On some equations concerning a new possible method for the calculation of the prime numbers: mathematical connections with various expressions of some sectors of String Theory and Number Theory

Roberto Servi, Michele Nardelli[1,2]

[1] Dipartimento di Scienze della Terra

Università degli Studi di Napoli Federico II, Largo S. Marcellino, 10

80138 Napoli, Italy

[2] Dipartimento di Matematica ed Applicazioni "R. Caccioppoli"

Università degli Studi di Napoli "Federico II" – Polo delle Scienze e delle Tecnologie

Monte S. Angelo, Via Cintia (Fuorigrotta), 80126 Napoli, Italy

Abstract

In this paper, in **Sections 1** and **2**, we have described some equations and theorems concerning and linked to the Riemann zeta function. In the **Section 3**, we have showed the fundamental equation of the Riemann zeta function and the some equations concerning a new possible method for the calculation of the prime numbers. In conclusion, in the **Section 4** we show the possible mathematical connections with various expressions of some sectors of String Theory and Number Theory and finally we suppose as the prime numbers can be identified as possible solutions to the some equations of the string theory (zeta string)

Section 1

On the real line with $x > 1$, the Riemann zeta function can be defined by the integral

$$\zeta(x) \equiv \frac{1}{\Gamma(x)} \int_0^\infty \frac{u^{x-1}}{e^u - 1} \, d\, u, \tag{1}$$

where $\Gamma(x)$ is the gamma function. If x is an integer n, then we have the identity

$$\frac{u^{n-1}}{e^u - 1} = \frac{e^{-u} \, u^{n-1}}{1 - e^{-u}} \tag{2}$$

$$= e^{-u} \, u^{n-1} \sum_{k=0}^{\infty} e^{-k\, u} \tag{3}$$

$$= \sum_{k=1}^{\infty} e^{-k\, u} \, u^{n-1}, \tag{4}$$

so

$$\int_0^\infty \frac{u^{n-1}}{e^u - 1} \, d\, u = \sum_{k=1}^{\infty} \int_0^\infty e^{-k\, u} \, u^{n-1} \, d\, u. \tag{5}$$

To evaluate $\zeta(n)$, let $y \equiv k\, u$ so that $d\, y = k\, d\, u$ and plug in the above identity to obtain

$$\zeta(n) = \frac{1}{\Gamma(n)} \sum_{k=1}^{\infty} \int_0^\infty e^{-k\, u} \, u^{n-1} \, d\, u \tag{6}$$

$$= \frac{1}{\Gamma(n)} \sum_{k=1}^{\infty} \int_0^\infty e^{-y} \left(\frac{y}{k}\right)^{n-1} \frac{d\, y}{k} \tag{7}$$

$$= \frac{1}{\Gamma(n)} \sum_{k=1}^{\infty} \frac{1}{k^n} \int_0^\infty e^{-y} \, y^{n-1} \, d\, y. \tag{8}$$

Integrating the final expression in (8) gives $\Gamma(n)$, which cancels the factor $1/\Gamma(n)$, and gives the most common form of the Riemann zeta function,

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}, \tag{9}$$

which is sometimes known as a *p*-series.

The Riemann zeta function can also be defined in terms of multiple integrals by

$$\zeta(n) = \underbrace{\int_0^1 \cdots \int_0^1}_{n} \frac{\prod_{i=1}^{n} d x_i}{1 - \prod_{i=1}^{n} x_i}, \tag{10}$$

and as a Mellin transform by

$$\int_0^\infty \operatorname{frac}\left(\frac{1}{t}\right) t^{s-1}\, d t = -\frac{\zeta(s)}{s} \tag{11}$$

for $0 < \Re[s] < 1$, where $\operatorname{frac}(x)$is the fractional part (Balazard and Saias 2000).

Section 2

The prime number theorem, asserts that the number of prime numbers less than a large number $x$is equal to

$$(1+o(1))\frac{x}{\log x}:$$

$$\sum_{p \leq x} 1 = (1+o(1))\frac{x}{\log x}. \qquad (1)$$

Here, of course, $o(1)$denotes a quantity that goes to zero as $x \to \infty$.

It is not hard to see (e.g. by summation by parts) that this is equivalent to the asymptotic

$$\sum_{n \leq x} \Lambda(n) = (1+o(1))x \qquad (2)$$

for the von Mangoldt function (the key point being that the squares, cubes, etc. of primes give a negligible contribution, so $\sum_{n\leq x}\Lambda(n)$ is essentially the same quantity as $\sum_{p\leq x}\log p$). Understanding the nature of the $o(1)$ term is a very important problem, with the conjectured optimal decay rate of $O(\sqrt{x}\log x)$ being equivalent to the Riemann hypothesis

Now, we give a highly non-rigorous *heuristic* elementary "proof" of the prime number theorem (2). Since we clearly have

$$\sum_{n\leq x} 1 = x + O(1)$$

one can view the prime number theorem as an assertion that the von Mangoldt function Λ "behaves like 1 on the average",

$$\Lambda(n) \approx 1, \qquad (3)$$

where we will be deliberately vague as to what the "$\approx$" symbol means. (One can think of this symbol as denoting some sort of proximity in the weak topology or vague topology, after suitable normalisation.)

To see why one would expect (3) to be true, we take divisor sums of (3) to heuristically obtain

$$\sum_{d|n}\Lambda(d) \approx \sum_{d|n} 1. \qquad (4)$$

By (1), the left-hand side is $\log n$; meanwhile, the right-hand side is the divisor function $\tau(n)$ of n, by definition. So we have a heuristic relationship between (3) and the informal approximation

$$\tau(n) \approx \log n. \qquad (5)$$

In particular, we expect

$$\sum_{n\leq x}\tau(n) \approx \sum_{n\leq x}\log n. \qquad (6)$$

The right-hand side of (6) can be approximated using the integral test as

$$\sum_{n\leq x}\log n = \int_1^x \log t\ dt + O(\log x) = x\log x - x + O(\log x) \qquad (7)$$

(one can also use Stirling's formula to obtain a similar asymptotic). As for the left-hand side, we write $\tau(n) = \sum_{d|n} 1$ and then make the substitution $n = dm$ to obtain

$$\sum_{n\leq x}\tau(n) = \sum_{d,m:dm\leq x} 1.$$

The right-hand side is the number of lattice points underneath the hyperbola $dm = x$, and can be counted using the Dirichlet hyperbola method:

The third sum is equal to $(\sqrt{x}+O(1))^2 = x + O(\sqrt{x})$. The second sum is equal to the first. The first sum can be computed as

meanwhile, from the integral test and the definition of Euler's constant $\gamma = 0.577\ldots$ one has

$$\sum_{d \leq y} \frac{1}{d} = \log y + \gamma + O(1/y) \qquad (8)$$

for any $y \geq 1$; combining all these estimates one obtains

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}). \qquad (9)$$

Comparing this with (7) we do see that $\tau(n)$ and $\log n$ are roughly equal "to top order" on average, thus giving some form of (5) and hence (4); if one could somehow invert the divisor sum operation, one could hope to get (3) and thus the prime number theorem.

(Looking at the next highest order terms in (7), (9), we see that we expect $\tau(n)$ to in fact be slightly larger than $\log n$ on the average, and so $\Lambda(n)$ should be slightly less than 1 on the average. There is indeed a slight effect of this form; for instance, it is possible (using the prime number theorem) to prove

$$\sum_{d \leq y} \frac{\Lambda(d)}{d} = \log y - \gamma + o(1),$$

which should be compared with (8).)

One can partially translate the above discussion into the language of Dirichlet series, by transforming various arithmetical functions $f(n)$ to their associated Dirichlet series

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

ignoring for now the issue of convergence of this series. By definition, the constant function 1 transforms to the Riemann zeta function $\zeta(s)$. Taking derivatives in s, we see (formally, at least) that if $f(n)$ has Dirichlet series $F(s)$, then $f(n) \log n$ has Dirichlet series $-F'(s)$; thus, for instance, $\log n$ has Dirichlet series $-\zeta'(s)$.

Most importantly, though, if $f(n), g(n)$ have Dirichlet series $F(s), G(s)$ respectively, then their Dirichlet convolution $f * g(n) := \sum_{d|n} f(d) g(\frac{n}{d})$ has Dirichlet series $F(s)G(s)$; this is closely related to the well-known ability of the Fourier transform to convert convolutions to pointwise multiplication. Thus, for instance, $\tau(n)$ has Dirichlet series $\zeta(s)^2$. Also, from (1) and the preceding discussion, we see that $\Lambda(n)$ has Dirichlet series $-\zeta'(s)/\zeta(s)$ (formally, at least). This already suggests that the von Mangoldt function will be sensitive to the zeroes of the zeta function.

An integral test computation closely related to (8) gives the asymptotic

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$$

for s close to one (and $\mathrm{Re}(s) > 1$, to avoid issues of convergence). This implies that the Dirichlet series $-\zeta'(s)/\zeta(s)$ for $\Lambda(n)$ has asymptotic

$$\frac{-\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \gamma + O(s-1)$$

thus giving support to (3); similarly, the Dirichlet series for $\log n$ and $\tau(n)$ have asymptotic

$$-\zeta'(s) = \frac{1}{(s-1)^2} + O(1)$$

and

$$\zeta(s)^2 = \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + O(1)$$

which gives support to (5) (and is also consistent with (7), (9)).

Remark 1 One can connect the properties of Dirichlet series $F(s)$ more rigorously to asymptotics of partial sums $\sum_{n \leq x} f(n)$ by means of various transforms in Fourier analysis and complex analysis, in particular contour integration or the Hilbert transform, but this becomes somewhat technical and we will not do so here. I will remark, though, that asymptotics of $F(s)$ for s close to 1 are not enough, by themselves, to get really precise asymptotics for the sharply truncated partial sums $\sum_{n \leq x} f(n)$, for reasons related to the uncertainty principle; in order to control such sums one also needs to understand the behaviour of F far away from $s = 1$, and in particular for $s = 1 + it$ for large real t. On the other hand, the asymptotics for $F(s)$ for s near 1 are just about all one needs to control *smoothly* truncated partial sums such as $\sum_n f(n)\eta(n/x)$ for suitable cutoff functions η. Also, while Dirichlet series are very powerful tools, particularly with regards to understanding Dirichlet convolution identities, and controlling everything in terms of the zeroes and poles of such series, they do have the drawback that they do not easily encode such fundamental "physical space" facts as the pointwise inequalities $|\mu(n)| \leq 1$ and $\Lambda(n) \geq 0$, which are also an important aspect of the theory.

Section 3

The Riemann zeta function $\zeta(s)$, defined for $\mathrm{Re}(s) > 1$ by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (1)$$

and then continued meromorphically to other values of s by analytic continuation, is a fundamentally important function in analytic number theory, as it is connected to the primes $p = 2, 3, 5, \ldots$ via the Euler product formula

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (2)$$

(for $\mathrm{Re}(s) > 1$, at least), where p ranges over primes. (The equivalence between (1) and (2) is essentially the generating function version of the fundamental theorem of arithmetic.) The function ζ has a pole at 1 and a number of zeroes ρ. A formal application of the factor theorem gives

$$\zeta(s) = \frac{1}{s-1} \prod_\rho (s - \rho) \times \ldots \qquad (3)$$

where ρ ranges over zeroes of ζ, and we will be vague about what the $\ldots$ factor is, how to make sense of the infinite product, and exactly which zeroes of ζ are involved in the product

We take the following sequence of numbers:

<u>1</u>*(17+9*0)=5^2- (2^2+<u>1</u>*4)
<u>2</u>*(17+9*1)=8^2- (2^2+<u>2</u>*4)
<u>3</u>*(17+9*2)=11^2-(2^2+<u>3</u>*4)
<u>4</u>*(17+9*3)=14^2-(2^2+<u>4</u>*4)
...

In this sequence it should be noted that products of the type x * [17 + 9 (x-1)] are proportionally distant from the square of the values associated with them, equivalent in this case to 5 + 3 (x-1).

N.B.
However, the demonstration of this sequence is already present in more detail in the previous papers.

The description of this first case also applies to the following:

<u>1</u>*(17+9*0)=5^2- (2^2+<u>1</u>*4) / <u>1</u>*(53+9*0)=11^2-(8^2+<u>1</u>*4) / <u>1</u>*(89+9*0)=17^2-(14^2+<u>1</u>*4) ...
<u>2</u>*(17+9*1)=8^2- (2^2+<u>2</u>*4) / <u>2</u>*(53+9*1)=14^2-(8^2+<u>2</u>*4) / <u>2</u>*(89+9*1)=20^2-(14^2+<u>2</u>*4) ...
<u>3</u>*(17+9*2)=11^2-(2^2+<u>3</u>*4) / <u>3</u>*(53+9*2)=17^3-(8^2+<u>3</u>*4) / <u>3</u>*(89+9*2)=23^2-(14^2+<u>3</u>*4) ...
<u>4</u>*(17+9*3)=14^2-(2^2+<u>4</u>*4) / <u>4</u>*(53+9*3)=20^2-(8^2+<u>4</u>*4) / <u>4</u>*(89+9*3)=26^2-(14^2+<u>4</u>*4) ...
… … …

At this point, we want to highlight the products for which the distance compared to the square of the values associated with them is in turn equivalent to a square. We obtain:

3 *(17+9*2) =11^2-(2^2+3*4) / 9 *(53+8*9) =35^2-(8^2+9*4) ...
8 *(17+9*7) =26^2-(2^2+8*4) / 20*(53+19*9)=68^2-(8^2+20*4) ...
15*(17+9*14)=47^2-(2^2+15*4) / 33*(53+32*9)=107^2-(8^2+33*4) ...
24*(17+9*23)=74^2-(2^2+24*4) / 48*(53+47*9)=152^2-(8^2+48*4) ...
… ...

In these cases, notice how each initial product is equal to a difference of squares. Indeed:
3 *35 =11^2-4^2 / 9 *125=35^2-10^2 / ...
8 *80 =26^2-6^2 / 20*224=68^2-12^2 / ...
15*143=47^2-8^2 / 33*341=107^2-14^2 / ...
24*224=74^2-10^2 / 48*476=152^2-16^2 / ...
… …

In particular, each odd product is equal to:

(1*3) *(5*7) / (1*9) *(5*25) / (1*15)*(5*43) / ...
(3*5) *(11*13) / (3*11)*(11*31) / (3*17)*(11*49) / ...
(5*7) *(17*19) / (5*13)*(17*37) / (5*19)*(17*55) / ...
(7*9) *(23*25) / (7*15)*(23*43) / (7*21)*(23*61) / ...
(9*11)*(29*31) / (9*17)*(29*49) / (9*27)*(29*67) / ...
… … …

It should be remembered that each column refers respectively to the following starting multiplications:

x*[17+9(x-1)] / x*[53+9(x-1)] / x*[89+9(x-1)] / x*[125+9(x-1)] / …

with x=1

equivalent to:

x*[17+9(x-1)] / x*{[17+(9*4)*1]+9(x-1)} / x*{[17+(9*4)*2]+9(x-1)] / x*{[17+(9*4)*3+9(x-1)]

with x=1

Then reporting each odd product of each column to the first starting multiplication, ie x * [17 + 9 (x-1)] with x = 1, and subdividing them neatly, we obtain:

(3+10*0)*(5*7) (15+22*0)*(11*13) (35+34*0)*(17*19) ...
(3+10*1)*(5*25) (15+22*1)*(11*31) (35+34*1)*(17*37) ...
(3+10*2)*(5*43) (15+22*2)*(11*49) (35+34*2)*(17*55) ...
(3+10*3)*(5*61) (15+22*3)*(11*67) (35+34*3)*(17*73) ...
(3+10*4)*(5*79) (15+22*4)*(11*85) (35+34*4)*(17*91) ...
… … ...

In conclusion, with reference to the products in brackets on the right side of each column, we can state that they represent exactly the “half” of every possible product of the type:
17 + 9 (x-1).

In fact, taking into account factors of the type 5 + 6y, we obtain that the only possible products for which the result is of the form 17 + 9 (x-1), with odd x, are:

(5+6*0)*(7+/-18z)
(5+6*1)*(13+/-18z)
(5+6*2)*(19+/-18z)
(5+6*3)*(25+/-18z)
(5+6*4)*(31+/-18z)
…

However, it will be clear later how the minus sign can be ignored. This represents an important peculiarity inherent to the final result that will make the scope more effective.

It should also be specified in this case how the odd x is necessary to obtain an odd number, being the number in question of the type:
17 + 9 (x-1)

We could indeed have that:

(5+6*0)*(7+9*1)=5*16=80 =17+9(8-1) Con N=80 e x=8
(5+6*0)*(7+9*3)=5*34=170=17+9(18-1) Con N=170 e x=18

Just to give some examples.
While:

(5+6*0)*(7+18*1)=5*25=125=17+9(13-1) Con N=125 e x=13
(5+6*0)*(7+18*2)=5*43=215=17+9(23-1) Con N=215 e x=23

The second half of every possible product equivalent to:
17 + 9 (x-1), it is possible to obtain as follows.

We take as reference the following sequences of numbers:

$\underline{1}$*(26+9*0) =6^2-(3^2+$\underline{1}$*1)
$\underline{2}$*(26+9*1) =9^2-(3^2+$\underline{2}$*1)
$\underline{3}$*(26+9*2)=12^2-(3^2+$\underline{3}$*1)
$\underline{4}$*(26+9*3)=15^2-(3^2+$\underline{4}$*1)
…

1*(44+9*0)=9^2-(6^2+1*1)
2*(44+9*1)=12^2-(6^2+2*1)
3*(44+9*2)=15^2-(6^2+3*1)
4*(44+9*3)=18^2-(6^2+4*1)
…

As for the initial sequence also in these, the products of the type x * [26 + 9 (x-1)] and x * [44 + 9 (x-1)] are proportionally distant from the square of the values associated with them, equivalent in these cases at 6 + 3 (x-1) and 9 + 3 (x-1).

The same obviously applies to the following cases:

1*(26+9*0)=6^2-(3^2+1*1) / 1*(62+9*0)=12^2-(9^2+1*1) / 1*(98+9*0)=18^2-(15^2+1*1) ...
2*(26+9*1)=9^2-(3^2+2*1) / 2*(62+9*1)=15^2-(9^2+2*1) / 2*(98+9*1)=21^2-(15^2+2*1) ...
3*(26+9*2)=12^2-(3^2+3*1) / 3*(62+9*2)=18^2-(9^2+3*1) / 3*(98+9*2)=24^2-(15^2+3*1) …
… ... ...

1*(44+9*0)=9^2-(6^2+1*1) / 1*(80+9*0)=15^2-(12^2+1*1) / 1*(116+9*0)=21^2-(18^2+1*1) ...
2*(44+9*1)=12^2-(6^2+2*1) / 2*(80+9*1)=18^2-(12^2+2*1) / 2*(116+9*1)=24^2-(18^2+2*1) ...
3*(44+9*2)=15^2-(6^2+3*1) / 3*(80+9*2)=21^2-(12^2+3*1) / 3*(116+9*2)=27^2-(18^2+3*1) ...
… ... …

Also now wanting to highlight the products for which the distance compared to the square of the values associated with them is in turn equivalent to a square, we obtain:

7*(26+6*9) =24^2-(3^2+7*1) / 19*(62+18*9)=66^2-(9^2+19*1) / ...
16*(26+15*9)=51^2-(3^2+16*1) / 40*(62+39*9)=129^2-(9^2+40*1) / ...
27*(26+26*9)=84^2-(3^2+27*1) / 63*(62+62*9)=198^2-(9^2+63*1) / ...
…

13*(44+12*9)=45^2-(6^2+13*1) / 25*(80+24*9)=87^2-(12^2+25*1) / ...
28*(44+27*9)=90^2-(6^2+28*1) / 52*(80+51*9)=168^2-(12^2+52*1) / ...
45*(44+44*9)=141^2-(6^2+45*1) / 81*(80+80*9)=255^2-(12^2+81*1) / ...
… …

We note, also in these cases how each initial product is equal to a difference in squares.

Thus, by isolating each multiplication, whose second factor is odd, we obtain:

(2*8) *(7*23) / (2*20) *(7*59) / (2*32) *(7*95) / ...
(4*10) *(13*29) / (4*22) *(13*65) / (4*34) *(13*101) / ...
(6*12) *(19*35) / (6*24) *(19*71) / (6*36) *(19*107) / ...
(8*14) *(25*41) / (8*26) *(25*77) / (8*38) *(25*113) / ...

...   …   …

(2*14)*(7*41)   / (2*26)*(7*77)   / (2*38)*(7*113)   / ...
(4*16)*(13*47) / (4*28)*(13*83) / (4*40)*(13*119) / ...
(6*18)*(19*53) / (6*30)*(19*89) / (6*42)*(19*125) / ...
(8*20)*(25*59) / (8*32)*(25*95) / (8*44)*(25*231) / ...
…   …   …

The starting multiplications to which these columns refer are:

x*[26+9(x-1)] / x*[62+9(x-1)] / x*[98+9(x-1)] / …   with x=1
E:
x*[44+9(x-1)] / x*[80+9(x-1)] / x*[116+9(x-19] / …   with x=1

equivalent to:

x*[26+9(x-1)] / x*{[26+(9*4)*1]+9(x-1)} / x*{[26+(9*4)*2]+9(x-1)} / …   with x=1
E:
x*[44+9(x-1)] / x*{[44+(9*4)*1]+9(x-1)} / x*{[44+(9*4)*2]+9(x-1)} / …   with x=1

Then reporting each multiplication, whose second factor is odd, of each column, to the first multiplication of departure above mentioned, namely x * [17 + 9 (x-1)] with x = 1, and subdividing them neatly, we finally get:

(17+14*0)*(7*23)   (41+26*0)*(13*29)   (73+38*0)*(19*35)   (113+50*0)*(25*41)   ...
(17+14*1)*(7*41)   (41+26*1)*(13*47)   (73+38*1)*(19*53)   (113+50*1)*(25*59)   ...
(17+14*2)*(7*59)   (41+26*2)*(13*65)   (73+38*2)*(19*71)   (113+50*2)*(25*77)   ...
…   …   …   ...

In conclusion, with reference to the products in brackets on the right side of each column, we can state for the same reasons as above, that they represent exactly the second “half” of every possible product of the type:
17 + 9 (x-1).

To summarize, it is deduced on the basis of the calculations that each odd number excluded from the following sequences must be a prime number.

17+9*[(3+10x)-1)]   ; 17+9*[(17+14x)-1]
17+9*[(15+22x)-1]   ; 17+9*[(41+26x)-1]
17+9*[(35+34x)-1]   ; 17+9*[(73+38x)-1]
17+9*[(63+46x9-1]   ; 17+9*[(113+50x)-1]
…   …

That can also be written as:

8+9*(3+10x) ; 8+9*(17+14x)
8+9*(15+22x) ; 8+9*(41+26x)
8+9*(35+34x) ; 8+9*(73+38x)
8+9*(63+46x) ; 8+9*(113+50x)
… …

Or how:

8+9*[(2^2-1)+5*2x] ; 8+9*[(5^2-8)+7*2x]
8+9*[(4^2-1)+11*2x] ; 8+9*[(7^2-8)+13*2x]
8+9*[(6^2-1)+17*2x] ; 8+9*[(9^2-8)+19*2x]
… …

Let us therefore want to identify the prime numbers in a given range.

Consider the following series of multiplications.

1*(8+9*1)
2*(8+9*2)
3*(8+9*3)
4*(8+9*4)
…

As showed, we will have at a certain point the following factors on the left side of the multiplication:

3+10x / 15+22x / 35+34x / 63+46x / 99+58x / 143+70x / 195+82x / 255+94x / 323+106x / …
17+14x / 41+26x / 73+38x / 113+50x / 161+62x / 217+74x / 281+86x / 353+98x / 433+110x / …

To these factors will correspond on the right side, the odd factors seen above.
As a consequence, to all the numbers on the left side of the series of multiplications taken into consideration, excluded from the indicated sequences, will correspond, on the right side, prime numbers.
In fact, wanting to consider a range up to 35:

We obtain:

1*17

…
5*53
…
7*71
…
9*89
…
11*107
...
19*179
…
21*197
...
25*233
…
27*251
…
29*269

In which all the right factors will be prime numbers.

The peculiarity of this result is, in our opinion, the possibility of excluding the minus sign in the number sequences taken into consideration.
If indeed they were of the type:

3 +/-10x
15+/-22x
35+/-34x
63+/-46x
…

the same effectiveness in operations would not have been obtained.
However, it is easy to see how this does not happen, although it is not easy to have a more rigorous verification

Section 4

We remember that the p-adic number is an extension of the field of rationals such that congruences modulo powers of a fixed prime p are related to proximity in the so called "p-adic metric."

Any nonzero rational number x can be represented by

$$x = \frac{p^a r}{s}, \qquad \text{(a)}$$

where is a prime number, and are integers not divisible by , and is a unique integer. Then define the p-adic norm of x by p r s p a

$$|x|_p = p^{-a}. \qquad \text{(b)}$$

Also define the p-adic norm

$$|0|_p = 0.$$

Like in the ordinary string theory, the starting point of p-adic strings is a construction of the corresponding scattering amplitudes. Recall that the ordinary crossing symmetric Veneziano amplitude can be presented in the following forms:

$$A_\infty(a,b) = g^2 \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} dx = g^2 \left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} + \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} + \frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)} \right] = g^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)} =$$

$$= g^2 \int DX \exp\left(-\frac{i}{2\pi} \int d^2\sigma \partial^\alpha X_\mu \partial_\alpha X^\mu \right) \prod_{j=1}^{4} \int d^2\sigma_j \exp\left(i k_\mu^{(j)} X^\mu \right), \quad (4.1-4.4)$$

where $\hbar = 1$, $T = 1/\pi$, and $a = -\alpha(s) = -1 - \frac{s}{2}$, $b = -\alpha(t)$, $c = -\alpha(u)$ with the condition $s+t+u=-8$, i.e. $a+b+c=1$.

The p-adic generalization of the above expression

$$A_\infty(a,b) = g^2 \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} dx,$$

is:

$$A_p(a,b) = g_p^2 \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} dx, \quad (4.5)$$

where $|...|_p$ denotes p-adic absolute value. In this case only string world-sheet parameter x is treated as p-adic variable, and all other quantities have their usual (real) valuation.

Now, we remember that the Gauss integrals satisfy adelic product formula

$$\int_R \chi_\infty\left(ax^2+bx\right)d_\infty x\prod_{p\in P}\int_{Q_p}\chi_p\left(ax^2+bx\right)d_p x=1, \quad a\in Q^\times, \quad b\in Q, \quad (4.6)$$

what follows from

$$\int_{Q_v}\chi_v\left(ax^2+bx\right)d_v x=\lambda_v(a)|2a|_v^{-\frac{1}{2}}\chi_v\left(-\frac{b^2}{4a}\right), \quad v=\infty,2,...,p.... \quad (4.7)$$

These Gauss integrals apply in evaluation of the Feynman path integrals

$$K_v(x'',t'';x',t')=\int_{x',t'}^{x'',t''}\chi_v\left(-\frac{1}{h}\int_{t'}^{t''}L(\dot{q},q,t)dt\right)D_v q, \quad (4.8)$$

for kernels $K_v(x'',t'';x',t')$ of the evolution operator in adelic quantum mechanics for quadratic Lagrangians. In the case of Lagrangian

$$L(\dot{q},q)=\frac{1}{2}\left(-\frac{\dot{q}^2}{4}-\lambda q+1\right),$$

for the de Sitter cosmological model one obtains

$$K_\infty(x'',T;x',0)\prod_{p\in P}K_p(x'',T;x',0)=1, \quad x'',x',\lambda\in Q, \ T\in Q^\times, \quad (4.9)$$

where

$$K_v(x'',T;x',0)=\lambda_v(-8T)|4T|_v^{-\frac{1}{2}}\chi_v\left(-\frac{\lambda^2T^3}{24}+[\lambda(x''+x')-2]\frac{T}{4}+\frac{(x''-x')^2}{8T}\right). \quad (4.10)$$

Also here we have the number 24 that correspond to the Ramanujan function that has 24 "modes", i.e., the physical vibrations of a bosonic string. Hence, we obtain the following mathematical connection:

$$K_v(x'',T;x',0)=\lambda_v(-8T)|4T|_v^{-\frac{1}{2}}\chi_v\left(-\frac{\lambda^2T^3}{24}+[\lambda(x''+x')-2]\frac{T}{4}+\frac{(x''-x')^2}{8T}\right)\Rightarrow$$

$$\Rightarrow\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (4.10b)$$

The adelic wave function for the simplest ground state has the form

$$\psi_A(x)=\psi_\infty(x)\prod_{p\in P}\Omega\left(|x|_p\right)=\begin{cases}\psi_\infty(x), x\in Z\\ 0, x\in Q\setminus Z\end{cases}, \quad (4.11)$$

where $\Omega\left(|x|_p\right)=1$ if $|x|_p\le 1$ and $\Omega\left(|x|_p\right)=0$ if $|x|_p>1$. Since this wave function is non-zero only in integer points it can be interpreted as discreteness of the space due to p-adic effects in adelic approach. The Gel'fand-Graev-Tate gamma and beta functions are:

$$\Gamma_\infty(a)=\int_R|x|_\infty^{a-1}\chi_\infty(x)d_\infty x=\frac{\zeta(1-a)}{\zeta(a)}, \quad \Gamma_p(a)=\int_{Q_p}|x|_p^{a-1}\chi_p(x)d_px=\frac{1-p^{a-1}}{1-p^{-a}}, \quad (4.12)$$

$$B_\infty(a,b)=\int_R|x|_\infty^{a-1}|1-x|_\infty^{b-1}d_\infty x=\Gamma_\infty(a)\Gamma_\infty(b)\Gamma_\infty(c), \quad (4.13)$$

$$B_p(a,b)=\int_{Q_p}|x|_p^{a-1}|1-x|_p^{b-1}d_px=\Gamma_p(a)\Gamma_p(b)\Gamma_p(c), \quad (4.14)$$

where $a,b,c \in C$ with condition $a+b+c=1$ and $\zeta(a)$ is the Riemann zeta function. With a regularization of the product of p-adic gamma functions one has adelic products:

$$\Gamma_\infty(u)\prod_{p\in P}\Gamma_p(u)=1, \quad B_\infty(a,b)\prod_{p\in P}B_p(a,b)=1, \quad u\neq 0,1, \quad u=a,b,c, \quad (4.15)$$

where $a+b+c=1$. We note that $B_\infty(a,b)$ and $B_p(a,b)$ are the crossing symmetric standard and p-adic Veneziano amplitudes for scattering of two open tachyon strings. Introducing real, p-adic and adelic zeta functions as

$$\zeta_\infty(a)=\int_R \exp\left(-\pi x^2\right)|x|_\infty^{a-1}d_\infty x=\pi^{-\frac{a}{2}}\Gamma\left(\frac{a}{2}\right), \quad (4.16)$$

$$\zeta_p(a)=\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|_p^{a-1}d_px=\frac{1}{1-p^{-a}}, \quad \operatorname{Re}a>1, \quad (4.17)$$

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a), \quad (4.18)$$

one obtains

$$\zeta_A(1-a)=\zeta_A(a), \quad (4.19)$$

where $\zeta_A(a)$ can be called adelic zeta function. We have also that

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a)=\int_R \exp\left(-\pi x^2\right)|x|_\infty^{a-1}d_\infty x\cdot\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|_p^{a-1}d_p x. \quad (4.19b)$$

Let us note that $\exp\left(-\pi x^2\right)$ and $\Omega\left(|x|_p\right)$ are analogous functions in real and p-adic cases. Adelic harmonic oscillator has connection with the Riemann zeta function. The simplest vacuum state of the adelic harmonic oscillator is the following Schwartz-Bruhat function:

$$\psi_A(x)=2^{\frac{1}{4}}e^{-\pi x_\infty^2}\prod_{p\in P}\Omega\left(|x_p|_p\right), \quad (4.20)$$

whose the Fourier transform

$$\psi_A(k)=\int\chi_A(kx)\psi_A(x)=2^{\frac{1}{4}}e^{-\pi k_\infty^2}\prod_{p\in P}\Omega\left(|k_p|_p\right) \quad (4.21)$$

has the same form as $\psi_A(x)$. The Mellin transform of $\psi_A(x)$ is

$$\Phi_A(a)=\int\psi_A(x)|x|^a d_A^\times x=\int_R\psi_\infty(x)|x|^{a-1}d_\infty x\prod_{p\in P}\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|^{a-1}d_p x=\sqrt{2}\Gamma\left(\frac{a}{2}\right)\pi^{-\frac{a}{2}}\zeta(a) \quad (4.22)$$

and the same for $\psi_A(k)$. Then according to the Tate formula one obtains (4.19).

The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p=\frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi+\frac{1}{p+1}\varphi^{p+1}\right], \quad (4.23)$$

where p is any prime number, $\Box=-\partial_t^2+\nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+...+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L=\sum_{n\geq 1}C_n\mathcal{L}_n=\sum_{n\geq 1}\frac{n-1}{n^2}\mathcal{L}_n=\frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq 1}n^{-\frac{\Box}{2}}\phi+\sum_{n\geq 1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (4.24)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}=\prod_p\frac{1}{1-p^{-s}}, \quad s=\sigma+i\tau, \quad \sigma>1. \quad (4.25)$$

Employing usual expansion for the logarithmic function and definition (4.25) we can rewrite (4.24) in the form

$$L=-\frac{1}{g^2}\left[\frac{1}{2}\phi\zeta\left(\frac{\Box}{2}\right)\phi+\phi+\ln(1-\phi)\right], \quad (4.26)$$

where $|\phi|<1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x)=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk, \quad -k^2=k_0^2-\vec{k}^2>2+\varepsilon, \quad (4.27)$$

where $\tilde{\phi}(k)=\int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. **When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string".** Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi} \quad (4.28)$$

which has an evident solution $\phi = 0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0 t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (4.29)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n, \quad (4.30)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right], \quad (4.31)$$

and one can easily see trivial solution $\phi = \theta = 0$

Now we take, for example:

$$5(8 + 9 \times 5) = 5(8 + 45) = 5 \times 53$$

Where 53 is a prime number and 5 is a Fibonacci's number.

We note that the number 8, is connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}, \quad (4.32)$$

Thence, we have that:

$$5\times(\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]})+9\times 5=5\times(8+45)=5\times 53, \quad (4.33)$$

Where 53 is a prime number.

We have also the following new mathematical connection:

$$\zeta\left(\frac{\square}{4}\right)\theta=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\widetilde{\theta}(k)dk=\sum_{n\geq 1}\left[\theta^{n^2}+\frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right]\Rightarrow$$

$$\Rightarrow 5\times(\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]})+9\times 5=5\times(8+45)=5\times 53. \quad (4.34)$$

Conclusion

The conclusion of this paper is that is possible that the prime numbers can be identified as particular solutions to the some equations of the string theory (zeta string)

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