Non-existence of odd harmonic divisor numbers

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April 21th, 2020

Abstract

Let n be a positive integer and let b be an odd harmonic divisor number. Let the prime factors of b which are different from each other be odd primes $p_1,p_2,\ldots,p_r$ and let the exponent of p_k be a positive integer q_k. If the product of the series of the prime factors is an integer a,

$$a = \prod_{k=1}^{r} ({p_k}^{q_k} + {p_k}^{q_k - 1} + \cdots + 1)$$

$$b = \prod_{k=1}^{r} {p_k}^{q_k}$$

If b is a harmonic divisor number, let m be an integer,

$$m = \prod\nolimits_{k=1}^{r} (q_k + 1)$$

$$an = bm$$

holds. By a research of this paper, let a_k be an integer and b_k be an odd integer and if

$$a_k = a/({p_k}^{q_k} + \cdots + 1)$$

$$b_k = b/{p_k}^{q_k}$$

holds, when $r \geqq 3$, by a proof which uses the primes and the greatest common divisor (GCD) contained in na_k/b_k, the following inequality was obtained.

$$b^{r-2} \leqq m/n$$

$$m/n < (3/2)^r$$

By these inequalities, we have obtained a conclusion that there are no odd harmonic divisor numbers other than 1.

Contents

1. Introduction

In mathematics, a harmonic divisor number, or Ore number (named after Øystein Ore who defined it in 1948), is a positive integer whose divisors have a harmonic mean that is an integer. For example, the harmonic divisor number 6 has the four divisors 1, 2, 3, and 6. Their harmonic mean is an integer:

$$4/(1 + 1/2 + 1/3 + 1/6) = 2$$

(Quoted from Wikipedia)

In this paper, we prove that there are no odd harmonic divisor numbers other than 1.

2. Proof

Let n be a positive integer and let b be an odd harmonic divisor number. Let the prime factors of b which are different from each other be odd primes p_1,p_2,…,p_r and let the exponent of p_k be a positive integer q_k. If the product of the series of the prime factors is an integer a,

$$a = \prod_{k=1}^{r} (p_k{}^{q_k} + p_k{}^{q_k-1} + \cdots + 1) \ \ldots ①$$

$$b = \prod_{k=1}^{r} p_k{}^{q_k} \ \ldots ②$$

If b is a harmonic divisor number, let m be an integer,

$$m = \prod\nolimits_{k=1}^{r} (q_k + 1)$$

$$an = bm \ \ldots ③$$

holds. Divide m and n by the greatest common divisor and assume that they are relatively prime. Even if this calculation is performed, generality is not lost.

Let $\mathrm{a_k}$ be an integer and $\mathrm{b_k}$ be an odd integer,

$\mathrm{a_k = a/(p_k{}^{q_k} + \cdots + 1)}$

$\mathrm{b_k = b/p_k{}^{q_k}}$

From the expression ③,

$\mathrm{na_k(p_k{}^{q_k} + \cdots + 1) = mb_kp_k{}^{q_k}}$ …④

Ⅰ. When $r = 1$

$n(p_1^{q_1} + \cdots + 1) = (q_1 + 1)p_1^{q_1}$

Let n' be an integer and if $n = n'p_1^{q_1}$ holds,

$n'(p_1^{q_1} + \cdots + 1) = q_1 + 1$

Since $n' \geqq 1$,

$(q_1 + 1)/(p_1^{q_1} + \cdots + 1) \geqq 1$

$q_1 + 1 \geqq p_1^{q_1} + \cdots + 1 \geqq p_1^{q_1} + 1$

$q_1 \geqq p_1^{q_1}$

When $q_1 \geqq 1$ and $p_1 \geqq 3$, this inequality does not hold obviously. Therefore, odd harmonic divisor numbers do not exist when $r = 1$.

Ⅱ. When $r \geqq 2$

From the equation ④,

$na_k(p_k^{q_k+1} - 1) = mb_k p_k^{q_k}(p_k - 1)$

$\left((na_k - mb_k)p_k + mb_k\right)p_k^{q_k} = na_k$

Since $na_k/p_k^{q_k}$ is an integer, let c_k be an integer.

$\left((na_k - mb_k)p_k + mb_k\right) = na_k/p_k^{q_k} = c_k$

When $p_k \geqq 3$,

$p_k^{q_k-1} + \cdots + 1 = (p_k^{q_k} - 1)/(p_k - 1) < p_k^{q_k}/2$

From the equation ④,

$mb_k - na_k = c_k(p_k^{q_k} + \cdots + 1) - c_k p_k^{q_k} = c_k(p_k^{q_k-1} + \cdots + 1)$

$mb_k - na_k < c_k p_k^{q_k}/2 = na_k/2$

$mb_k < 3na_k/2$

$a_k/b_k > 2m/(3n) > 2/3$ …⑤

When $r = 2$,

$a_1 = p_2^{q_2} + \cdots + 1$

$b_1 = p_2^{q_2}$

$a_1/b_1 = (p_2^{q_2} + \cdots + 1)/p_2^{q_2} = (p_2^{q_2+1} - 1)/(p_2^{q_2}(p_2 - 1)) < p_2/(p_2 - 1)$

If $p_1 < p_2$, since $p_2 \geqq 5$ holds,

$a_1/b_1 < 5/4$

This inequality contradicts the inequality ⑤. Therefore, there are no odd harmonic divisor numbers when $r = 2$.

Ⅲ. When $r \geqq 3$

From the equation ④,

$na_k/b_k = mp_k^{q_k}/(p_k^{q_k} + \cdots + 1)$ …⑥

When m is divided by $p_k^{q_k} + \cdots + 1$, let m' be an integer,

$m' = mp_k^{q_k}/(p_k^{q_k} + \cdots + 1)$

$a_k = m'/n \times b_k$

hold.

The equation ⑥ is an equation for obtaining m'/n-multiperfect numbers. When m is divisible by $p_k^{q_k} + \cdots + 1$ with respect to a plurality of q_k with the same p_k, n is divided by the number of one of the q_k. By repeating this operation, it is possible to prevent m from being divisible by $p_k^{q_k} + \cdots + 1$ for all k.

A case where m cannot be divided by $p_k^{q_k} + \cdots + 1$ for all k is considered. At this time, the right side is not an integer. $p_k^{q_k} + \cdots + 1$ is the product of the prime factors p_1 to p_r excluding p_k and the divisor of m. Let d_k be an integer and if $(p_k^{q_k} + \cdots + 1) = d_k b_k$ holds, $c_k = m/d_k$ is established. d_k must be the divisor of m since c_k is an integer. Therefore, if $c_k = m/d_k$ does not hold, after the denominators of both sides are divided by its greatest common divisor C, at least one of the prime factors p_1 to p_r excluding p_k remains in the denominator on the left side.

When the denominators of both sides are divided by the GCD C, if nC is a multiple of the denominator of the left side, let s be an integer,

$mC = s(p_k^{q_k} + \cdots + 1)$

holds. The value of the left side of the equation ⑥ is $sp_k^{q_k}/C$. If this value is assumed to be an integer, s is a multiple of C since p_k does not exist as the prime factor of C. However, this contradicts the condition that m is not divided by $p_k^{q_k} + \cdots + 1$. Therefore, when the C is transposed from the denominator of the left side to the right side, the right side does not become an integer.

After this transposition, the product of the prime factors remaining in the denominator is assumed to be an odd integer P. If the numerator of the left side is a multiple of P, it becomes inconsistent since the left side is an integer and the right side is not an integer. Thereby, when the left side is reduced, at least one of the prime factors p_s of P remains in the denominator. At this time, it becomes a contradiction since p_s does not exist in the denominator of the right side. From the above, the equation ⑥ does not hold when $c_k = m/d_k$ does not hold.

When $c_k = m/d_k$ holds, since $c_k \leqq m$ is established,

$a_k = c_k {p_k}^{q_k}/n \leqq m{p_k}^{q_k}/n$

$\prod_{k=1}^{r} a_k \leqq \prod_{k=1}^{r} (m{p_k}^{q_k}/n)$

$a^{r-1} \leqq b(m/n)^r$

From the expression ③,

$(bm/n)^{r-1} \leqq b(m/n)^r$

$b^{r-2} \leqq m/n$ …⑦

From the inequality ⑤,

$\prod_{k=1}^{r} (a_k/b_k) > (2m/(3n))^r$

$(a/b)^{r-1} > (2m/(3n))^r$

$(m/n)^{r-1} > (2m/(3n))^r$

$m/n < (3/2)^r$

From the inequality ⑦,

$b^{r-2} < (3/2)^r$

$b < (3/2)^{r/(r-2)}$

This inequality does not hold since the right side is a monotonically decreasing function in the range of $r > 2$ and the maximum value is $27/8$ when $r \geqq 3$. Therefore, there are no odd harmonic divisor numbers when $r \geqq 3$.

From the above Ⅰ, Ⅱ, and Ⅲ, there are no odd harmonic divisor numbers other than 1.

3. Acknowledgement

We would like to thank the family members who sustained the research environment and the mathematicians who reviewed this research.

4. References

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