}{\partial x}\right] = 0$$

where$x = x_0 + h$

We now have,

$$lim_{h\to 0} \frac{\partial^2 f(x)}{\partial x^2} = 0 \ (13)$$

$$\left[\frac{\partial^2 f(x)}{\partial x^2}\right]_{x=x_0} = 0$$

But $x = x_0$ could be any arbitrary point.

By differentiating (10) we obtain the expected result

$$\frac{\partial^2 f(x_0+h)}{\partial h^2} = f''(x_0)(14)$$

which contradicts the earlier result given by (13)unless $f''(x_0) = 0$

Direct Calculations

We write the Taylor series

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots \ldots (15).$$

$$\frac{\partial}{\partial x}f(x+h) = f'(x) + \frac{h}{1!}f''(x) + \frac{h^2}{2!}f'''(x) + \frac{h^3}{3!}f''''(x) + \cdots \ldots (16).$$

$$\frac{\partial}{\partial h}f(x+h) = f'^{(x)} + \frac{h}{1!}f''(x) + \frac{h^2}{2!}f'''(x) + \frac{h^3}{3!}f''''(x) + \cdots (17)$$

From (10) and (11) we have,

$$\frac{\partial}{\partial x}f(x+h) = \frac{\partial}{\partial h}f(x+h)(18)$$

Differentiating (10) with respect to $x+h$[holding x as constant]

$$\frac{\partial}{\partial x}f(x+h) = \frac{d}{d(x+h)}f(x+h)\ (19)$$

$$\left[\frac{\partial}{\partial x}f(y)\right]_{y=x} = \left[\frac{\partial}{\partial h}f(y)\right]_{y=y}\ \ (20)$$

$\frac{\partial}{\partial x}f(y) = \frac{df(x)}{dx}$is a constant on $(x, x+h)$. This notion may be considered to show that$\frac{df(x)}{dx}$ is constant everywhere.[we take $(x, x+h)$, $(x+h, x+2h)$, $(x+2h, x+3h)$ ....and consider the proof given over and over again]

$$\frac{\partial}{\partial x}f(x) = const \Rightarrow \frac{\partial^2}{\partial x^2}f(x) = 0$$

which we got earlier

Now [treating f as a function of x and h we may write

$$df(x+h) = \frac{\partial}{\partial x}f(x+h)dx + \frac{\partial}{\partial h}f(x+h)dh\ (21)$$

Again

$$df(x+h) = \frac{\partial}{\partial(x+h)}f(x+h)d(x+h)\ \ (22)$$

$$\Rightarrow df(x+h) = \frac{\partial}{\partial(x+h)}f(x+h)dx + \frac{\partial}{\partial(x+h)}f(x+h)dh\ \ (23)$$

From (21) and (22) we have,

$$\left[\frac{\partial}{\partial(x+h)}f(x+h)-\frac{\partial}{\partial x}f(x+h)\right]dx+\left[\frac{\partial}{\partial(x+h)}f(x+h)-\frac{\partial}{\partial h}f(x+h)\right]dh=0$$

$$\frac{\partial}{\partial(x+h)}f(x+h)=\frac{\partial}{\partial x}f(x+h)=\frac{\partial}{\partial h}f(x+h)\ \ (24)$$

We clearly see that the function $\frac{df}{dx}$is a constant function that is $\frac{d^2f}{dx^2}=0$

Further Considerations

We recall (9)

$$f(x_0+h)=f(x_0)+\frac{h}{1!}f'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+\cdots\ldots(9)$$

We differentiate the above with respect to $x=x_0+h'; h'<h$

$$\left[\frac{df(x_0+h)}{dh}\right]_{h=h'}=f'(x_0)+h'f''(x_0)+\frac{h'^2}{2!}f'''(x_0)+\cdots..=f'(x_0+h')\ (25)$$

$$\frac{df(x_0+h)}{d(x_0+h)}=\frac{df(x_0+h)}{dh}\frac{dh}{d(x_0+h)}=\frac{df(x_0+h)}{dh}$$

$$\frac{df(x_0+h)}{d(x_0+h)}=\frac{df(x_0+h)}{dh}\ \ (26)$$

We obtain an indication of constancy of $\frac{df(x_0+h)}{dh}$ from (26) and keeping in mind equation (18) we have

$$\frac{\partial}{\partial x}f(x+h)=\frac{\partial}{\partial h}f(x+h)=\frac{df(x+h)}{d(x+h)}$$

Next we consider a truncated Taylor series which has been approximated with 'n' terms. Now we have an equation and not an identity and there are discrete solutions for h. Since we have taken an approximation to the Taylor series it is least likely the corresponding roots will cause a divergence of the infinite series in the Taylor expansion. It would be better to take a truncation which is not an approximation but the infinite Taylor series is convergent for it. These solutions for 'h' will not satisfy the entire Taylor series with an infinite number of terms. Suppose one solution of 'h' from approximated equation[equation with finite number of terms] satisfied the infinite Taylor series, we will have (9) as well as a truncated (9)[approximated up to 'n' terms. The situation has been delineated below

We no0wcconsider the Maclaurin expansion for e^x

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots..$$

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\epsilon_n(x)\ \ (26)$$

$\epsilon_n(x)$: Remainder after the nth term count starting from zero: n=0,1,2…….

$$\epsilon_0(x) = e^x - 1$$

Differentiating (26) with respect to ‘x’ for a fixed ‘n’we obtain

$$\frac{de^x}{dx} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{d\epsilon_n(x)}{dx} \quad (27)$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{d\epsilon_n(x)}{dx} \quad (28)$$

$$\frac{d\epsilon_n(x)}{dx} - \epsilon_n(x) \ = \frac{x^n}{n!} (29)$$

$$\frac{d^2e^x}{dx^2} = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-2}}{(n-1)!} + \frac{d^2\epsilon_n(x)}{dx^2}$$

We consider a positive interval $(x_1, x_2)$and make $n \to \infty$.For such an interval

$$lim_{n\to\infty} \frac{x^n}{n!} = 0$$

$$lim_{n\to\infty} \left[\frac{d\epsilon_n(x)}{dx} - \epsilon_n(x)\right] \ = lim_{n\to\infty} \frac{x^n}{n!}$$

$$lim_{n\to\infty} \left[\frac{d\epsilon_n(x)}{dx} - \epsilon_n(x)\right] = 0$$

For sufficiently large ,n$\left|\frac{d\epsilon_n(x)}{dx} - \epsilon_n(x)\right|$ can be made arbitrarily close to zero

For the concerned interval we have in the limit n tending to infinity [for the interval (x_1, x_2)]the following[rigorous]equation

$$\frac{d\epsilon_\infty(x)}{dx} - \epsilon_\infty(x) = 0 \ (30.1)$$

$$ln\epsilon_\infty(x) = \mathrm{x} + C' \ \ (30.2)$$

If C’=0

$$\epsilon_\infty(x) = e^x \ (31.1)$$

If $C' \neq 0, C' = lnC$

$$\epsilon_\infty(x) = Ce^x (31.2)$$

Again

$$C = 0 \Rightarrow C' = -\infty \ (32)$$

That means we used $-\infty$ as the constant of integration in equation (30.2).Suppose we take $|C'| \gg 0; C' < 0$ so that C is a very small fraction:

$$Ce^{x_1} < \epsilon_\infty(x) < Ce^{x_2}$$

But the point is that once we decide on the value of C[C cannot be minus infinityby itself if a value is considered] we cannot vary it. Though $\epsilon_\infty(x)$ will be very small we cannot take it arbitrarily close to zero

Next we consider a much larger the interval (x_1, x_2') which contains the interval $(x_1, x_2); x_2' \gg x_2[x_2'$ remaining finite. We have the same equation as given by (30.1) and the same solution $\epsilon_\infty(x) = Ce^x$.This time we cannot change the value of the constant. If we changed it to C_{new}then we have untenable results li8ke $\epsilon_\infty(x_1) = C_{new}e^{x_1}$ and$\epsilon_\infty(x_2) = C_{new}e^{x_2}$. But with the old constant $\epsilon_\infty(x_2') = Ce^{x_2'} \gg 0$ since $x_2' \gg x_2$

It is not possible to cover the entire x axisor the semi x axis:$(0, \infty)$ by a single constant having a numerical value.

The discrepancy we have found should not surprise us in view of the earlier discrepancies , for example those notified through case1 and case2.

Further Investigation

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \epsilon_n(x)$$

We define $f(x, n)$ as follows:

$$f(x, n) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \ (33)$$

$$f(x, n) - f(x, n-1) = \frac{x^n}{n!}$$

We make r $f(x, n)$ a smooth[obviously continuous) by interpolation with a suitable curve where n isposotive everywhere .

$$\frac{\partial f(x,n)}{\partial n} - \frac{\partial f(x,n-1)}{\partial n} = \frac{x^n lnx}{n!} + x^n \frac{d}{dn}\left(\frac{1}{n!}\right) \ (34)$$

$$\frac{\partial f(x,n)}{\partial n} - \frac{\partial f(x,n-1)}{\partial n} = \frac{x^n lnx}{n!} - x^n \frac{1}{n!}\left[\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots 1\right] \ (35)$$

In the above $n! = n(n-1) \ldots . up\ to\ |n|$ *terms . The equation considers right handed derivatives exclusively.*

For large 'n' the left side of (35) is zero: $f(x, n) \approx f(x, n-1) \approx e^x$

Therefore,

$$lim_{n\to\infty}\left[\frac{x^n lnx}{n!}-x^n\frac{1}{n!}\left[\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\cdots 1\right]\right]=0$$

That is to say that in the limit irrespective of the value of x---x very large or very small

$$\frac{lnx}{n!}\approx\frac{1}{n!}\left[\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\cdots 1\right]$$

$$\ln x\approx\left[\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\cdots 1\right]$$

We have a strange result

$$x\approx e^{\left[\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\cdots 1\right]}; n{:}\, very\ large\ but\ constant\ in\ value$$

NB: For testing ,in the above, w have taken a very large but fixed value of x so as to ensure that on the left side of (35) we have an approximately.

Looking Directly into Sources of Error

Assume that the formula for e^x holds simultaneously over the entire real axis $x\in(-\infty,\infty)$. We4 partition he real axis into accountably infinite number of closed intervals indexed by the natural numbers. The remainder term[Cauchy form],$R_n=\frac{x^n}{n!}e^{\theta x}$ should tend to zero for all intervals we have in the partition. For any preassigned $\epsilon>0$ no matter how small we have for

Interval 1: $N_1>0$ such that for all $n>\ N_1$ we have $|R_n|<\epsilon$

Interval 2: $N_2>0$ such that for all $n>\ N_2$ we have $|R_n|<\epsilon$

Interval 3: $N_3>0$ such that for all $n>\ N_3$ we have $|R_n|<\epsilon$

Interval k: $N_k>0$ such that for all $n>\ N_k$ we have $|R_n|<\epsilon$

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Fort the largest N_k, $|R_n|<\epsilon$ for all intervals. There is no such largest N_k[we cannot denote it numerically]

Therefore our usual formula for e^x will not hold for the entire real axis at one stroke..

Analogous conclusions may follow from all non terminating instances of the Taylor series.

Conclusions

As claimed we have arrived at some inconsistent aspects of the Taylor expansion

References

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