On the Lebesgue integral and the Lebesgue measure: mathematical applications in some sectors of Chern-Simons theory and Yang-Mills gauge theory and mathematical connections with some sectors of String Theory and Number Theory

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Abstract

In this paper, in the **Section 1**, we have described some equations and theorems concerning the Lebesgue integral and the Lebesgue measure. In the **Section 2**, we have described the possible mathematical applications, of Lebesgue integration, in some equations concerning various sectors of Chern-Simons theory and Yang-Mills gauge theory, precisely the two dimensional quantum Yang-Mills theory. In conclusion, in the **Section 3**, we have described also the possible mathematical connections with some sectors of String Theory and Number Theory, principally with some equations concerning the Ramanujan's modular equations that are related to the physical vibrations of the bosonic strings and of the superstrings, some Ramanujan's identities concerning π and the zeta strings.

1. On the Lebesgue integral and the Lebesgue measure [1] [2] [3] [4]

In this paper we use Lebesgue measure to define the $\int_{R^d} f(x)dx$ of functions $f: R^d \to C \cup \{\infty\}$, where R^d is the Euclidean space.

If $f = c_1 1_{E_1} + \cdots + c_k 1_{E_k}$ is an unsigned simple function, the integral $\int_{R^d} f(x)dx$ is defined by the formula

$$Simp \int_{R^d} f(x)dx := c_1 m(E_1) + \cdots + c_k m(E_k), \quad (1)$$

thus $\int_{R^d} f(x)dx$ will take values in $[0, +\infty]$.

Let $k, k' \geq 0$ be natural numbers, $c_1, \ldots, c_k, c'_1, \ldots, c'_{k'} \in [0, +\infty]$, and let $E_1, \ldots, E_k, E'_1, \ldots, E'_{k'} \subset R^d$ be Lebesgue measurable sets such that the identity

$$c_1 1_{E_1} + \cdots + c_k 1_{E_k} = c'_1 1_{E'_1} + \cdots + c'_{k'} 1_{E'_{k'}} \quad (2)$$

holds identically on R^d. Then one has

$$c_1 m(E_1) + \cdots + c_k m(E_k) = c'_1 m(E'_1) + \cdots + c'_{k'} m(E'_{k'}). \quad (3)$$

A complex-valued simple function $f: R^d \to C$ is said to be absolutely integrable of $\int_{R^d} |f(x)| dx < \infty$. If f is absolutely integrable, the integral $\int_{R^d} f(x)dx$ is defined for real signed f by the formula

$$Simp \int_{R^d} f(x)dx := Simp \int_{R^d} f_+(x)dx - Simp \int_{R^d} f_-(x)dx \quad (4)$$

where $f_+(x) := max(f(x), 0)$ and $f_-(x) := max(-f(x), 0)$ (we note that these are unsigned simple functions that are pointwise dominated by $|f|$ and thus have finite integral), and for complex-valued f by the formula

$$Simp \int_{R^d} f(x)dx := Simp \int_{R^d} Ref(x)dx + iSimp \int_{R^d} Imf(x)dx. \quad (5)$$

Let $f: R^d \to [0, +\infty]$ be an unsigned function (not necessarily measurable). We define the *lower unsigned Lebesgue integral* $\underline{\int_{R^d}} f(x)dx$

$$\underline{\int_{R^d}} f(x)dx := sup_{0 \leq g \leq f; g simple} Simp \int_{R^d} g(x)dx \quad (6)$$

where g ranges over all unsigned simple functions $g: R^d \to [0, +\infty]$ that are pointwise bounded by f. One can also define the *upper unsigned Lebesgue integral*

$$\overline{\int_{R^d}} f(x)dx := inf_{h \geq f; h simple} Simp \int_{R^d} h(x)dx \quad (7)$$

but we will use this integral much more rarely. Note that both integrals take values in $[0, +\infty]$, and that the upper Lebesgue integral is always at least as large as the lower Lebesgue integral.

Let $f: R^d \to [0,+\infty]$ be measurable. Then for any $0 < \lambda < \infty$, one has

$$m(\{x \in R^d: f(x) \geq \lambda\}) \leq \frac{1}{\lambda}\int_{R^d} f(x)dx, \quad (7b)$$

that is the Markov's inequality.

An almost everywhere defined measurable function $f: R^d \to C$ is said be *absolutely integrable* if the unsigned integral

$$\|f\|_{L^1(R^d)} := \int_{R^d}|f(x)|dx \quad (8)$$

is finite. We refer to this quantity $\|f\|_{L^1(R^d)}$ as the $L^1(R^d)$ *norm* of f, and use $L^1(R^d)$ or $L^1(R^d \to C)$ to denote the space of absolutely integrable functions. If f is real-valued and absolutely integrable, we define the Lebesgue integral $\int_{R^d} f(x)dx$ by the formula

$$\int_{R^d} f(x)dx := \int_{R^d} f_+(x)dx - \int_{R^d} f_-(x)dx \quad (9)$$

where $f_+ := max(f,0)$ and $f_- := max(-f,0)$ are the positive and negative components of f. If f is complex-valued and absolutely integrable, we define the Lebesgue integral $\int_{R^d} f(x)dx$ by the formula

$$\int_{R^d} f(x)dx := \int_{R^d} Ref(x)dx + i\int_{R^d} Imf(x)dx \quad (10)$$

where the two integrals on the right are interpreted as real-valued absolutely integrable Lebesgue integrals.

Let $f \in L^1(R^d \to C)$. Then

$$\left|\int_{R^d} f(x)dx\right| \leq \int_{R^d}|f(x)|dx. \quad (11)$$

This is the **Triangle inequality.**

If f is real-valued, then $|f| = f_+ + f_-$. When f is complex-value one cannot argue quite so simply; a naive mimicking of the real-valued argument would lose a factor of 2, giving the inferior bound

$$\left|\int_{R^d} f(x)dx\right| \leq 2\int_{R^d}|f(x)|dx. \quad (12)$$

To do better, we exploit the phase rotation invariance properties of the absolute value operation and of the integral, as follows. Note that for any complex number z, one can write $|z|$ as $ze^{i\theta}$ for some real θ. In particular, we have

$$\left|\int_{R^d} f(x)dx\right| = e^{i\theta}\int_{R^d} f(x)dx = \int_{R^d} e^{i\theta} f(x)dx, \quad (13)$$

for some real θ. Taking real parts of both sides, we obtain

$$\left|\int_{R^d} f(x)dx\right| = \int_{R^d} Re\left(e^{i\theta} f(x)\right)dx. \quad (14)$$

Since

$$Re\left(e^{i\theta} f(x)\right) \leq \left|e^{i\theta} f(x)\right| = |f(x)|,$$

we obtain the eq. (11)

Let (X, B, μ) be a measure space, and let $f, g: X \to [0, +\infty]$ be measurable. Then

$$\int_X (f+g)d\mu = \int_X fd\mu + \int_X gd\mu. \quad (15)$$

It suffices to establish the sub-additivity property

$$\int_X (f+g)d\mu \leq \int_X fd\mu + \int_X gd\mu. \quad (15b)$$

We establish this in stages. We first deal with the case when μ is a finite measure (which means that $\mu(X) < \infty$) and f, g are bounded. Pick an $\epsilon > 0$ and f_ϵ be f rounded down to the nearest integer multiple of ϵ, and f^ϵ be f rounded up to nearest integer multiple. Clearly, we have the pointwise bound

$$f_\epsilon(x) \leq f(x) \leq f^\epsilon(x) \quad (16)$$

and

$$f^\epsilon(x) - f_\epsilon(x) \leq \epsilon. \quad (17)$$

Since f is bounded, f_ϵ and f^ϵ are simple. Similarly define g_ϵ, g^ϵ. We then have the pointwise bound

$$f + g \leq f^\epsilon + g^\epsilon \leq f_\epsilon + g_\epsilon + 2\epsilon, \quad (18)$$

hence, from the properties of the simple integral,

$$\int_X f + gd\mu \leq \int_X f_\epsilon + g_\epsilon + 2\epsilon d\mu = Simp\int_X f_\epsilon + g_\epsilon + 2\epsilon d\mu =$$

$$= Simp\int_X f_\epsilon d\mu + Simp\int_X g_\epsilon d\mu + 2\epsilon\mu(X). \quad (19)$$

From the following equation

$$\int_X f d\mu := sup_{0\leq g\leq f; g simple} Simp\int_X g d\mu, \quad (19b)$$

we conclude that

$$\int_X f + g d\mu \leq \int_X f d\mu + \int_X g d\mu + 2\epsilon\mu(X). \quad (20)$$

Letting $\epsilon \to 0$ and using the assumption that $\mu(X)$ is finite, we obtain the claim. Now we continue to assume that μ is a finite measure, but now do not assume that f, g are bounded. Then for any natural number (also the primes) n, we can use the previous case to deduce that

$$\int_X min(f,n) + min(g,n) d\mu \leq \int_X min(f,n) d\mu + \int_X min(g,n) d\mu. \quad (21)$$

Since $min(f+g,n) \leq min(f,n) + min(g,n)$, we conclude that

$$\int_X min(f+g,n) \leq \int_X min(f,n) d\mu + \int_X min(g,n) d\mu. \quad (22)$$

Taking limits as $n \to \infty$ using horizontal truncation, we obtain the claim.

Finally, we no longer assume that μ is of finite measure, and also do not require f, g to be bounded. By Markov's inequality, we see that for each natural number (also the primes) n, the set

$$E_n := \left\{x \in X : f(x) > \frac{1}{n}\right\} \cup \left\{x \in X : g(x) > \frac{1}{n}\right\},$$

has finite measure. These sets are increasing in n, and $f, g, f+g$ are supported on $\cup_{n=1}^{\infty} E_n$, and so by vertical truncation

$$\int_X (f+g) d\mu = \lim_{n\to\infty} \int_X (f+g) 1_{E_n} d\mu. \quad (23)$$

From the previous case, we have

$$\int_X (f+g) 1_{E_n} d\mu \leq \int_X f 1_{E_n} d\mu + \int_X g 1_{E_n} d\mu. \quad (24)$$

Let (X, B, μ) be a measure space, and let $0 \leq f_1 \leq f_2 \leq \cdots$ be a monotone non-decreasing sequence of unsigned measurable functions on X. Then we have

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X \lim_{n\to\infty} f_n d\mu. \quad (25)$$

Write $f \coloneqq \lim_{n\to\infty} f_n = sup_{n\to\infty} f_n$, then $f: X \to [0, +\infty]$ is measurable. Since the f_n are non decreasing to f, we see from monotonicity that $\int_X f_n d\mu$ are non decreasing and bounded above by $\int_X f d\mu$, which gives the bound

$$\lim_{n\to\infty} \int_X f_n d\mu \leq \int_X f d\mu. \quad (26)$$

It remains to establish the reverse inequality

$$\int_X f d\mu \leq \lim_{n\to\infty} \int_X f_n d\mu. \quad (27)$$

By definition, it suffices to show that

$$\int_X g d\mu \leq \lim_{n\to\infty} \int_X f_n d\mu, \quad (28)$$

whenever g is a simple function that is bounded pointwise by f. By horizontal truncation we may assume without loss of generality that g also is finite everywhere, then we can write

$$g = \sum_{i=1}^{k} c_i 1_{A_i} \quad (29)$$

for some $0 \leq c_i < \infty$ and some disjoint B-measurable sets $A_1, \dots, A_k$, thus

$$\int_X g d\mu = \sum_{i=1}^{k} c_i \mu(A_i). \quad (30)$$

Let $0 < \epsilon < 1$ be arbitrary (also $\frac{\sqrt{5}-1}{2} = 0{,}61803398\dots$). Then we have

$$f(x) = sup_n f_n(x) > (1-\epsilon)c_i \quad (31)$$

for all $x \epsilon A_i$. Thus, if we define the sets

$$A_{i,n} \coloneqq \{x \in A_i : f_n(x) > (1-\epsilon)c_i\} \quad (32)$$

then the $A_{i,n}$ increase to A_i and are measurable. By upwards monotonicity of measure, we conclude that

$$\lim_{n\to\infty} \mu\left(A_{i,n}\right) = \mu(A_i). \quad (33)$$

On the other hand, observe the pointwise bound

$$f_n \geq \sum_{i=1}^{k} (1-\epsilon) c_i 1_{A_{i,n}} \quad (34)$$

for any n; integrating this, we obtain

$$\int_X f_n d\mu \geq (1-\epsilon)\sum_{i=1}^{k} c_i \mu(A_{i,n}). \quad (35)$$

Taking limits as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \int_X f_n d\mu \geq (1-\epsilon)\sum_{i=1}^{k} c_i \mu(A_i); \quad (36)$$

sending $\epsilon \to 0$ we then obtain the claim.

Let (X, B, μ) be a measure space, and let $f_1, f_2, \ldots : X \to C$ be a sequence of measurable functions that converge pointwise μ-almost everywhere to a measurable limit $f: X \to C$. Suppose that there is an unsigned absolutely integrable function $G: X \to [0, +\infty]$ such that $|f_n|$ are pointwise μ-almost everywhere bounded by G for each n. Then we have

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu. \quad (37)$$

By modifying f_n, f on a null set, we may assume without loss of generality that the f_n converge to f pointwise everywhere rather than μ-almost everywhere, and similarly we can assume that $|f_n|$ are bounded by G pointwise everywhere rather than μ-almost everywhere. By taking real and imaginary parts we may assume without loss of generality that f_n, f are real, thus $-G \leq f_n \leq G$ pointwise. Of course, this implies that $-G \leq f \leq G$ pointwise also. If we apply Fatou's lemma to the unsigned functions $f_n + G$, we see that

$$\int_X f + G d\mu \leq \lim_{n\to\infty} inf \int_X f_n + G d\mu, \quad (38)$$

which on subtracting the finite quantity $\int_X G d\mu$ gives

$$\int_X f d\mu \leq \lim_{n\to\infty} inf \int_X f_n d\mu. \quad (39)$$

Similarly, if we apply that lemma to the unsigned functions $G - f_n$, we obtain

$$\int_X G - f d\mu \leq \lim_{n\to\infty} inf \int_X G - f_n d\mu; \quad (40)$$

negating this inequality and then cancelling $\int_X G d\mu$ again we conclude that

$$\lim_{n\to\infty} sup \int_X f_n d\mu \leq \int_X f d\mu. \quad (41)$$

The claim then follows by combining these inequalities.

A probability space is a measure space (Ω, F, P) of total measure1: $P(\Omega) = 1$. The measure P is known as a probability measure. If Ω is a (possibility infinite) non-empty set with the discrete σ-algebra 2^{Ω}, and if $(p_\omega)_{\omega \in \Omega}$ are a collection of real numbers in $[0,1]$ with $\sum_{\omega \in \Omega} p_\omega = 1$, then the probability measure P defined by $P := \sum_{\omega \in \Omega} p_\omega \delta_\omega$, or in other words

$$P(E) := \sum_{\omega \in E} p_\omega , \quad (42)$$

is indeed a probability measure, and $(\Omega, 2^{\Omega}, P)$ is a probability space. The function $\omega \mapsto p_\omega$ is known as the (discrete) probability distribution of the state variable ω. Similarly, if Ω is a Lebesgue measurable subset of R^d of positive (and possibly infinite) measure, and $f : \Omega \to [0, +\infty]$ is a Lebesgue measurable function on Ω (where of course we restrict the Lebesgue measure space on R^d to Ω in the usual fashion) with $\int_\Omega f(x)dx = 1$, then $(\Omega, \mathcal{L}[R^d]\downarrow_\Omega, P)$ is a probability space, where $P := m_f$ is the measure

$$P(E) := \int_\Omega 1_E(x) f(x) dx = \int_E f(x) dx . \quad (43)$$

The function f is known as the (continuous) probability density of the state variable ω.

Theorem 1

(Connes' Trace Theorem) *Let M be a compact n-dimensional manifold, ξ a complex vector bundle on M, and P a pseudo-differential operator of order $-n$ acting on sections of ξ. Then the corresponding operator P in $H = L^2(M, \xi)$ belongs to $\mathcal{L}^{1,\infty}(H)$ and one has*:

$$Tr_\omega(P) = \frac{1}{n}\operatorname{Res}(P) \quad (44)$$

for any ω.

Here Res is the restriction of the Adler-Manin-Wodzicki residue to pseudo-differential operators of order $-n$. Let ξ be the exterior bundle on a (closed) compact Riemannian manifold M, $|vol|$ the 1-density of M, $f \in C^\infty(M)$, M_f the operator given by f acting by multiplication on smooth sections of ξ, Δ the Hodge Laplacian on smooth sections of ξ, and $P = M_f(1+\Delta)^{-n/2}$, which is a pseudo-differential operator of order $-n$. Using Theorem 1, we have that

$$\phi_\omega(M_f) = Tr_\omega(M_f T_\Delta) = \frac{1}{2^{(n-1)}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)} \int_M f(x)|vol|(x) , \quad f \in C^\infty(M) \quad (45)$$

where we set $T_\Delta := (1+\Delta)^{-n/2} \in \mathcal{L}^{1,\infty}$. This has become the standard way to identify ϕ_ω with the Lebesgue integral for $f \in C^\infty(M)$.

Corollary 1

Let M be a n-dimensional (closed) compact Riemannian manifold with Hodge Laplacian Δ. Set $T_\Delta := (1+\Delta)^{-n/2} \in \mathcal{L}^{1,\infty}(L^2(M))$. Then

$$\phi_\omega(M_f) := Tr_\omega(M_f T_\Delta) = c\int_M f(x)|vol|(x), \quad \forall f \in L^\infty(M) \quad (46)$$

where $c > 0$ is a constant independent of $\omega \in DL_2$.

Theorem 2

Let M, Δ, T_Δ be as in Corollary 1. Then, $\langle M_f \rangle_{T_\Delta^s} = T_\Delta^{s/2} M_f T_\Delta^{s/2} \in \mathcal{L}^1(L^2(M))$ for all $s > 1$ if and only if $f \in L^1(M)$. Moreover, setting

$$\psi_\xi(M_f) := \xi\left(\frac{1}{k} Tr\left(\langle M_f \rangle_{T_\Delta^{1+\frac{1}{k}}}\right)\right) \quad (47)$$

for any $\xi \in BL$,

$$\psi_\xi(M_f) := \lim_{k\to\infty} \frac{1}{k} Tr\left(\langle M_f \rangle_{T_\Delta^{1+\frac{1}{k}}}\right) = c\int_M f(x)|vol|(x), \quad \forall f \in L^1(M) \quad (48)$$

for a constant $c > 0$ independent of $\xi \in BL$.

Thus ψ_ξ, as the residue of the zeta function $Tr\left(T_\Delta^{s/2} M_f T_\Delta^{s/2}\right)$ at $s = 1$, is the value of the Lebesgue integral of the integrable function f on M. This is the most general form of the identification between the Lebesgue integral and an algebraic expression involving M_f, the compact operator $(1+\Delta^2)^{-1}$ and a trace.

Theorem 3

Let $0 < G(D) \in \mathcal{L}^{1,\infty}$ and $\xi \in BL \cap DL$. Then

$$\phi_{\mathcal{L}(\xi)}(T_f) := Tr_{\mathcal{L}(\xi)}(T_f G(D)) = \xi\left(\frac{1}{k}\int_F f(x) d\mu_{1+\frac{1}{k}}(x)\right), \quad \forall f \in L_0^2(F, \mu_{1,\infty}). \quad (49)$$

Moreover, if $\lim_{k\to\infty} k^{-1}\int_F h(x) d\mu_{1+k^{-1}}(x)$ exists for all $h \in L^\infty(F, \mu_{1,\infty})$, then

$$\phi_\omega(T_f) := Tr_\omega(T_f G(D)) = \lim_{k\to\infty} \frac{1}{k}\int_F f(x) d\mu_{1+\frac{1}{k}}(x), \quad \forall f \in L_0^2(F, \mu_{1,\infty}) \quad (50)$$

and all $\omega \in DL_2$.

We note that it is possible to identify ϕ_ω with the Lebesgue integral.

Now we consider an arbitrary manifold X with a fixed continuous non-negative finite Borel measure m. The construction of the integral models of representations of the current groups G^X is based on the existence, in the space $D(X)$ of Schwartz distributions on X, of a certain measure $\mathcal{L}$ which is an infinite-dimensional analogue of the Lebesgue measure. Furthermore, we have that ξ runs over the points of the cone

$$l_+^1(X) = \left\{ \xi = \sum_{k=1}^{\infty} r_k \delta_{x_k} \middle| r_k > 0, \sum_k r_k < \infty, x_k \in X \right\},$$

on which the infinite-dimensional Lebesgue measure $\mathcal{L}$ is concentrated. With each finite partition of X into measurable sets,

$$\alpha : X = \bigcup_{k=1}^{n} X_k, \quad m(X_k) = \lambda_k, \quad k = 1,\dots,n,$$

we associate the cone $\mathcal{F}_\alpha = R_+^n$ of piecewise constant positive functions of the form

$$f(x) = \sum_{k=1}^{n} f_k \chi_k(x), \quad f_k > 0,$$

where χ_k is the characteristic function of X_k, and we denote by $\Phi_\alpha = (R_+^n)'$ the dual cone in the space distributions. We define a measure $\mathcal{L}_\alpha$ on Φ_α by

$$d\mathcal{L}_\alpha(x_1,\dots,x_n) = \prod_{k=1}^{n} \frac{x_k^{\lambda_k - 1} dx_k}{\gamma(\lambda_k)}, \quad \text{where} \quad \lambda_k = m(X_k). \quad (51)$$

Let $D_+(X) \subset D(X)$ be the set (cone) of non-negative Schwartz distributions on X, and let $l_+^1(X) \subset D_+(X)$ be the subset (cone) of discrete finite (non-negative) measures on X, that is,

$$l_+^1(X) = \left\{ \xi = \sum_{k=1}^{\infty} r_k \delta_{x_k} \middle| r_k > 0, \sum_k r_k < \infty, x_k \in X \right\}.$$

There is a natural projection $D_+(X) \to \Phi_\alpha$.

Theorem-definition

There is a σ-finite (infinite) measure $\mathcal{L}$ on the cone $D_+(X)$ that is finite on compact sets, concentrated on the cone $l_+^1(X)$, and such that for every partition α of the space X its projection on the subspace Φ_α has the form (51). *This measure is uniquely determined by its Laplace transform*

$$F(f) \equiv \int_{l_+^1(X)} \exp\left(- \sum_k r_k f(x_k) \right) d\mathcal{L}(\xi) = \exp\left(- \int_X \log f(x) dm(x) \right), \quad (52)$$

where f *is an arbitrary non-negative measurable function on* (X, m) *which satisfies* $\int_X \log f(x) dm(x) < \infty$.

Elements of $l_+^1(X)$ will be briefly denoted by $\xi = \{r_k, x_k\}_{k=1}^{\infty}$, or even just $\xi = \{r_k, x_k\}$ (sequences that differ only by the order of elements are regarded as identical).

Let us apply the properties of the measure $\mathcal{L}$ to computing the integral

$$I = \int_{l_+^1(X)} \left(\prod_{k=1}^{\infty} \varphi(r_k, x_k) \right) d\mathcal{L}(\xi), \quad (53)$$

where $\varphi(r, x)$ is a function on $R_+^* \times X$ satisfying the conditions

$$\varphi(0, x) \equiv 1 \quad \text{and} \quad \int_X \int_0^{\infty} \left(\varphi(r, x) - e^{-r} \right) r^{-1} dr dm(x) < \infty . \quad (54)$$

Theorem

The following equality holds:

$$\int_{l_+^1(X)} \left(\prod_{k=1}^{\infty} \varphi(r_k, x_k) \right) d\mathcal{L}(\xi) = (\exp \int_X \int_0^{\infty} \left(\varphi(r, x) - e^{-r} \right) r^{-1} dr dm(x)) . \quad (55)$$

Proof. Under the projection $D_+(X) \to \Phi_\alpha$ (recall that Φ_α is the finite-dimensional space associated with a partition $\alpha : X = \bigcup_{k=1}^n X_k$) the left-hand side of (53) takes the form

$$I_\alpha = \prod_{k=1}^{n} I_\alpha^k , \quad I_\alpha^k = \frac{1}{\Gamma(\lambda_k)} \int_0^{\infty} \varphi_{\alpha,k}(r_k) r_k^{\lambda_k - 1} dr_k , \quad (56)$$

where $\lambda_k = m(X_k)$ and $\varphi_{\alpha,k}(r_k) = \lambda_k^{-1} \int_{X_k} \varphi(r_k, x) dm(x)$. Thence the eq. (56) can be rewritten also as follows

$$I_\alpha = \prod_{k=1}^{n} \frac{1}{\Gamma(\lambda_k)} \int_0^{\infty} \lambda_k^{-1} \int_{X_k} \varphi(r_k, x) dm(x) r_k^{\lambda_k - 1} dr_k . \quad (56b)$$

The original integral I is the inductive limit of the integrals I_α over the set of partitions α . Since $\frac{1}{\Gamma(\lambda_k)} \int_0^{\infty} e^{-t} r_k^{\lambda_k - 1} dr_k = 1$, the integral I_α^k can be written in the form

$$I_\alpha^k = 1 + \frac{1}{\Gamma(\lambda_k)} \int_0^{\infty} \left(\varphi_{\alpha,k}(r_k) - e^{-r_k} \right) r_k^{\lambda_k - 1} dr_k . \quad (57)$$

It follows that

$$I_\alpha^k = 1 + \lambda_k \int_0^\infty \left(\varphi_{\alpha,k}(r_k) - e^{-r_k}\right) r_k^{-1} dr_k + O\left(\lambda_k^2\right), \quad (58)$$

whence

$$I_\alpha^k = \exp\left(\lambda_k \int_0^\infty \left(\varphi_{\alpha,k}(r_k) - e^{-r_k}\right) r_k^{-1} dr_k\right) + O\left(\lambda_k^2\right).$$

The eq. (58) can be rewritten also as follows

$$I_\alpha^k = 1 + \lambda_k \int_0^\infty \left(\varphi_{\alpha,k}(r_k) - e^{-r_k}\right) r_k^{-1} dr_k + O\left(\lambda_k^2\right) = \exp\left(\lambda_k \int_0^\infty \left(\varphi_{\alpha,k}(r_k) - e^{-r_k}\right) r_k^{-1} dr_k\right) + O\left(\lambda_k^2\right). \quad (58b)$$

Thus, up to terms of order greater than 1 with respect to λ_k,

$$I_\alpha \cong \exp\left(\sum_{k=1}^n \lambda_k \int_0^\infty \left(\varphi_{\alpha,k}(r) - e^{-r}\right) r^{-1} dr\right).$$

Since

$$\sum_{k=1}^n \left(\lambda_k \left(\varphi_{\alpha,k}(r) - e^{-r}\right)\right) = \int_X \left(\varphi(r,x) - e^{-r}\right) dm(x),$$

the expression obtained can be written in the following form

$$I_\alpha \cong \exp\left(\int_X \int_0^\infty \left(\varphi(r,x) - e^{-r}\right) r^{-1} dr dm(x)\right). \quad (59)$$

The proof is completed by taking the inductive limit over the set of partitions α .

Corollary

If $\varphi(r,x) = \sum_{i=1}^n c_i \varphi_i(r,x)$, *where* $c_i > 0$, $\sum c_i = 1$, *and the functions* $\varphi_i(r,x)$ *satisfy (54), then*

$$\int_{l_+^1(X)} \left(\prod_{k=1}^\infty \varphi(r_k, x_k)\right) d\mathcal{L}(\xi) = \prod_{i=1}^n \exp\left(c_i \int_X \int_0^\infty \left(\varphi_i(r,x) - e^{-r}\right) r^{-1} dr dm(x)\right). \quad (60)$$

Let $\varphi(r,x) = e^{-r^\sigma a(x)}$, where $\sigma \geq 1$ and $\operatorname{Re} a(x) > 0$. In this case we obtain

$$\int_{l_+^1(X)} \left(\prod_{k=1}^\infty e^{-r_k^\sigma a(x_k)}\right) d\mathcal{L}(\xi) = \exp\left(\int_X \int_0^\infty \left(e^{-r^\sigma a(x)} - e^{-r}\right) r^{-1} dr dm(x)\right). \quad (61)$$

Let us integrate with respect to r. We have

$$\int_0^\infty \left(e^{-r^\sigma a(x)} - e^{-r}\right) r^{-1} dr = \lim_{\lambda \to 0}\left(\int_0^\infty \left(e^{-r^\sigma a(x)} - e^{-r}\right) r^{\lambda-1} dr\right) = \lim_{\lambda \to 0}\left(\sigma^{-1} \Gamma\left(\frac{\lambda}{\sigma}\right) a^{-\lambda/\sigma}(x) - \Gamma(\lambda)\right). \quad (62)$$

Since $\Gamma(\lambda) \approx \lambda^{-1} + \gamma$ as $\lambda \to 0$, where γ is the Euler constant, it follows that

$$\int_0^\infty \left(\exp\left(-r^\sigma a(x)\right) - \exp(-r)\right) r^{-1} dr = -\sigma^{-1} \log a(x) + \left(\sigma^{-1} - 1\right)\gamma .$$

Hence,

$$\int_{l_+^1(X)} \left(\prod_{k=1}^{\infty} \exp\left(-r_k^\sigma a(x_k)\right) \right) d\mathcal{L}(\xi) = \exp\left(\left(\sigma^{-1} - 1\right)\gamma\right) \exp\left(-\sigma^{-1} \int_X \log a(x) dm(x)\right). \quad (63)$$

In particular, for $\sigma = 1$ we recover the original formula for the Laplace transform of the measure $\mathcal{L}$:

$$\int_{l_+^1(X)} \prod_{k=1}^{\infty} \exp\left(-\sum r_k a(x_k)\right) d\mathcal{L}(\xi) = \exp\left(-\int_X \log a(x) dm(x)\right). \quad (63b)$$

2. **Mathematical applications in some equations concerning various sectors of Chern-Simons theory and Yang-Mills gauge theory**

2.1 Chern-Simons theory **[5]**

The typical functional integral arising in quantum field theory has the form

$$Z^{-1} \int_{\mathcal{A}} f(A) e^{\beta S(A)} DA \quad (64)$$

where $S(\cdot)$ is an action functional, β a physical constant (real or complex), f is some function of the field A of interest, DA signifies "Lebesgue integration" on an infinite-dimensional space $\mathcal{A}$ of field configurations, and Z a "normalizing constant". Now we shall describe the Chern-Simons theory over R^3, with gauge group a compact matrix group G whose Lie algebra is denoted $L(G)$. The formal Chern-Simons functional integral has the form

$$\frac{1}{Z} \int_{\mathcal{A}} f(A) e^{iCS(A)} DA \quad (65)$$

where f is a function of interest on the linear space $\mathcal{A}$ of all $L(G)$-valued 1-forms A on R^3, and $CS(\cdot)$ is the Chern-Simons action given by

$$CS(A) = \frac{\kappa}{4\pi} \int_{R^3} Tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (66)$$

involving a parameter κ. We choose a gauge in which one component of $A = a_0 dx_1 + a_1 dx_1 + a_2 dx_2$ vanishes, say $a_2 = 0$. This makes the triple wedge term $A \wedge A \wedge A$ disappear, and we end up with a quadratic expression

$$CS(A) = \frac{\kappa}{4\pi}\int_{R^3} Tr(A \wedge dA) \qquad \text{for } A = a_0dx_0 + a_1dx_1. \quad (67)$$

Then the functional integral has the form

$$\frac{1}{Z}\int_{\mathrm{A}_0} \phi(A) e^{i\frac{\kappa}{4\pi}\int_{R^3} Tr(A\wedge dA)} DA \quad (68)$$

where A_0 consists of all A for which $a_2 = 0$. As in the two dimensional case, the integration element remains DA after gauge fixing. The map

$$\phi \mapsto \langle \phi \rangle_{CS} = \frac{1}{Z}\int_{\mathrm{A}_0} \phi(A) e^{i\frac{\kappa}{4\pi}\int_{R^3} Tr(A\wedge dA)} DA, \quad (69)$$

whatever rigorously, would be a linear functional on a space of functions ϕ over A_0. Now for $A = a_0dx_0 + a_1dx_1 \in \mathrm{A}_0$, decaying fast enough at infinity, we have, on integrating by parts,

$$CS(a_0dx_0 + a_1dx_1) = -\frac{\kappa}{2\pi}\int_{R^3} Tr(a_0f_1)dx_0dx_1dx_2 \quad (70)$$

where

$$f_1 = \partial_2 a_1. \quad (71)$$

So now the original functional integral is reformulated as an integral of the form

$$\langle \phi \rangle_{CS} = \frac{1}{Z}\int e^{i\frac{\kappa}{4\pi}\langle a_0, f_1\rangle} \phi(a_0, f_1) Da_0 Df_1 \quad (72)$$

where

$$\langle a, f\rangle = -\int_{R^3} Tr(af)dvol, \quad (73)$$

and Z always denotes the relevant formal normalizing constant. Taking ϕ to be of the special form

$$\phi_j(a_0, f_1) = e^{ia_0(j_0) + if_1(j_1)} \quad (74)$$

where j_0 and j_1 are, say, rapidly decreasing $L(G)$-valued smooth functions on R^3, we find, from a formal calculation

$$\langle \phi_j \rangle_{CS} = e^{-i\frac{2\pi}{\kappa}(j_0, j_1)}. \quad (75)$$

In the paper "Non-Abelian localization for Chern-Simons theory" of Beasley and Witten (2005), the Chern-Simons partition function is

$$Z(k) = \frac{1}{Vol(\mathcal{G})}\left(\frac{k}{4\pi^2}\right)^{\Delta \mathcal{G}} \int \mathfrak{D}A \exp\left[i\frac{k}{4\pi}\int_X Tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right]. \quad (76)$$

We note that, in this equation, $\mathfrak{D}A$ signifies "Lebesgue integration" on an infinite-dimensional space of field configurations. If X is assumed to carry the additional geometric structure of a Seifert manifold, then the partition function of eq. (76) does admit a more conventional interpretation in terms of the cohomology of some classical moduli space of connections. Using the additional Seifert structure on X, decouple one of the components of a gauge field A, and introduce a new partition function

$$\bar{Z}(k) = K\cdot\int \mathfrak{D}A\mathfrak{D}\Phi\ \exp\left[i\frac{k}{4\pi}\left(\int_X Tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) - \int_X 2\kappa\wedge Tr(\Phi F_A) + \int_X \kappa\wedge d\kappa Tr(\Phi^2)\right)\right],$$

(77), then give a heuristic argument showing that the partition function computed using the alternative description of eq. (77) should be the same as the Chern-Simons partition function of eq. (76). In essence, it is possible to show that

$$Z(k) = \bar{Z}(k), \quad (78)$$

by gauge fixing $\Phi = 0$ using the shift symmetry. The Φ dependence in the integral can be eliminated by simply performing the Gaussian integral over Φ in eq. (77) directly. We obtain the alternative formulation

$$Z(k) = \bar{Z}(k) = K'\cdot\int \mathfrak{D}A \exp\left[i\frac{k}{4\pi}\left(\int_X Tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) - \int_X \frac{1}{\kappa\wedge d\kappa} Tr\left[(\kappa\wedge F_A)^2\right]\right)\right], \quad (79)$$

where $K' := \frac{1}{Vol(\mathcal{G})}\frac{1}{Vol(S)}\left(\frac{-ik}{4\pi^2}\right)^{\Delta\mathcal{G}/2}$. thence, we can rewrite the eq. (79) also as follows:

$$Z(k) = \bar{Z}(k) = \frac{1}{Vol(\mathcal{G})}\frac{1}{Vol(S)}\left(\frac{-ik}{4\pi^2}\right)^{\Delta\mathcal{G}/2}\cdot$$

$$\cdot\int \mathfrak{D}A \exp\left[i\frac{k}{4\pi}\left(\int_X Tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) - \int_X \frac{1}{\kappa\wedge d\kappa} Tr\left[(\kappa\wedge F_A)^2\right]\right)\right]. \quad (79b)$$

We restrict to the gauge group $U(1)$ so that the action is quadratic and hence the stationary phase approximation is exact. A salient point is that the group $U(1)$ is not simple, and therefore may have non-trivial principal bundles associated with it. This makes the $U(1)$-theory very different from the $SU(2)$-theory in that one must now incorporate a sum over bundle classes in a definition of the $U(1)$-partition function. As an analogue of eq. (76), our basic definition of the partition function for $U(1)$-Chern-Simons theory is now

$$Z_{U(1)}(X,k) = \sum_{p\in TorsH^2(X;Z)} Z_{U(1)}(X,p,k) \quad (80)$$

where

$$Z_{U(1)}(X,p,k)=\frac{1}{Vol(\mathcal{G}_P)}\int_{\mathcal{A}_P}\mathfrak{D}Ae^{\pi ikS_{X,P}(A)}. \quad (81)$$

Thence, the eq. (80) can be rewritten also as follows

$$Z_{U(1)}(X,k)=\sum_{p\in TorsH^2(X;Z)}\frac{1}{Vol(\mathcal{G}_P)}\int_{\mathcal{A}}\mathfrak{D}Ae^{\pi ikS_{X,P}(A)}. \quad (80b)$$

The main result is the following:

Proposition 1

Let (X,ϕ,ξ,κ,g) be a closed, *quasi-regular* K*-contact* three manifold. If,

$$\bar{Z}_{U(1)}(X,p,k)=k^{n_X}e^{\pi ikS_{X,P}(A_0)}e^{\frac{\pi i}{4}\left(\eta(-*D)+\frac{1}{512}\int_X R^2\kappa\wedge d\kappa\right)}\int_{\mathcal{M}_P}\left(T_C^d\right)^{1/2} \quad (82)$$

where $R\in C^\infty(X)=$ the Tanaka-Webster scalar curvature of X, and

$$Z_{U(1)}(X,p,k)=k^{m_X}e^{\pi ikS_{X,P}(A_0)}e^{\pi i\left(\frac{\eta(-*d)}{4}+\frac{1}{12}\frac{CS(A^g)}{2\pi}\right)}\int_{\mathcal{M}_P}\left(T_{RS}^d\right)^{1/2} \quad (83)$$

then

$$Z_{U(1)}(X,k)=\bar{Z}_{U(1)}(X,k)$$

as topological invariants.

Now our starting point is the analogue of eq. (79) for the $U(1)$-Chern-Simons partition function:

$$\bar{Z}_{U(1)}(X,p,k)=\frac{e^{\pi ikS_{X,P}(A_0)}}{Vol(S)Vol(\mathcal{G}_P)}\int_{\mathcal{A}_P}DA\exp\left[\frac{ik}{4\pi}\left(\int_X A\wedge dA-\int_X\frac{(\kappa\wedge dA)^2}{\kappa\wedge d\kappa}\right)\right] \quad (84)$$

where $S_{X,P}(A_0)$ is the Chern-Simons invariant associated to P for A_0 a flat connection on P. Also here, DA signifies "Lebesgue integration" on an infinite-dimensional space $\mathcal{A}_P$ of field configurations. The eq. (84) is obtained by expanding the $U(1)$ analogue of eq. (79) around a critical point A_0 of the action. Note that the critical points of this action, up to the action of the shift symmetry, are precisely the flat connections. In our notation, $A\in T_{A_0}\mathcal{A}_P$. Let us define the notation

$$S(A):=\int_X A\wedge dA-\int_X\frac{(\kappa\wedge dA)^2}{\kappa\wedge d\kappa} \quad (85)$$

for the new action that appears in the partition function. Also define

$$\bar{S}(A) := \int_X \frac{(\kappa \wedge dA)^2}{\kappa \wedge d\kappa} \quad (86)$$

so that we may write

$$S(A) = CS(A) - \bar{S}(A). \quad (87)$$

Thence, we can rewrite the eq. (87) also as follows:

$$\int_X A \wedge dA - \int_X \frac{(\kappa \wedge dA)^2}{\kappa \wedge d\kappa} = \int_X Tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) - \int_X \frac{(\kappa \wedge dA)^2}{\kappa \wedge d\kappa}. \quad (87b)$$

The primary virtue of eq. (84) above is that it is exactly equal to the original Chern-Simons partition function of eq. (81) and yet it is expressed in such a way that the action $S(A)$ is invariant under the shift symmetry. This means that $S(A + \sigma\kappa) = S(A)$ for all tangent vectors $A \in T_{A_0}(\mathcal{A}_P) \cong \Omega^1(X)$ and $\sigma \in \Omega^0(X)$. We may naturally view $A \in \Omega^1(H)$, the sub-bundle of $\Omega^1(X)$ restricted to the contact distribution $H \subset TX$. Equivalently, if ξ denotes the Reeb vector field of κ, then $\Omega^1(H) = \{\omega \in \Omega^1(X) | \iota_\xi \omega = 0\}$. The remaining contributions to the partition function come from the orbits of $\boldsymbol{S}$ in $\mathcal{A}_P$, which turn out to give a contributing factor of $Vol(\boldsymbol{S})$. We thus reduce our integral to an integral over $\bar{\mathcal{A}}_P := \mathcal{A}_P / \boldsymbol{S}$ and obtain:

$$Z_{U(1)}(X, p, k) = \frac{e^{\pi i k S_{X,P}(A_0)}}{Vol(\mathcal{G}_P)} \int_{\bar{\mathcal{A}}_P} \bar{D}A \exp\left[\frac{ik}{4\pi}\left(\int_X A \wedge dA - \int_X \frac{(\kappa \wedge dA)^2}{\kappa \wedge d\kappa}\right)\right] =$$

$$= \frac{e^{\pi i k S_{X,P}(A_0)}}{Vol(\mathcal{G}_P)} \int_{\bar{\mathcal{A}}_P} \bar{D}A \exp\left[\frac{ik}{4\pi} S(A)\right], \quad (88)$$

where $\bar{D}A$ denotes an appropriate quotient measure on $\bar{\mathcal{A}}_P$, i.e. the “Lebesgue integration” on an infinite-dimensional space $\bar{\mathcal{A}}_P$ of field configurations.

We now have

$$Z_{U(1)}(X, p, k) = \frac{e^{\pi i k S_{X,P}(A_0)}}{Vol(\mathcal{G}_P)} \int_{\bar{\mathcal{A}}_P} \bar{D}A e^{\left[\frac{ik}{4\pi} S(A)\right]} = \frac{Vol(\mathcal{G}_P)}{Vol(H)} \frac{e^{\pi i k S_{X,P}(A_0)}}{Vol(\mathcal{G}_P)} \int_{\bar{\mathcal{A}}_P / \mathcal{G}_P} e^{\left[\frac{ik}{4\pi} S(A)\right]} \left[\det{}'(d_H^* d_H)\right]^{1/2} \mu =$$

$$= \frac{e^{\pi i k S_{X,P}(A_0)}}{Vol(H)} \int_{\bar{\mathcal{A}}_P / \mathcal{G}_P} e^{\left[\frac{ik}{4\pi} S(A)\right]} \left[\det{}'(d_H^* d_H)\right]^{1/2} \mu \quad (89)$$

where μ is the induced measure on the quotient space $\bar{\mathcal{A}}_P / \mathcal{G}_P$ and det' denotes a regularized determinant. Since $S(A) = \langle A, - *_H D^1 A \rangle_\kappa^1$ is quadratic in A, we may apply the method of stationary phase to evaluate the oscillatory integral (89) exactly. We obtain,

$$Z_{U(1)}(X,p,k)=\frac{e^{\pi ikS_{X,P}(A_0)}}{Vol(H)}\int_{\mathrm{M}_P}e^{\frac{\pi i}{4}\mathrm{sgn}(-*_H D^1)}\frac{[\det'(d_H^* d_H)]^{1/2}}{[\det'(-k*_H D^1)]^{1/2}}\nu \qquad (90)$$

We will use the following to define the regularized determinant of $-k*_H D^1$:

$$\det'(-k*_H D^1):=C(k,J)\cdot\frac{[\det'(S^2+TT^*)]^{1/2}}{[\det'(TT^*)]^{1/2}} \qquad (90b)$$

where $S^2+TT^*=k^2\left((D^1)*D^1+(d_H d_H^*)^2\right)$, $TT^*=k^2(d_H d_H^*)^2$ and

$$C(k,J):=k^{\left(-\frac{1}{1024}\int_X R^2\kappa\wedge d\kappa\right)} \qquad (90c)$$

is a function of $R\in C^\infty(X)$, the Tanaka-Webster scalar curvature of X, which in turn depends only on a choice of a compatible complex structure $J\in \mathrm{End}(H)$. The operator

$$\Delta:=(D^1)^* D^1+(d_H d_H^*)^2 \qquad (91)$$

is equal to the middle degree Laplacian and is maximally hypoelliptic and invertible in the Heisenberg symbolic calculus. We define the regularized determinant of Δ via its zeta function

$$\zeta(\Delta)(s):=\sum_{\lambda\in spec^*(\Delta)}\lambda^{-s} \qquad (92)$$

Also, $\zeta(\Delta)(s)$ admits a meromorphic extension to C that is regular at $s=0$. Thus, we define the regularized determinant of Δ as

$$\det'(\Delta):=e^{-\zeta'(\Delta)(0)}. \qquad (93)$$

Let $\Delta_0:=(d_H^* d_H)^2$ on $\Omega^0(X)$, $\Delta_1:=\Delta$ on $\Omega^1(H)$ and define $\zeta_i(s):=\zeta(\Delta_i)(s)$. We claim the following

Proposition 2

For any real number $0<c\in R$,

$$\det'(c\Delta_i):=c^{\zeta_i(0)}\det'(\Delta_i) \qquad (94)$$

for $i=0,1$.

Proposition 3

For $\Delta_0:=(d_H^* d_H)^2$ on $\Omega^0(X)$, $\Delta_1:=\Delta$ on $\Omega^1(H)$ defined as above and $\zeta_i(s):=\zeta(\Delta_i)(s)$, we have

$$\zeta_0(0)-\zeta_1(0)=\left(-\frac{1}{8^3}\int_X R^2\kappa\wedge d\kappa\right)+\dim Ker\Delta_1-\dim Ker\Delta_0, \qquad (95)$$

$$\zeta_0(0) - \zeta_1(0) = \left(-\frac{1}{512}\int_X R^2\kappa \wedge d\kappa \right) + \dim H^1(X,d_H) - \dim H^0(X,d_H), \quad (96)$$

where $R \in C^\infty(X)$ is the Tanaka-Webster scalar curvature of X and $\kappa \in \Omega^1(X)$ is our chosen contact form as usual. Let

$$\hat{\zeta}_0(s) := \dim Ker\Delta_0 + \zeta_0(s), \quad \hat{\zeta}_1(s) := \dim Ker\Delta_1 + \zeta_1(s) \quad (97)$$

denote the zeta functions. We have that $\hat{\zeta}_1(0) = 2\hat{\zeta}_0(0)$ for all 3-dimensional contact manifolds. We know that on CR-Seifert manifolds that

$$\hat{\zeta}_0(0) = \hat{\zeta}(\Delta_0)(0) = \hat{\zeta}(\Delta_0^2)(0) = \frac{1}{512}\int_X R^2\kappa \wedge d\kappa . \quad (98)$$

Thus,

$$\hat{\zeta}_1(0) = \frac{1}{256}\int_X R^2\kappa \wedge d\kappa . \quad (99)$$

By our definition of the zeta functions, we therefore have

$$\zeta_0(0) = \frac{1}{512}\int_X R^2\kappa \wedge d\kappa - \dim Ker\Delta_0, \quad \zeta_1(0) = \frac{1}{256}\int_X R^2\kappa \wedge d\kappa - \dim Ker\Delta_1 . \quad (100)$$

Hence,

$$\zeta_0(0) - \zeta_1(0) = \left[\frac{1}{512}\int_X R^2\kappa \wedge d\kappa - \dim Ker\Delta_0\right] - \left[\frac{1}{256}\int_X R^2\kappa \wedge d\kappa - \dim Ker\Delta_1\right] =$$

$$= \left(-\frac{1}{512}\int_X R^2\kappa \wedge d\kappa \right) + \dim Ker\Delta_1 - \dim Ker\Delta_0 =$$

$$= \left(-\frac{1}{512}\int_X R^2\kappa \wedge d\kappa \right) + \dim H^1(X,d_H) - \dim H^0(X,d_H). \quad (101)$$

2.2 Yang-Mills Gauge Theory **[6]**

Let Σ be an oriented closed Riemann surface of genus g. Let E be an H bundle over Σ. The adjoint vector bundle associated with E will be called $\mathrm{ad}(E)$. Let $\mathcal{A}$ be the space of connections on E, and G be the group of gauge transformations on E. The Lie algebra $\mathcal{G}$ of G is the space of $\mathrm{ad}(E)$-valued two-forms. G acts symplectically on $\mathcal{A}$, with a moment map given by the map

$$\mu(A) = -\frac{F}{4\pi^2}, \quad (102)$$

from the connection A to its $\mathrm{ad}(E)$-valued curvature two-form $F = dA + A \wedge A$. $\mu^{-1}(0)$ therefore consists of flat connections, and $\mu^{-1}(0)/G$ is the moduli space $\mathcal{M}$ of flat connections on E up to gauge transformation. $\mathcal{M}$ is a component of the moduli space of homomorphisms $\rho : \pi_1(\Sigma) \to H$, up to conjugation. The partition function of two dimensional quantum Yang-Mills theory on the surface Σ is formally given by the Feynman path integral

$$Z(\varepsilon) = \frac{1}{vol(G)} \int_{\mathrm{A}} DA \exp\left(-\frac{1}{2\varepsilon}(F,F) \right), \quad (103)$$

where ε is a real constant, DA is the symplectic measure on the infinite dimensional function space A, i.e. the "Lebesgue integration", and $vol(G)$ is the volume of G.

For any BRST invariant operator O (we want remember that the BRST (i.e. the Becchi-Rouet-Stora-Tyutin) invariance is a nilpotent symmetry of Faddeev-Popov gauge-fixed theories, which encodes the information contained in the original gauge symmetry) , let $\langle \mathrm{O} \rangle$ be the expectation value of O computed with the following equation concerning the cohomological theory

$$L = -i\{Q,V\} = \frac{1}{h^2} \int_{\Sigma} d\mu\, Tr\left(\frac{1}{2}(H-f)^2 - \frac{1}{2} f^2 - i\chi * D\psi + iD_i\eta\, \psi^i + D_i\lambda D^i\phi + \frac{i}{2}\chi[\chi,\phi] + i[\psi_i,\lambda]\psi^i \right),$$
(104)

and let $\langle \mathrm{O} \rangle'$ be the corresponding expectation value concerning the following equation

$$L''(u) = \frac{i}{h^2 u} \int_{\Sigma} d\mu\, Tr\left(D_i f D^i\phi + if[\psi_i, \psi^i] - iD_i\psi_l \varepsilon^{ij} D_i\psi_j \right). \quad (105)$$

We will describe a class of O's such that the higher critical points do not contribute, and hence $\langle \mathrm{O} \rangle = \langle \mathrm{O} \rangle'$. Two particular BRST invariant operators will play an important role. The first, related to the symplectic structure of $\mathcal{M}$, is

$$\omega = \frac{1}{4\pi^2} \int_{\Sigma} Tr\left(i\phi F + \frac{1}{2}\psi \wedge \psi \right). \quad (106)$$

The second is

$$\Theta = \frac{1}{8\pi^2} \int_{\Sigma} d\mu\, Tr\phi^2. \quad (107)$$

We wish to compute

$$\langle \exp(\omega + \varepsilon\Theta) \cdot \beta \rangle' \quad (108)$$

with ε a positive real number, and β an arbitrary observable with at most a polynomial dependence on ϕ. This is

$$\langle \exp(\omega + \varepsilon\Theta)\cdot\beta\rangle' = \frac{1}{vol(G)}\int DAD\psi D\phi\cdot\beta\cdot\exp\left(\frac{1}{h^2 u}\left\{Q,\int_\Sigma d\mu\,\psi^k D_k f\right\} + \right.$$

$$\left. + \frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F + \frac{1}{2}\psi\wedge\psi\right) + \frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu\,Tr\phi^2\right). \quad (109)$$

Thus, we can simply set $u = \infty$ in eq. (109), discarding the terms of order $1/u$, and reducing to

$$\langle \exp(\omega + \varepsilon\Theta)\cdot\beta\rangle' = \frac{1}{vol(G)}\int DAD\psi D\phi\,\exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F + \frac{1}{2}\psi\wedge\psi\right) + \frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu\,Tr\phi^2\right)\cdot\beta\,. \quad (110)$$

Thence, we have passed from "cohomological" to "physical" Yang-Mills theory. Also here DA is the "Lebesgue integration". With regard the eq. (110), if assuming that $\beta = 1$, in this case, the only ψ dependent factors are in

$$DAD\psi\,\exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\psi\wedge\psi\right). \quad (111)$$

Let us generalize to $\varepsilon \neq 0$, but for simplicity $\beta = 1$. In this case, by integrating out ϕ, we get

$$\langle \exp(\omega + \varepsilon\Theta)\rangle' = \frac{1}{vol(G)}\int DA\exp\left(\frac{2\pi^2}{\varepsilon}\int_\Sigma d\mu\,Trf^2\right). \quad (112)$$

This is the path integral of conventional two dimensional Yang-Mills theory. Now, at $\varepsilon \neq 0$, we cannot claim that the $\langle\ \rangle$ and $\langle\ \rangle'$ operations coincide, since the higher critical components $\mathcal{M}_\alpha$ contribute. However, their contributions are exponentially small, involving the relevant values of $I = -\int_\Sigma d\mu\,Trf^2$. So we get

$$\langle \exp(\omega + \varepsilon\Theta)\rangle = \frac{1}{vol(G)}\int DA\exp\left(\frac{2\pi^2}{\varepsilon}\int_\Sigma d\mu\,Trf^2\right) + O\left(\exp\left(-2\pi^2 c/\varepsilon\right)\right), \quad (113)$$

where c is the smallest value of the Yang-Mills action I on one of the higher critical points. We consider the topological field theory with Lagrangian

$$L = -\frac{i}{4\pi^2}\int_\Sigma Tr\phi F\,, \quad (114)$$

which is related to Reidemeister-Ray-Singer torsion. The partition function is defined formally by

$$Z(\Sigma) = \frac{1}{Vol(G)}\int DAD\phi\,\exp(-L). \quad (115)$$

Here if E is trivial, G is the group of maps of Σ to H; in general G is the group of gauge transformations. Now we want to calculate the H' partition function

$$\widetilde{Z}(\Sigma;u) = \frac{1}{Vol(G')}\int DA'D\phi\, \exp(-L). \quad (116)$$

Thence, with the eq. (114) we can rewrite the eq. (116) also as follows

$$\widetilde{Z}(\Sigma;u) = \frac{1}{Vol(G')}\int DA'D\phi\, \exp\left(\frac{i}{4\pi^2}\int_\Sigma Tr\phi F\right). \quad (116b)$$

First we calculate the corresponding H partition function for connections on $\Sigma - P$ with monodromy u around P. This is

$$Z(\Sigma;u) = \frac{1}{Vol(G)}\int DAD\phi\, \exp(-L). \quad (117)$$

Also here, we can rewrite the above equation also as follows

$$Z(\Sigma;u) = \frac{1}{Vol(G')}\int DAD\phi\, \exp\left(\frac{i}{4\pi^2}\int_\Sigma Tr\phi F\right). \quad (117b)$$

A is the lift of A'. From what we have just said, this is given by the same formula as the following

$$Z(\Sigma) = \left(\frac{Vol(H)}{(2\pi)^{\dim H}}\right)^{2g-2} \cdot \sum_\alpha \frac{1}{(\dim\alpha)^{2g-2}}, \quad (118)$$

but weighting each representation by an extra factor of $\lambda_\alpha(u^{-1})$. So

$$Z(\Sigma;u) = \left(\frac{Vol(H)}{(2\pi)^{\dim H}}\right)^{2g-2} \cdot \sum_\alpha \frac{\lambda_\alpha(u^{-1})}{(\dim\alpha)^{2g-2}}. \quad (119)$$

We now use the following equation

$$Vol(G) = \#\Gamma^{1-2g} Vol(G'), \quad (120)$$

to relate $Z(\Sigma;u)$ to $\widetilde{Z}(\Sigma;u)$, and also

$$Vol(H) = \#\Gamma \cdot Vol(H'); \quad \#Z(H) = \#\Gamma \cdot \#Z(H'); \quad \#\pi_1(H') = \#\Gamma; \quad (121)$$

to express the result directly in terms of properties of H'. Using also (121), we get

$$\widetilde{Z}(\Sigma;u) = \frac{1}{\#\pi_1(H')}\left(\frac{Vol(H')}{(2\pi)^{\dim H'}}\right)^{2g-2} \cdot \sum_\alpha \frac{\lambda_\alpha(u^{-1})}{(\dim\alpha)^{2g-2}}. \quad (122)$$

Note that in this formula, the sums runs over all isomorphism classes of irreducible representations of the universal cover H of H'. We can immediately write down the partition function, with gauge group H', for connections on a bundle $E'(u)$, generalizing (122) to $\varepsilon \neq 0$. We get

$$\widetilde{Z}(\Sigma,\varepsilon;u)=\frac{1}{\#\pi_1(H')}\left(\frac{Vol(H')}{(2\pi)^{\dim H'}}\right)^{2g-2}\cdot\sum_{\alpha}\frac{\lambda_\alpha(u^{-1})\cdot\exp\left(-\varepsilon'\left(\frac{C_2(\alpha)}{2}+t\right)\right)}{(\dim\alpha)^{2g-2}}. \quad (122a)$$

Furthermore, with the (119) and (122) we can rewrite the eqs. (116b) and (117b) also as follows:

$$\widetilde{Z}(\Sigma;u)=\frac{1}{Vol(G')}\int DA'D\phi\,\exp\left(\frac{i}{4\pi^2}\int_\Sigma Tr\phi F\right)=\frac{1}{\#\pi_1(H')}\left(\frac{Vol(H')}{(2\pi)^{\dim H'}}\right)^{2g-2}\cdot\sum_\alpha\frac{\lambda_\alpha(u^{-1})}{(\dim\alpha)^{2g-2}}; \quad (122b)$$

$$Z(\Sigma;u)=\frac{1}{Vol(G')}\int DAD\phi\,\exp\left(\frac{i}{4\pi^2}\int_\Sigma Tr\phi F\right)=\left(\frac{Vol(H)}{(2\pi)^{\dim H}}\right)^{2g-2}\cdot\sum_\alpha\frac{\lambda_\alpha(u^{-1})}{(\dim\alpha)^{2g-2}}. \quad (122c)$$

We consider the case of $H=SU(2)$. Then $Vol(SU(2))=2^{5/2}\pi^2$ with our conventions, and so

$$Z(\Sigma,\varepsilon)=\frac{1}{(2\pi^2)^{g-1}}\sum_{n=1}^{\infty}\frac{\exp(-\varepsilon' n^2/4)}{n^{2g-2}}. \quad (123)$$

On the other hand, for a non-trivial $SO(3)$ bundle with $u=-1$, we have $\lambda_n(u^{-1})=(-1)^{n+1}$, $\#\pi_1(H')=2$ and $Vol(SO(3))=2^{3/2}\pi^2$, so

$$\widetilde{Z}(\Sigma,\varepsilon;-1)=\frac{1}{2\cdot(8\pi^2)^{g-1}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\exp(-\varepsilon' n^2/4)}{n^{2g-2}}. \quad (124)$$

We will now show how (124) and (123) can be written as a sum over critical points. First we consider the case of a non-trivial $SO(3)$ bundle. It is convenient to look at not $\widetilde{Z}$ but

$$\frac{\partial^{g-1}\widetilde{Z}}{\partial\varepsilon'^{g-1}}=\frac{(-1)^g}{2\cdot(32\pi^2)^{g-1}}\sum_{n=1}^{\infty}(-1)^n\exp(-\varepsilon' n^2/4). \quad (125)$$

We write

$$\sum_{n=1}^{\infty}(-1)^n\exp(-\varepsilon' n^2/4)=-\frac{1}{2}+\frac{1}{2}\sum_{n\in Z}(-1)^n\exp(-\varepsilon' n^2/4). \quad (126)$$

The sum on the right hand side of (126) is a **theta function**, and in the standard way we can use the Poisson summation formula to derive the Jacobi inversion formula:

$$\sum_{n\in Z}(-1)^n\exp(-\varepsilon' n^2/4)=\sum_{m\in Z}\int_{-\infty}^{\infty}dn\exp(2\pi inm+i\pi n-\varepsilon' n^2/4)=\sqrt{\frac{4\pi}{\varepsilon'}}\sum_{m\in Z}\exp\left(-\frac{(2\pi(m+1/2))^2}{\varepsilon'}\right) \quad (127)$$

Putting the pieces together,

$$\frac{\partial^{g-1}\widetilde{Z}}{\partial\varepsilon'^{g-1}}=\frac{(-1)^g}{4\cdot(32\pi^2)^{g-1}}\cdot\left(-1+\sqrt{\frac{4\pi}{\varepsilon'}}\sum_{m\in Z}\exp\left(-\frac{(2\pi(m+1/2))^2}{\varepsilon'}\right)\right). \quad (128)$$

The eq. (128) shows that $\partial^{g-1}\widetilde{Z}/\partial\varepsilon'^{g-1}$ is a constant up to exponentially small terms, and hence $\widetilde{Z}(\varepsilon)$ is a polynomial of degree $g-1$ up to exponentially small terms. The terms of order ε^k, $k \leq g-2$ that have been annihilated by differentiating $g-1$ times with respect to ε' are most easily computed by expanding (124) in powers of ε :

$$\widetilde{Z}(\varepsilon) = \frac{1}{2(8\pi^2)^{g-1}} \sum_{k=0}^{g-2} \frac{(-\pi^2\varepsilon)^k}{k!} \left(1 - 2^{3-2g+2k}\right) \zeta(2g-2-2k) + O(\varepsilon^{g-1}). \quad (129)$$

Using **Euler's formula** expressing $\zeta(2n)$ for positive integral n in terms of the **Bernoulli number** B_{2n},

$$\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1} B_{2n}}{2(2n)!}, \quad (130)$$

eq. (129) implies

$$\int_{\mathcal{M}'(-1)} \exp(\omega + \varepsilon\Theta) = (-1)^{g+1} \sum_{k=0}^{g-1} \frac{\varepsilon^k}{k!} \frac{\left(2^{2g-2-2k} - 2\right) B_{2g-2-2k}}{2^{3g-1}(2g-2-2k)!}. \quad (131)$$

Thence, we obtain the following relationship:

$$\widetilde{Z}(\varepsilon) = \frac{1}{2(8\pi^2)^{g-1}} \sum_{k=0}^{g-2} \frac{(-\pi^2\varepsilon)^k}{k!} \left(1 - 2^{3-2g+2k}\right) \zeta(2g-2-2k) + O(\varepsilon^{g-1}) \Rightarrow$$

$$\Rightarrow \int_{\mathcal{M}'(-1)} \exp(\omega + \varepsilon\Theta) = (-1)^{g+1} \sum_{k=0}^{g-1} \frac{\varepsilon^k}{k!} \frac{\left(2^{2g-2-2k} - 2\right) B_{2g-2-2k}}{2^{3g-1}(2g-2-2k)!}. \quad (132)$$

With regard the link between the **Bernoulli number** and the Riemann's zeta function, we remember that

$$S_k(x) = \frac{B_{k+1}(x+1) - B_{k+1}(1)}{k+1}.$$

As $B'_m(x) = mB_{m-1}(x)$ for all m, we see that

$$\int_0^1 S_k(x-1)dx = \int_0^1 \frac{B_{k+1}(x+1) - B_{k+1}(1)}{k+1} = (-1)^k \frac{B_{k+1}}{k+1}.$$

Thence, we have that:

$$\zeta(-k) = \int_0^1 S_k(x-1)dx = (-1)^k \frac{B_{k+1}}{k+1}.$$

The cohomology of the smooth $SO(3)$ moduli space $\mathcal{M}'(-1)$ is known to be generated by the classes ω and Θ , whose intersection pairings have been determined in equation (131) above, along with certain non-algebraic cycles, which we will now incorporate. The basic formula that we will use is equation (110):

$$\left\langle \exp(\omega + \varepsilon\Theta)\cdot\beta\right\rangle' = \frac{1}{vol(G)}\int DAD\psi\, D\phi\, \exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F + \frac{1}{2}\psi\wedge\psi\right) + \frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu\, Tr\phi^2\right)\cdot\beta\,. \quad (133)$$

We recall that $\langle\ \rangle'$ coincides with integration over moduli space, up to terms that vanish exponentially for $\varepsilon\to 0$. Note that ψ is a free field, with a Gaussian measure, and the "trivial" propagator

$$\left\langle \psi_i^a(x)\psi_j^b(y)\right\rangle = -4\pi^2\varepsilon_{ij}\delta^{ab}\delta^2(x-y)\,. \quad (134)$$

For every circle $C\subset\Sigma$ there is a quantum field operator

$$V_C = \frac{1}{4\pi^2}\int_C Tr\phi\,\psi\,. \quad (135)$$

It represents a three dimensional class on moduli space; this class depends only on the homology class of C. As the algebraic cycles are even dimensional, non-zero intersection pairings are possible only with an even number of the V_C's. The first case is $\left\langle \exp(\omega+\varepsilon\Theta)\cdot V_{C_1}V_{C_2}\right\rangle'$, with two oriented circles C_1, C_2 that we can suppose to intersect transversely in finitely many points. So we consider

$$\left\langle \exp(\omega + \varepsilon\Theta)\cdot V_{C_1}V_{C_2}\right\rangle' = \frac{1}{vol(G)}\int DAD\psi\, D\phi\, \exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F + \frac{1}{2}\psi\wedge\psi\right) + \frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu\, Tr\phi^2\right)\cdot$$

$$\cdot\frac{1}{4\pi^2}\int_{C_1} Tr\phi\,\psi\cdot\frac{1}{4\pi^2}\int_{C_2} Tr\phi\,\psi\,. \quad (136)$$

Upon performing the ψ integral, using (134), we see that this is equivalent to

$$\frac{1}{vol(G)}\int DAD\psi\, D\phi\, \exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F + \frac{1}{2}\psi\wedge\psi\right) + \frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu\, Tr\phi^2\right)\cdot\sum_{P\in C_1\cap C_2}\frac{1}{4\pi^2} - \sigma(P)Tr\phi^2(P) \quad (137)$$

Here P runs over all intersection points of C_1 and C_2, and $\sigma(P)=\pm 1$ is the oriented intersection number of C_1 and C_2 at P. Since the cohomology class of $Tr\phi^2(P)$ is independent of P, and equal to that of $\int_\Sigma d\mu\, Tr\phi^2$, eq. (137) implies

$$\left\langle \exp(\omega + \varepsilon\Theta)\cdot V_{C_1}V_{C_2}\right\rangle' = \frac{1}{vol(G)}\int DAD\psi\, D\phi\, \exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F + \frac{1}{2}\psi\wedge\psi\right) + \frac{\varepsilon}{8\pi^2}\int_\Sigma Tr\phi^2\right)\cdot$$

$$\cdot\left(-\frac{\#(C_1\cap C_2)}{4\pi^2}\int_\Sigma Tr\phi^2\right), \quad (138)$$

with $\#(C_1 \cap C_2) = \sum_P \sigma(P)$ the algebraic intersection number of C_1 and C_2. The eq. (138) is equivalent to

$$\left\langle \exp(\omega + \varepsilon\Theta) V_{C_1} V_{C_2} \right\rangle' = -2\#(C_1 \cap C_2) \cdot \frac{\partial}{\partial\varepsilon} \left\langle \exp(\omega + \varepsilon\Theta) \right\rangle', \quad (139)$$

which interpreted in terms of intersection numbers gives in particular

$$\int_{\mathcal{M}'(-1)} \exp(\omega + \varepsilon\Theta) V_{C_1} V_{C_2} = -2\#(C_1 \cap C_2) \frac{\partial}{\partial\varepsilon} \int_{\mathcal{M}'(-1)} \exp(\omega + \varepsilon\Theta). \quad (140)$$

Of course, the right hand side is known from (131). Indeed, we can to obtain the following relationship:

$$\int_{\mathcal{M}'(-1)} \exp(\omega + \varepsilon\Theta) V_{C_1} V_{C_2} = -2\#(C_1 \cap C_2) \frac{\partial}{\partial\varepsilon} \int_{\mathcal{M}'(-1)} \exp(\omega + \varepsilon\Theta) =$$

$$= -2\#(C_1 \cap C_2) \frac{\partial}{\partial\varepsilon} \left[(-1)^{g+1} \sum_{k=0}^{g-1} \frac{\varepsilon^k}{k!} \frac{\left(2^{2g-2-2k} - 2\right) B_{2g-2-2k}}{2^{3g-1}(2g-2-2k)!} \right], \quad (140b)$$

where we remember that B represent the **Bernoulli number**. The generalization to an arbitrary number of V 's is almost immediate. Consider oriented circles C_σ, $\sigma = 1...2g$, representing a basis of $H_1(\Sigma, Z)$. Let $\gamma_{\sigma\tau} = \#(C_\sigma \cap C_\tau)$ be the matrix of intersection numbers. Introduce anticommuting parameters $\eta_\sigma, \sigma = 1...2n$. It is possible to claim that

$$\int_{\mathcal{M}'(-1)} \exp\left(\omega + \varepsilon\Theta + \sum_{\sigma=1}^{2g} \eta_\sigma V_{C_\sigma} \right) = \int_{\mathcal{M}'(-1)} \exp(\omega + \hat{\varepsilon}\Theta), \quad (141)$$

with

$$\hat{\varepsilon} = \varepsilon - 2\sum_{\sigma<\tau} \eta_\sigma \eta_\tau \gamma_{\sigma\tau}. \quad (142)$$

The computation leading to this formula is a minor variant of the one we have just done. The left hand side of (141) is equal to

$$\frac{1}{vol(G)} \int DAD\psi D\phi \exp\left(\frac{1}{4\pi^2} \int_\Sigma Tr\left(i\phi F + \frac{1}{2}\psi \wedge \psi \right) + \frac{\varepsilon}{8\pi^2} \int_\Sigma d\mu\, Tr\phi^2 + \frac{1}{4\pi^2} \sum_{\sigma=1}^{2n} \eta_\sigma \int_{C_\sigma} Tr\phi\psi \right). \quad (143)$$

Shifting ψ to complete the square, and then performing the Gaussian integral over ψ, this becomes

$$\frac{1}{vol(G)} \int DAD\psi D\phi \exp\left(\frac{1}{4\pi^2} \int_\Sigma Tr\left(i\phi F + \frac{1}{2}\psi \wedge \psi \right) + \frac{\hat{\varepsilon}}{8\pi^2} \int_\Sigma d\mu\, Tr\phi^2 \right). \quad (144)$$

The polynomial part of this is the right hand side of (141). Thence, we can rewrite the eq. (141) also as follows:

$$\int_{\mathcal{M}'(-1)} \exp\left(\omega + \varepsilon\Theta + \sum_{\sigma=1}^{2g} \eta_\sigma V_{C_\sigma}\right) =$$

$$= \frac{1}{vol(G)} \int DAD\psi\, D\phi \,\exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F + \frac{1}{2}\psi \wedge \psi\right) + \frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu\, Tr\phi^2 + \frac{1}{4\pi^2}\sum_{\sigma=1}^{2n} \eta_\sigma \int_{C_\sigma} Tr\phi\,\psi\right). \quad (144b)$$

We now want to evaluate the generalization of the following conventional Yang-Mills path integral

$$\int DAD\phi \,\exp\left(\frac{i}{4\pi^2}\int_\Sigma Tr\phi F + \frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu\, Tr\phi^2\right), \quad (145)$$

i.e.:

$$\int DAD\phi \,\exp\left(\frac{i}{4\pi^2}\int_\Sigma Tr\phi F + \int_\Sigma Q(\widetilde{\phi})\right), \quad (146)$$

with $Q(\widetilde{\phi})$ an arbitrary invariant polynomial on $\mathcal{H}$. The path integral can be evaluated by summing over the same physical states. The Hamiltonian is now: $H = -\hat{Q}$. With our normal-ordering recipe, the generalization of the following equation

$$\widetilde{Z}(\Sigma,\varepsilon;u) = \frac{1}{\#\pi_1(H')}\cdot\left(\frac{Vol(H')}{(2\pi)^{\dim H'}}\right)^{2g-2}\cdot\sum_\alpha \frac{\lambda_\alpha(u^{-1})\cdot\exp\left(-\varepsilon'\left(\frac{C_2(\alpha)}{2}+t\right)\right)}{(\dim\alpha)^{2g-2}} \quad (147)$$

is then

$$\widetilde{Z}(\Sigma,Q;u) = \frac{1}{\#\pi_1(H')}\cdot\left(\frac{Vol(H')}{(2\pi)^{\dim H'}}\right)^{2g-2}\cdot\sum_h \frac{\lambda_h(u^{-1})\cdot\exp(Q(h+\delta))}{d(h)^{2g-2}}. \quad (148)$$

With regard the intersection ring of the moduli space, the basic formula that we will use is (110):

$$\langle\exp(\omega+\varepsilon\Theta)\cdot\beta\rangle' = \frac{1}{vol(G')}\int DAD\psi\, D\phi\,\exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F+\frac{1}{2}\psi\wedge\psi\right)+\frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu\, Tr\phi^2\right)\cdot\beta. \quad (149)$$

In eq. (149), β is supposed to be an equivariant differential form with a polynomial dependence on ϕ. We aim to compute

$$\left\langle\exp\left(Q_{(2)}+T_{(0)}+\sum_\rho S^\rho_{(1)}(C_\rho)\right)\right\rangle'. \quad (150)$$

It is convenient to introduce

$$\hat{\phi}_a = 4\pi^2\frac{\partial Q}{\partial\phi^a}. \quad (151)$$

We will first evaluate (150) under the restriction

$$\det\left(\frac{\partial \hat{\phi}^a}{\partial \phi^b}\right) = 1. \quad (152)$$

The basic formula (149) equates (150) with the following path integral:

$$\frac{1}{vol(G')}\int DAD\psi\, D\phi\, \exp\left(\int_\Sigma \left(i\frac{\partial Q}{\partial \phi^a}F^a + \frac{1}{2}\frac{\partial^2 Q}{\partial \phi^a \partial \phi^b}\psi^a \wedge \psi^b\right) - \sum_\sigma \oint_{C_\sigma} \frac{\partial S^\sigma}{\partial \phi^a}\psi^a + \int_\Sigma d\mu\, T(\phi)\right). \quad (153)$$

First we carry out the integral over ψ. Because of (152), the ψ determinant coincides with what it would be if $Q = Tr\phi^2/8\pi^2$. As we have discussed in connection with (111), this determinant just produces the standard symplectic measure on the space $\mathcal{A}$ of connections; this measure we conventionally call DA (it is always the "Lebesgue integration"). Let $\left(\partial^2 Q\right)^{-1}$ be the inverse matrix to the matrix $\partial^2 Q/\partial\phi^a\partial\phi^b$, and let

$$\hat{T}(\phi) = T(\phi) - \sum_{\sigma<\tau}\gamma_{\sigma\tau}\frac{\partial S^\sigma}{\partial\phi^a}\frac{\partial S^\tau}{\partial\phi^b}\left(\partial^2 Q\right)^{-1}_{ab}. \quad (154)$$

The second term arises, as in the derivation of (138), in shifting ψ to complete the square in (153). Then integrating out ψ gives

$$\frac{1}{Vol(G')}\int DAD\phi\, \exp\left(i\int_\Sigma \frac{\partial Q}{\partial\phi^a}F^a + \int_\Sigma d\mu\, \hat{T}(\phi)\right). \quad (155)$$

Now change variables from ϕ to $\hat{\phi}$, defined in (151). The Jacobian for this change of variables is 1 because of (152). Because the δ_i are nilpotent, the transformation is invertible; the inverse is given by some functions $\phi^a = W^a\left(\hat{\phi}\right)$. After the change of variables, (155) becomes

$$\frac{1}{Vol(G')}\int DAD\hat{\phi}\, \exp\left(\frac{i}{4\pi^2}\int_\Sigma Tr\hat{\phi}F + \int_\Sigma d\mu\, \hat{T}\circ W\left(\hat{\phi}\right)\right). \quad (156)$$

This is a path integral of the type that we evaluated in equation (148). In canonical quantization, $\hat{\phi}^a/4\pi^2$ is identified with the group generator $-iT^a$. To avoid repeated factors of $4\pi^2$, define an invariant function V by $W\left(\hat{\phi}\right) = V\left(\hat{\phi}/4\pi^2\right)$. The invariant function $\hat{T}\circ W\left(\hat{\phi}\right)$ corresponds in the quantum theory to the operator that on a representation of highest weight h is equal to $\hat{T}\circ V(h+\delta)$, with δ equal to half the sum of the positive roots. Borrowing the result of (148), the explicit evaluation of (156) gives

$$\frac{1}{\#\pi_1(H')}\cdot\left(\frac{Vol(H')}{(2\pi)^{\dim H'}}\right)^{2g-2}\cdot\sum_h \frac{\lambda_h\left(u^{-1}\right)\cdot\exp\left(\hat{T}\circ V(h+\delta)\right)}{d(h)^{2g-2}}, \quad (157)$$

with h running over dominant weights and δ as above. The determinant in the ψ integral would be formally, if (152) is not assumed,

$$\prod_{x\in\Sigma}\det\left(\frac{\partial^2 Q'}{\partial\phi^a\partial\phi^b}\right), \quad (158)$$

times the determinant for $Q = Tr\phi^2/8\pi^2$. We have set $Q' = 4\pi^2 Q$. The factors in (158) are all equal up to coboundaries (since more generally, for any invariant function U on $\mathcal{H}$, $U(\phi(P))$ is cohomologous to $U(\phi(P'))$, for $P, P' \in \Sigma$, according to the following equation: $d0_T^{(0)} = -i\{Q, O_T^{(1)}\}$). This infinite product of essentially equal factors diverges unless (152) is assumed. The Jacobian in the changes of variables from ϕ to $\hat{\phi}$ is formally

$$\prod_{x\in\Sigma}\left(\det\left(\frac{\partial^2 Q'}{\partial\phi^a\partial\phi^b}\right)\right)^{-1}. \quad (159)$$

Formally, these two factors appear to cancel, but this cancellation should be taken to mean only that the result is finite, not that it equals one. The number of factors in (158) should be interpreted as $N_1/2$, half the dimension of the space of one-forms. The number of factors in (159) should be interpreted as N_0, the dimension of the space of zero-forms. The difference $N_1/2 - N_0$ is $-1/2$ the Euler characteristic of Σ, or $g-1$. Thus the product of (158) and (159) should be interpreted as $\det\left(\partial^2 Q'/\partial\phi^a\partial\phi^b\right)^{g-1}$. A convenient function cohomologous to this is

$$\exp\left(\int_\Sigma (g-1)\ln\det\left(\frac{\partial^2 Q'}{\partial\phi^a\partial\phi^b}\right)\right).$$

The sole result of relaxing (152) is accordingly that (156) becomes

$$\frac{1}{Vol(G')}\int DAD\hat{\phi}\exp\left(\frac{i}{4\pi^2}\int_\Sigma Tr\hat{\phi}F + \int_\Sigma d\mu\,\widetilde{T}\circ W(\hat{\phi})\right), \quad (160)$$

with

$$\widetilde{T} = \hat{T} + (g-1)\ln\det\left(\frac{\partial^2 Q'}{\partial\phi^a\partial\phi^b}\right) = T - \sum_{\sigma<\tau}\gamma_{\sigma\tau}\frac{\partial S^\sigma}{\partial\phi^a}\frac{\partial S^\tau}{\partial\phi^b}\left(\partial^2 Q\right)^{-1}_{ab} + (g-1)\ln\det\left(\frac{\partial^2 Q'}{\partial\phi^a\partial\phi^b}\right). \quad (161)$$

The evaluation of the path integral therefore leaves in general not quite (157) but

$$\frac{1}{\#\pi_1(H')}\left[\frac{Vol(H')}{(2\pi)^{\dim(H')}}\right]^{2g-2}\sum_h\frac{\lambda_h(u^{-1})\exp\left(\widetilde{T}\circ V(h+\delta)\right)}{d(h)^{2g-2}}. \quad (162)$$

Furthermore, we can rewrite the expression (160) also as follows:

$$\frac{1}{Vol(G')}\int DAD\hat{\phi}\exp\left(\frac{i}{4\pi^2}\int_\Sigma Tr\hat{\phi}F+\int_\Sigma d\mu\cdot T-\sum_{\sigma<\tau}\gamma_{\sigma\tau}\frac{\partial S^\sigma}{\partial\phi^a}\frac{\partial S^\tau}{\partial\phi^b}\left(\partial^2 Q\right)^{-1}_{ab}+(g-1)\ln\det\left(\frac{\partial^2 Q'}{\partial\phi^a\partial\phi^b}\right)\circ W\left(\hat{\phi}\right)\right).$$

(162b)

2.3 On some equations concerning the large N 2D Yang-Mills Theory and Topological String Theory **[7]**

The partition function of two-dimensional Yang-Mills theory on an orientable closed manifold Σ_T of genus G is

$$Z(SU(N),\Sigma_T)=\int\left[DA^\mu\right]\exp\left[-\frac{1}{4e^2}\int_{\Sigma_T}d^2x\sqrt{\det G_{ij}}TrF_{ij}F^{ij}\right]=\sum_R(\dim R)^{2-2G}e^{-\frac{\lambda A}{2N}C_2(R)}\qquad(163)$$

where the gauge coupling $\lambda=e^2N$ is held fixed in the large N limit, the sum runs over all unitary irreducible representations R of the gauge group $\mathcal{G}=SU(N)$, $C_2(R)$ is the second casimir, and A is the area of the spacetime in the metric G_{ij}. Also here DA is always the "Lebesgue integration". Let $\mathcal{F}^{(1)}$ be the simple Hurwitz space of maps with B simple branch points. Denote these simple branch points by P_I with corresponding ramification points of index 2 at R_I: these are the unique ramification points above P_I. We can choose a basis $\{G_I\}_{I=1,\dots,2B}$ for $T\mathcal{F}$, such that G_{2I-1} and G_{2I} have support only at the I^{th} ramification point. The analogue in ordinary string theory is a choice of Beltrami differentials which have support only at punctures. This is a well-defined choice away from the boundary of moduli space. Now consider the curvature insertions in these local coordinates:

$$\int\mathcal{D}\left[\tilde{A}\right]\exp\left[-\frac{1}{4}\tilde{A}^I\mathcal{R}_{IJ}\tilde{A}^J\right]=\frac{(-1)^B}{2^B}\mathrm{Pfaff}\left(\mathcal{R}_{IJ}\right),\quad(164)$$

where B is even and the matrix, $\mathcal{R}_{IJ}$, takes the following form in an oriented orthonormal basis

$$\mathcal{R}_{IJ}=\begin{pmatrix}0&\mathcal{R}_{12}&&\\-\mathcal{R}_{12}&0&\dots\dots\dots\dots&\\&&0&\mathcal{R}_{2B-12B}\\&&-\mathcal{R}_{2B-12B}&0\end{pmatrix},\quad(165)$$

so that

$$\mathrm{Pfaff}\left(\mathcal{R}_{IJ}\right)=\prod_{I=1}^{B}\mathcal{R}_{2I-12I}\left[G^{2I-1},G^{2I}\right]\left(R_I\right)\quad(166)$$

and the full measure for the topological string theory is

$$\frac{(-1)^B}{(2\pi)^B}\int_{\mathcal{F}^{(1)}}\mathcal{D}[F,G]\prod_{I=1}^{B}\mathcal{R}_{2I-12I}\left[G^{2I-1},G^{2I}\right](R_I)\exp\left[-\frac{1}{2}\int_{\Sigma_W}f^*\omega\right]=$$

$$\frac{1}{(2\pi)^B}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{\mathcal{F}^{(1)}}\mathcal{D}[F,G]\prod_{I=1}^{B}\mathcal{R}_{2I-12I}\left[G^{2I-1},G^{2I}\right]\left[\frac{1}{2}\int_{\Sigma_W}f^*\omega\right]^k=\frac{1}{(2\pi)^B}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k k!}\left\langle\left\langle \mathrm{A}^{(2)}\ldots\mathrm{A}^{(2)}\right\rangle\right\rangle_{\mathcal{F}(B,k)}.$$

(167)

In the last line we have introduced a space $\mathcal{F}(B,k)$, which is the product space

$$\mathcal{F}(B,k)=\mathcal{F}^{(1)}\times(\Sigma_W)^k. \quad (168)$$

The integral over this space, $\langle\langle\ldots\rangle\rangle_{\mathcal{F}(B,k)}$ is formally defined by the eq. (167). Furthermore, we have the following expression:

$$\left\langle\left\langle \mathrm{A}^{(2)}\ldots\mathrm{A}^{(2)}\right\rangle\right\rangle_{\mathcal{F}(B,k)}=\sum_{l=0}^{k}\binom{k}{l}\frac{2^l B!(-1)^l}{(B-l)!}(nA)^{k-l}\left\langle\left\langle \mathrm{A}^{(0)}(R_1)\ldots\mathrm{A}^{(0)}(R_l)\right\rangle\right\rangle_{\mathcal{F}(B,0;B-l)}. \quad (169)$$

When $l>B$ it is clear that the correlation function on the right vanishes, by ghost number counting. So that altogether

$$\frac{1}{(2\pi)^B}\int_{\mathcal{F}^{(1)}}\mathcal{D}[F,G]\prod_{I=1}^{B}\mathcal{R}_{2I-12I}\left[G^{2I-1},G^{2I}\right](Q_I)\exp-\frac{1}{2}\int_{\Sigma_W}f^*\omega=$$

$$=\sum_{k=0}^{\infty}\frac{B!}{k!(B-k)!}\sum_{l=0}^{\min[k,B]}\binom{k}{l}\left(-\frac{1}{2}nA\right)^{k-l}\left\langle\left\langle \mathrm{A}^{(0)}(R_1)\ldots\mathrm{A}^{(0)}(R_l)\right\rangle\right\rangle_{\mathcal{F}(B,0;B-l)}. \quad (170)$$

Substituting in the right-hand side of (170) we obtain

$$e^{-\frac{1}{2}nA}\sum_{k=0}^{B}\frac{B!}{k!(B-k)!}\left\langle\left\langle \mathrm{A}^{(0)}(R_1)\ldots\mathrm{A}^{(0)}(R_k)\right\rangle\right\rangle_{\mathcal{F}(B,0;r=B-k)}. \quad (171)$$

So we are left with the integral

$$\int_{\mathcal{F}^{(1)}}\mathcal{D}[F,G]\prod_{I=1}^{B-k}\mathcal{R}_{2I-12I}\left[G^{2I-1},G^{2I}\right](R_I)\mathrm{A}^{(0)}(R_{B-k+1})\ldots\mathrm{A}^{(0)}(R_B). \quad (172)$$

We are only interested in the contribution of simple Hurwitz space. This space is a bundle over $C_{0,B}/S_B$ with discrete fiber the set $\Psi(n,B,G,L=B)$. Further the measure on Hurwitz space inherited from the path integral divides out by diffeomorphisms. Therefore the correlator in (171) is:

$$\sum_{\psi\in\Psi(n,B,G,L=B)}\frac{1}{|C(\psi)|}\times\frac{1}{B!}\int_{C_{0,B}}\left\langle\prod_{I=1}^{B-k}R_{2I-12I}\left[G^{2I-1},G^{2I}\right](R_I)\mathrm{A}^{(0)}(R_{B-k+1})\ldots\mathrm{A}^{(0)}(R_B)\right\rangle. \quad (173)$$

In isolating the contributions of simple Hurwitz space we must ignore the singularities from the collisions of R_I, $I \leq B-k$ with R_J, $J \geq B-k+1$. Thus we replace (173) by the expression:

$$\sum_{\psi \in \Psi(n,B,G,L=B)} \frac{1}{|C(\psi)|} \times \frac{1}{B!} \times \int_{C_{0,B-k} \times (\Sigma_T)^k} \left\langle \prod_{I=1}^{B-k} R_{2I-12I}\left[G^{2I-1}, G^{2I}\right](R_I) \right\rangle \wedge \omega(P_{B-k+1}) \wedge \ldots \wedge \omega(P_B), \quad (174)$$

where $P_J \in \Sigma_T$ are the images of the simple ramification points R_J. Thence, from the eqs. (170) and (174), we obtain the following expression:

$$\frac{1}{(2\pi)^B} \int_{\mathcal{F}^{(1)}} \mathcal{D}[F,G] \prod_{I=1}^{B} \mathcal{R}_{2I-12I}\left[G^{2I-1}, G^{2I}\right](Q_I) \exp - \frac{1}{2} \int_{\Sigma_W} f^* \omega =$$

$$= \sum_{\psi \in \Psi(n,B,G,L=B)} \frac{1}{|C(\psi)|} \times \frac{1}{B!} \times \int_{C_{0,B-k} \times (\Sigma_T)^k} \left\langle \prod_{I=1}^{B-k} R_{2I-12I}\left[G^{2I-1}, G^{2I}\right](R_I) \right\rangle \wedge \omega(P_{B-k+1}) \wedge \ldots \wedge \omega(P_B). \quad (174b)$$

3. Ramanujan's equations, zeta strings and mathematical connections

Now we describe some mathematical connections with some sectors of String Theory and Number Theory, principally with some equations concerning the Ramanujan's modular equations that are related to the physical vibrations of the bosonic strings and of the superstrings, the Ramanujan's identities concerning π and the zeta strings.

3.1 Ramanujan's equations **[8] [9]**

With regard the Ramanujan's modular functions, we note that the number 8, and thence the numbers $64 = 8^2$ and $32 = 2^2 \times 8$, are connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8 = \frac{1}{3} \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} . \quad (175)$$

Furthermore, with regard the number 24 (12 = 24 / 2 and 32 = 24 + 8) this is related to the physical vibrations of the bosonic strings by the following Ramanujan function:

$$24 = \frac{4\left[anti\log \frac{\int_0^{\infty} \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')} \right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4} \right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4} \right)} \right]} \quad (176)$$

It is well-known that the series of Fibonacci's numbers exhibits a fractal character, where the forms repeat their similarity starting from the reduction factor $1/\phi = 0{,}618033 = \frac{\sqrt{5}-1}{2}$ (Peitgen et al. 1986). Such a factor appears also in the famous fractal Ramanujan identity (Hardy 1927):

$$0{,}618033 = 1/\phi = \frac{\sqrt{5}-1}{2} = R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2} \exp\left(\frac{1}{\sqrt{5}} \int_0^q \frac{f^5(-t)}{f(-t^{1/5})} \frac{dt}{t^{4/5}} \right)}, \quad (177)$$

and

$$\pi = 2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2} \exp\left(\frac{1}{\sqrt{5}} \int_0^q \frac{f^5(-t)}{f(-t^{1/5})} \frac{dt}{t^{4/5}} \right)} \right], \quad (178)$$

where

$$\Phi = \frac{\sqrt{5}+1}{2}.$$

Furthermore, we remember that π arises also from the following identities (Ramanujan's paper: "Modular equations and approximations to π" Quarterly Journal of Mathematics, 45 (1914), 350-372.):

$$\pi = \frac{12}{\sqrt{130}}\log\left[\frac{\left(2+\sqrt{5}\right)\left(3+\sqrt{13}\right)}{\sqrt{2}}\right], \quad (178a) \quad \text{and} \quad \pi = \frac{24}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right].$$
(178b)

From (178b), we have that

$$24 = \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (178c)$$

Let $u(q)$ denote the Rogers-Ramanujan continued fraction, defined by the following equation

$$u := u(q) := \frac{q^{1/5}}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+ \ldots}, \quad |q| < 1 \quad (179)$$

and set $v = u(q^2)$. Recall that $\psi(q)$ is defined by the following equation

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \quad (180)$$

Then

$$\frac{8}{5}\int\frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = \log(u^2v^3) + \sqrt{5}\log\left(\frac{1+\left(\sqrt{5}-2\right)uv^2}{1-\left(\sqrt{5}+2\right)uv^2}\right). \quad (181)$$

We note that $1+\left(\sqrt{5}-2\right) = 2\cdot 0{,}61803398$ and that $1-\left(\sqrt{5}+2\right) = 2\cdot 1{,}61803398$, where $\phi = 0{,}61803398$ and $\Phi = 1{,}61803398$ are the aurea section and the aurea ratio respectively. Let $k := k(q) := uv^2$. Then from page 326 of Ramanujan's second notebook, we have

$$u^5 = k\left(\frac{1-k}{1+k}\right)^2 \quad \text{and} \quad v^5 = k^2\left(\frac{1+k}{1-k}\right). \quad (182)$$

It follows that

$$\log(u^2v^3) = \frac{1}{5}\log\left(k^8\frac{1-k}{1+k}\right). \quad (183)$$

If we set $\varepsilon = \left(\sqrt{5}+1\right)/2 = 1{,}61803398$, i.e. the aurea ratio, we readily find that $\varepsilon^3 = \sqrt{5}+2$ and $\varepsilon^{-3} = \sqrt{5}-2$. Then, with the use of (183), we see that (181) is equivalent to the equality

$$\frac{8}{5}\int\frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q}=\frac{1}{5}\log\left(k^8\frac{1-k}{1+k}\right)+\sqrt{5}\log\left(\frac{1+\varepsilon^{-3}k}{1-\varepsilon^3 k}\right). \quad (184)$$

Now from Entry 9 (vi) in Chapter 19 of Ramanujan's second notebook,

$$\frac{\psi^5(q)}{\psi(q^5)}=25q^2\psi(q)\psi^3(q^5)+1-5q\frac{d}{dq}\log\frac{f(q^2,q^3)}{f(q,q^4)}. \quad (185)$$

By the Jacobi triple product identity

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad (186)$$

we have

$$\frac{f(q^2,q^3)}{f(q,q^4)}=\frac{(-q^2;q^5)_\infty(-q^3;q^5)_\infty}{(-q;q^5)_\infty(-q^4;q^5)_\infty}=\frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^4;q^{10})_\infty(q^6;q^{10})_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^2;q^{10})_\infty(q^8;q^{10})_\infty}=q^{1/5}\frac{u(q)}{v(q)}, \quad (187)$$

by the following expression

$$u(q)=q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \quad (188)$$

Using (187) in (185), we find that

$$\frac{8}{5}\int\frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q}=40\int q\psi(q)\psi^3(q^5)dq+\int\frac{8}{5q}dq-8\int\frac{d}{dq}\log(q^{1/5}u/v)dq=$$

$$=40\int q\psi(q)\psi^3(q^5)dq-8\log(u/v)=40\int q\psi(q)\psi^3(q^5)dq+\frac{8}{5}\log k-\frac{24}{5}\log\frac{1-k}{1+k}, \quad (189)$$

where (182) has been employed. We note that we can rewrite the eq. (189) also as follows:

$$\frac{8}{5}\int\frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q}=40\int q\psi(q)\psi^3(q^5)dq+\frac{8}{5}\log k-\frac{24}{5}\log\frac{1-k}{1+k}. \quad (190)$$

In the Ramanujan's notebook part IV in the Section "Integrals" are examined various results on integrals appearing in the 100 pages at the end of the second notebook, and in the 33 pages of the third notebook. Here, we have showed some integrals that can be related with some arguments above described.

$$\int_0^\infty e^{-2a^2n}\psi(n)dn=\frac{1}{8\pi a^2}+4a^2\sum_{k=1}^\infty\frac{k}{(e^{2\pi k}-1)(4a^4+k^4)}-4a^2\int_0^\infty\frac{xdx}{(e^{2\pi x}-1)(4a^4+x^4)}; \quad (191)$$

$$4a^2\int_0^\infty\frac{xdx}{(e^{2\pi x}-1)(4a^4+x^4)}=\frac{1}{4a}-\frac{\pi}{4}+a\sum_{k=1}^\infty\frac{1}{a^2+(a+k)^2}. \quad (192)$$

Let $n\geq 0$. Then

$$\int_0^\infty \frac{\sin(2nx)dx}{x(\cosh(\pi x)+\cos(\pi x))} = \frac{\pi}{4} - 2\sum_{k=0}^{\infty} \frac{(-1)^k e^{-(2k+1)n}\cos\{(2k+1)n\}}{(2k+1)\cosh\{(2k+1)\pi/2\}}; \quad (193)$$

Now we analyze the following integral:

$$I := \int_0^1 \frac{\log\left(\frac{1+\sqrt{1+4x}}{2}\right)}{x} dx = \frac{\pi^2}{15}; \quad (194)$$

Let $u = \left(1+\sqrt{1+4x}\right)/2$, so that $x = u^2 - u$. Then integrating by parts, setting $u = 1/v$ and using the following expression $Li_2(z) = -\int_0^z \frac{\log(1-w)}{w} dw$, $z \in C$, and employing the value $Li_2\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{10} - \log^2\left(\frac{\sqrt{5}-1}{2}\right)$, we find that

$$I = \int_1^{(\sqrt{5}+1)/2} \frac{\log u}{u^2-u}(2u-1)du = -\int_1^{(\sqrt{5}+1)/2} \frac{\log(u^2-u)}{u} du = -\int_1^{(\sqrt{5}+1)/2} \left(\frac{\log u}{u} + \frac{\log(u-1)}{u}\right) du =$$

$$= -\frac{1}{2}\log^2\left(\frac{\sqrt{5}+1}{2}\right) + \int_1^{(\sqrt{5}+1)/2} \frac{\log(1-v) - \log v}{v} dv =$$

$$= -\frac{1}{2}\log^2\left(\frac{\sqrt{5}+1}{2}\right) - Li_2\left(\frac{\sqrt{5}-1}{2}\right) + Li_2(1) - \frac{1}{2}\log^2\left(\frac{\sqrt{5}+1}{2}\right) = -\frac{\pi^2}{10} + \frac{\pi^2}{6} = \frac{\pi^2}{15}. \quad (195)$$

Thence, we obtain the following equation:

$$I = \int_0^1 \frac{\log\left(\frac{1+\sqrt{1+4x}}{2}\right)}{x} dx = \int_1^{(\sqrt{5}+1)/2} \frac{\log u}{u^2-u}(2u-1)du = \frac{\pi^2}{15}. \quad (196)$$

In the work of Ramanujan, [i.e. the modular functions,] the number 24 (8 x 3) appears repeatedly. This is an example of what mathematicians call magic numbers, which continually appear where we least expect them, for reasons that no one understands. Ramanujan‘s function also appears in string theory. Modular functions are used in the mathematical analysis of Riemann surfaces. Riemann surface theory is relevant to describing the behavior of strings as they move through space-time. When strings move they maintain a kind of symmetry called "conformal invariance". Conformal invariance (including "scale invariance") is related to the fact that points on the surface of a string's world sheet need not be evaluated in a particular order. As long as all points on the surface are taken into account in any consistent way, the physics should not change. Equations of how strings must behave when moving involve the Ramanujan function. When a string moves in space-time by splitting and recombining a large number of mathematical identities must be satisfied. These are the identities of Ramanujan's modular function. The KSV loop diagrams of interacting strings can be described using modular functions. The "Ramanujan function" (an elliptic modular function that satisfies the need for "conformal symmetry") has 24 "modes" that correspond to the physical vibrations of a bosonic string. When the Ramanujan function is generalized, 24 is replaced by 8 (8 + 2 = 10) for fermionic strings.

3.2 Zeta Strings **[10]**

The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (197)$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+\ldots+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L = \sum_{n\geq 1} C_n \mathcal{L}_n = \sum_{n\geq 1}\frac{n-1}{n^2}\mathcal{L}_n = \frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq 1} n^{-\frac{\Box}{2}}\phi + \sum_{n\geq 1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (198)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1. \quad (199)$$

Employing usual expansion for the logarithmic function and definition (199) we can rewrite (198) in the form

$$L = -\frac{1}{g^2}\left[\frac{1}{2}\phi\,\zeta\left(\frac{\Box}{2}\right)\phi + \phi + \ln(1-\phi)\right], \quad (200)$$

where $|\phi| < 1$. $\zeta\left(\frac{\square}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\square}{2}\right)\phi(x) = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk \quad , \quad -k^2 = k_0^2 - \vec{k}^2 > 2+\varepsilon \quad , \quad (201)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string". Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \quad (202)$$

which has an evident solution $\phi = 0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)} . \quad (203)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n \quad , \quad (204)$$

$$\zeta\left(\frac{\square}{4}\right)\theta = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right] \quad , \quad (205)$$

and one can easily see trivial solution $\phi = \theta = 0$.

3.3 Mathematical connections.

With regard the mathematical connections with the Lebesgue measure, Lebesgue integrals and some equations concerning the Chern-Simons theory and the Yang-Mills theory, we have the following expressions:

$$\phi_\omega\left(M_f\right) = Tr_\omega\left(M_f T_\Delta\right) = \frac{1}{2^{(n-1)}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)}\int_M f(x)|vol|(x) \Rightarrow$$

$$\Rightarrow \frac{e^{\pi ikS_{X,P}(A_0)}}{Vol(\mathcal{S})Vol(\mathcal{G}_P)}\int_{\mathcal{A}_P} DA\exp\left[\frac{ik}{4\pi}\left(\int_X A\wedge dA - \int_X \frac{(\kappa\wedge dA)^2}{\kappa\wedge d\kappa}\right)\right], \quad (206)$$

thence between the eq. (45) and the eq. (84).

$$\phi_\omega\left(M_f\right) = Tr_\omega\left(M_f T_\Delta\right) = \frac{1}{2^{(n-1)}\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)}\int_M f(x)|vol|(x) \Rightarrow$$

$$\Rightarrow \frac{1}{vol(G)}\int DAD\psi D\phi \exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F + \frac{1}{2}\psi\wedge\psi\right) + \frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu\, Tr\phi^2\right)\cdot\beta, \quad (207)$$

thence between the eq. (45) and the eq. (110).

$$\int_{l_+^1(X)}\left(\prod_{k=1}^{\infty}\exp\left(-r_k^\sigma a(x_k)\right)\right)d\mathcal{L}(\xi) = \exp\left(\left(\sigma^{-1}-1\right)\gamma\right)\exp\left(-\sigma^{-1}\int_X \log a(x)dm(x)\right) \Rightarrow$$

$$\Rightarrow \frac{e^{\pi ikS_{X,P}(A_0)}}{Vol(\mathcal{S})Vol(\mathcal{G}_P)}\int_{\mathcal{A}_P} DA\exp\left[\frac{ik}{4\pi}\left(\int_X A\wedge dA - \int_X \frac{(\kappa\wedge dA)^2}{\kappa\wedge d\kappa}\right)\right], \quad (208)$$

thence between the eq. (63) and the eq. (84).

$$\int_{l_+^1(X)}\left(\prod_{k=1}^{\infty}\exp\left(-r_k^\sigma a(x_k)\right)\right)d\mathcal{L}(\xi) = \exp\left(\left(\sigma^{-1}-1\right)\gamma\right)\exp\left(-\sigma^{-1}\int_X \log a(x)dm(x)\right) \Rightarrow$$

$$\Rightarrow \frac{1}{vol(G)}\int DAD\psi D\phi \exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F + \frac{1}{2}\psi\wedge\psi\right) + \frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu\, Tr\phi^2\right)\cdot\beta, \quad (209)$$

thence between the eq. (63) and the eq. (110)

With regard the Ramanujan's equations we now describe various mathematical connections with some equations concerning the Chern-Simons theory and the Yang-Mills theory. With regard the Chern-Simons theory, we have:

$$Z(k) = \frac{1}{Vol(\mathcal{G})}\left(\frac{k}{4\pi^2}\right)^{\Delta\mathcal{G}} \int \mathcal{D}A \exp\left[i\frac{k}{4\pi}\int_X Tr\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right)\right] \Rightarrow$$

$$\Rightarrow \; 4a^2\int_0^\infty \frac{xdx}{\left(e^{2\pi x}-1\right)\left(4a^4+x^4\right)} = \frac{1}{4a} - \frac{\pi}{4} + a\sum_{k=1}^{\infty}\frac{1}{a^2+(a+k)^2}, \quad (210)$$

thence, between the eq. (76) and the eq. (192).

$$\bar{Z}_{U(1)}(X,p,k) = \frac{e^{\pi ikS_{X,P}(A_0)}}{Vol(S)Vol(\mathcal{G}_P)}\int_{A_P} DA \exp\left[\frac{ik}{4\pi}\left(\int_X A \wedge dA - \int_X \frac{(\kappa \wedge dA)^2}{\kappa \wedge d\kappa}\right)\right] \Rightarrow$$

$$\Rightarrow \; 4a^2\int_0^\infty \frac{xdx}{\left(e^{2\pi x}-1\right)\left(4a^4+x^4\right)} = \frac{1}{4a} - \frac{\pi}{4} + a\sum_{k=1}^{\infty}\frac{1}{a^2+(a+k)^2}, \quad (211)$$

thence, between the eq. (84) and the eq. (192).

$$Z_{U(1)}(X,p,k) = \frac{e^{\pi ikS_{X,P}(A_0)}}{Vol(\mathcal{G}_P)}\int_{\bar{A}_P} \bar{D}A \exp\left[\frac{ik}{4\pi}\left(\int_X A \wedge dA - \int_X \frac{(\kappa \wedge dA)^2}{\kappa \wedge d\kappa}\right)\right] =$$

$$= \frac{e^{\pi ikS_{X,P}(A_0)}}{Vol(\mathcal{G}_P)}\int_{\bar{A}_P} \bar{D}A \exp\left[\frac{ik}{4\pi}S(A)\right] \Rightarrow$$

$$\Rightarrow \; 4a^2\int_0^\infty \frac{xdx}{\left(e^{2\pi x}-1\right)\left(4a^4+x^4\right)} = \frac{1}{4a} - \frac{\pi}{4} + a\sum_{k=1}^{\infty}\frac{1}{a^2+(a+k)^2} \Rightarrow$$

$$\Rightarrow \; \int_0^\infty \frac{\sin(2nx)dx}{x(\cosh(\pi x)+\cos(\pi x))} = \frac{\pi}{4} - 2\sum_{k=0}^{\infty}\frac{(-1)^k e^{-(2k+1)n}\cos\{(2k+1)n\}}{(2k+1)\cosh\{(2k+1)\pi/2\}}, \quad (212)$$

thence, between the eq. (88) and the eqs. (192), (193).

With regard the Yang-Mills theory, we have:

$$\langle \exp(\omega + \varepsilon\Theta) \cdot \beta \rangle' = \frac{1}{vol(G)}\int DAD\psi D\phi \exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F + \frac{1}{2}\psi \wedge \psi\right) + \frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu \, Tr\phi^2\right) \cdot \beta \Rightarrow$$

$$\Rightarrow \; \frac{8}{5}\int \frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = 40\int q\psi(q)\psi^3(q^5)dq + \frac{8}{5}\log k - \frac{24}{5}\log\frac{1-k}{1+k}, \quad (213)$$

Thence, between the eq. (110) and the eq. (190), where 8 and 24 are connected with the physical vibrations of the superstrings and of the bosonic strings respectively.

$$Z(\Sigma;u)=\frac{1}{Vol(G')}\int DAD\phi\,\exp\left(\frac{i}{4\pi^2}\int_\Sigma Tr\phi F\right)=\left(\frac{Vol(H)}{(2\pi)^{\dim H}}\right)^{2g-2}\cdot\sum_\alpha\frac{\lambda_\alpha(u^{-1})}{(\dim\alpha)^{2g-2}}\Rightarrow$$

$$\Rightarrow\ 4a^2\int_0^\infty\frac{xdx}{(e^{2\pi x}-1)(4a^4+x^4)}=\frac{1}{4a}-\frac{\pi}{4}+a\sum_{k=1}^\infty\frac{1}{a^2+(a+k)^2}\Rightarrow$$

$$\Rightarrow\ \int_0^\infty\frac{\sin(2nx)dx}{x(\cosh(\pi x)+\cos(\pi x))}=\frac{\pi}{4}-2\sum_{k=0}^\infty\frac{(-1)^k e^{-(2k+1)n}\cos\{(2k+1)n\}}{(2k+1)\cosh\{(2k+1)\pi/2\}},\quad(214)$$

thence, between the eq. (122c) and the eqs. (192), (193).

$$\left\langle\exp(\omega+\varepsilon\Theta)\cdot V_{C_1}V_{C_2}\right\rangle'=\frac{1}{vol(G)}\int DAD\psi D\phi\,\exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F+\frac{1}{2}\psi\wedge\psi\right)+\frac{\varepsilon}{8\pi^2}\int_\Sigma Tr\phi^2\right)\cdot$$

$$\cdot\left(-\frac{\#(C_1\cap C_2)}{4\pi^2}\int_\Sigma Tr\phi^2\right)\Rightarrow$$

$$\Rightarrow\ \int_0^\infty e^{-2a^2n}\psi(n)dn=\frac{1}{8\pi a^2}+4a^2\sum_{k=1}^\infty\frac{k}{(e^{2\pi k}-1)(4a^4+k^4)}-4a^2\int_0^\infty\frac{xdx}{(e^{2\pi x}-1)(4a^4+x^4)},\quad(215)$$

thence, between the eq. (138) and the eq. (191).

$$\int_{\mathcal{M}'(-1)}\exp\left(\omega+\varepsilon\Theta+\sum_{\sigma=1}^{2g}\eta_\sigma V_{C_\sigma}\right)=$$

$$=\frac{1}{vol(G)}\int DAD\psi D\phi\,\exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F+\frac{1}{2}\psi\wedge\psi\right)+\frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu\,Tr\phi^2+\frac{1}{4\pi^2}\sum_{\sigma=1}^{2n}\eta_\sigma\int_{C_\sigma}Tr\phi\psi\right)\Rightarrow$$

$$\Rightarrow\ \int_0^\infty e^{-2a^2n}\psi(n)dn=\frac{1}{8\pi a^2}+4a^2\sum_{k=1}^\infty\frac{k}{(e^{2\pi k}-1)(4a^4+k^4)}-4a^2\int_0^\infty\frac{xdx}{(e^{2\pi x}-1)(4a^4+x^4)},\quad(216)$$

thence, between the eq. (144b) and the eq. (191).

$$\frac{1}{Vol(G')}\int DAD\hat{\phi}\,\exp$$

$$\left(\frac{i}{4\pi^2}\int_\Sigma Tr\hat{\phi}F+\int_\Sigma d\mu\cdot T-\sum_{\sigma<\tau}\gamma_{\sigma\tau}\frac{\partial S^\sigma}{\partial\phi^a}\frac{\partial S^\tau}{\partial\phi^b}(\partial^2Q)^{-1}_{ab}+(g-1)\ln\det\left(\frac{\partial^2Q'}{\partial\phi^a\partial\phi^b}\right)\circ W(\hat{\phi})\right)\Rightarrow$$

$$\Rightarrow\ 4a^2\int_0^\infty\frac{xdx}{(e^{2\pi x}-1)(4a^4+x^4)}=\frac{1}{4a}-\frac{\pi}{4}+a\sum_{k=1}^\infty\frac{1}{a^2+(a+k)^2}\Rightarrow$$

$$\Rightarrow\ \int_0^\infty\frac{\sin(2nx)dx}{x(\cosh(\pi x)+\cos(\pi x))}=\frac{\pi}{4}-2\sum_{k=0}^\infty\frac{(-1)^k e^{-(2k+1)n}\cos\{(2k+1)n\}}{(2k+1)\cosh\{(2k+1)\pi/2\}},\quad(217)$$

thence, between the eq. (162b) and the eqs. (192), (193).

Furthermore, we have the following mathematical connections:

$$\langle \exp(\omega + \varepsilon\Theta)\cdot\beta \rangle' = \frac{1}{vol(G)}\int DAD\psi D\phi \exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F + \frac{1}{2}\psi \wedge \psi\right) + \frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu \, Tr\phi^2\right)\cdot\beta \Rightarrow$$

$$\Rightarrow \frac{8}{5}\int \frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = 40\int q\psi(q)\psi^3(q^5)dq + \frac{8}{5}\log k - \frac{24}{5}\log\frac{1-k}{1+k} \Rightarrow$$

$$\Rightarrow \int_0^1 \frac{\log\left(\frac{1+\sqrt{1+4x}}{2}\right)}{x}dx = \int_1^{(\sqrt{5}+1)/2}\frac{\log u}{u^2-u}(2u-1)du = \frac{\pi^2}{15}, \quad (218)$$

thence, between the eq. (110) and the eqs. (190), (196).

$$\int_{\mathcal{M}'(-1)} \exp\left(\omega + \varepsilon\Theta + \sum_{\sigma=1}^{2g}\eta_\sigma V_{C_\sigma}\right) =$$

$$= \frac{1}{vol(G)}\int DAD\psi D\phi \exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F + \frac{1}{2}\psi \wedge \psi\right) + \frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu \, Tr\phi^2 + \frac{1}{4\pi^2}\sum_{\sigma=1}^{2n}\eta_\sigma \int_{C_\sigma} Tr\phi\psi\right) \Rightarrow$$

$$\Rightarrow \frac{8}{5}\int \frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = 40\int q\psi(q)\psi^3(q^5)dq + \frac{8}{5}\log k - \frac{24}{5}\log\frac{1-k}{1+k} \Rightarrow$$

$$\Rightarrow \int_0^1 \frac{\log\left(\frac{1+\sqrt{1+4x}}{2}\right)}{x}dx = \int_1^{(\sqrt{5}+1)/2}\frac{\log u}{u^2-u}(2u-1)du = \frac{\pi^2}{15}, \quad (219)$$

thence, between the eq. (144b) and the eqs. (190), (196).

With regard the mathematical connections between the fundamental equation of the Yang-Mills theory that we have described in this paper and the topological string theory, we have the following relationship.

$$\frac{(-1)^B}{(2\pi)^B}\int_{\mathcal{F}(1)} \mathcal{D}[F,G]\prod_{I=1}^{B}\mathcal{R}_{2I-12I}\left[G^{2I-1},G^{2I}\right](R_I)\exp\left[-\frac{1}{2}\int_{\Sigma_W} f^*\omega\right] =$$

$$\frac{1}{(2\pi)^B}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{\mathcal{F}(1)} \mathcal{D}[F,G]\prod_{I=1}^{B}\mathcal{R}_{2I-12I}\left[G^{2I-1},G^{2I}\right]\left[\frac{1}{2}\int_{\Sigma_W} f^*\omega\right]^k = \frac{1}{(2\pi)^B}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k k!}\left\langle\left\langle A^{(2)}\ldots A^{(2)}\right\rangle\right\rangle_{\mathcal{F}(B,k)} \Rightarrow$$

$$\Rightarrow Z(SU(N),\Sigma_T) = \int\left[DA^\mu\right]\exp\left[-\frac{1}{4e^2}\int_{\Sigma_T} d^2x\sqrt{\det G_{ij}}\,TrF_{ij}F^{ij}\right] = \sum_R (\dim R)^{2-2G} e^{-\frac{\lambda A}{2N}C_2(R)} \quad (220)$$

thence, between the eq. (167) and the eq. (163).

With regard the zeta strings, it is possible to obtain some interesting mathematical connections that we now go to describe.

$$\zeta_0(0)-\zeta_1(0)=\left[\frac{1}{512}\int_X R^2\kappa\wedge d\kappa-\dim Ker\Delta_0\right]-\left[\frac{1}{256}\int_X R^2\kappa\wedge d\kappa-\dim Ker\Delta_1\right]=$$

$$=\left(-\frac{1}{512}\int_X R^2\kappa\wedge d\kappa\right)+\dim Ker\Delta_1-\dim Ker\Delta_0=$$

$$=\left(-\frac{1}{512}\int_X R^2\kappa\wedge d\kappa\right)+\dim H^1(X,d_H)-\dim H^0(X,d_H)\Rightarrow$$

$$\Rightarrow\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t)=\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}, \quad (221)$$

thence, between the eq. (101) and the eq. (203).

$$\tilde{Z}(\varepsilon)=\frac{1}{2(8\pi^2)^{g-1}}\sum_{k=0}^{g-2}\frac{(-\pi^2\varepsilon)^k}{k!}\left(1-2^{3-2g+2k}\right)\zeta(2g-2-2k)+O(\varepsilon^{g-1})\Rightarrow$$

$$\Rightarrow\int_{\mathcal{M}'(-1)}\exp(\omega+\varepsilon\Theta)=(-1)^{g+1}\sum_{k=0}^{g-1}\frac{\varepsilon^k}{k!}\frac{\left(2^{2g-2-2k}-2\right)B_{2g-2-2k}}{2^{3g-1}(2g-2-2k)!}\Rightarrow$$

$$\Rightarrow\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t)=\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}, \quad (222)$$

thence, between the eq. (132) and the eq. (203).

$$\int_{\mathcal{M}'(-1)}\exp(\omega+\varepsilon\Theta)V_{C_1}V_{C_2}=-2\#(C_1\cap C_2)\frac{\partial}{\partial\varepsilon}\int_{\mathcal{M}'(-1)}\exp(\omega+\varepsilon\Theta)=$$

$$=-2\#(C_1\cap C_2)\frac{\partial}{\partial\varepsilon}\left[(-1)^{g+1}\sum_{k=0}^{g-1}\frac{\varepsilon^k}{k!}\frac{\left(2^{2g-2-2k}-2\right)B_{2g-2-2k}}{2^{3g-1}(2g-2-2k)!}\right]\Rightarrow$$

$$\Rightarrow\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t)=\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}, \quad (223)$$

Thence, between the eq. (140b) and the eq. (203).

We note also that the eqs. (101) and (132) can be connected with the Ramanujan's equation (175) concerning the number 8, corresponding to the physical vibrations of the superstring. Indeed, we have:

$$\zeta_0(0)-\zeta_1(0)=\left[\frac{1}{512}\int_X R^2\kappa\wedge d\kappa-\dim Ker\Delta_0\right]-\left[\frac{1}{256}\int_X R^2\kappa\wedge d\kappa-\dim Ker\Delta_1\right]=$$

$$=\left(-\frac{1}{512}\int_X R^2\kappa\wedge d\kappa\right)+\dim Ker\Delta_1-\dim Ker\Delta_0=$$

$$=\left(-\frac{1}{512}\int_X R^2\kappa\wedge d\kappa\right)+\dim H^1(X,d_H)-\dim H^0(X,d_H)\Rightarrow$$

$$\Rightarrow\quad 8=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\quad.\quad(224)$$

$$\tilde{Z}(\varepsilon)=\frac{1}{2(8\pi^2)^{g-1}}\sum_{k=0}^{g-2}\frac{(-\pi^2\varepsilon)^k}{k!}\left(1-2^{3-2g+2k}\right)\zeta(2g-2-2k)+O\left(\varepsilon^{g-1}\right)\Rightarrow$$

$$\Rightarrow\int_{\mathcal{M}'(-1)}\exp(\omega+\varepsilon\Theta)=(-1)^{g+1}\sum_{k=0}^{g-1}\frac{\varepsilon^k}{k!}\frac{\left(2^{2g-2-2k}-2\right)B_{2g-2-2k}}{2^{3g-1}(2g-2-2k)!}\Rightarrow$$

$$\Rightarrow\quad 8=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\quad.\quad(225)$$

In conclusion, also the eqs. (110) and (218) can be related to the Ramanujan's equation (175), obtaining the following mathematical connections:

$$\langle\exp(\omega+\varepsilon\Theta)\cdot\beta\rangle'=\frac{1}{vol(G)}\int DAD\psi D\phi\exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F+\frac{1}{2}\psi\wedge\psi\right)+\frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu\, Tr\phi^2\right)\cdot\beta\Rightarrow$$

$$\Rightarrow\quad 8=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\quad,\quad(226)$$

$$\langle \exp(\omega + \varepsilon\Theta)\cdot\beta\rangle' = \frac{1}{vol(G)}\int DAD\psi D\phi \exp\left(\frac{1}{4\pi^2}\int_\Sigma Tr\left(i\phi F + \frac{1}{2}\psi\wedge\psi\right) + \frac{\varepsilon}{8\pi^2}\int_\Sigma d\mu Tr\phi^2\right)\cdot\beta \Rightarrow$$

$$\Rightarrow \frac{8}{5}\int\frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = 40\int q\psi(q)\psi^3(q^5)dq + \frac{8}{5}\log k - \frac{24}{5}\log\frac{1-k}{1+k} \Rightarrow$$

$$\Rightarrow \int_0^1 \frac{\log\left(\frac{1+\sqrt{1+4x}}{2}\right)}{x}dx = \int_1^{(\sqrt{5}+1)/2}\frac{\log u}{u^2-u}(2u-1)du = \frac{\pi^2}{15} \Rightarrow$$

$$\Rightarrow \quad 8 = \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} . \quad (227)$$

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