# Non-existence of odd n-multiperfect numbers

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# Abstract

Let $n$ be an integer greater than 1 and let $b$ be an odd $n$-multiperfect number. Let the prime factors of $b$ which are different from each other be odd primes $p_1, p_2, \ldots, p_r$ and let the exponent of $p_k$ be an integer $q_k$. If the product of the series of the prime factors is an integer $a$,

$$a = \prod_{k=1}^{r} ({p_k}^{q_k} + {p_k}^{q_k - 1} + \cdots + 1)$$

$$b = \prod_{k=1}^{r} {p_k}^{q_k}$$

If $b$ is a $n$-multiperfect number,

$$a = nb$$

holds. By a research of this paper, let $a_k$ be an integer and $b_k$ be an odd integer and if

$$a_k = a/({p_k}^{q_k} + \cdots + 1)$$

$$b_k = b/{p_k}^{q_k}$$

holds, when $r \geqq 3$, by a proof which uses the primes and the greatest common divisor (GCD) contained in $a_k/b_k$, the following inequality was obtained.

$$b^{r-2} \leqq n$$

$$n < (3/2)^r$$

By these inequalities, we have obtained a conclusion that there are no odd $n$-perfect numbers when $n > 1$.


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1. Introduction

A multiperfect number is a natural number whose divisor sum is an integral multiple of the original number. 2-multiperfect number is simply called a perfect number. For example, the sum of the divisors of 120 is

$$1+2+3+4+5+6+8+10+12+15+20+24+30+40+60+120=360$$
$$=3\times 120$$

holds. Since it is three times 120, 120 is 3-multiperfect number. (Quoted from Wikipedia)

In this paper, we prove that there are no odd n-multiperfect numbers when $n>1$.

2. Proof

Let $n$ be an integer greater than 1 and let $b$ be an odd $n$-multiperfect number. Let the prime factors of $b$ which are different from each other be odd primes $p_1, p_2, \dots, p_r$ and let the exponent of $p_k$ be an integer $q_k$. If the product of the series of the prime factors is an integer $a$,

$$a=\prod_{k=1}^{r}(p_k{}^{q_k}+p_k{}^{q_k-1}+\cdots+1)\ \ldots ①$$

$$b=\prod_{k=1}^{r}p_k{}^{q_k}\ \ldots ②$$

If $b$ is a $n$-multiperfect number,

$$a=nb\ \ldots ③$$

holds.

Let $a_k$ be an integer and $b_k$ be an odd integer,

$a_k=a/(p_k{}^{q_k}+\cdots+1)$

$b_k=b/p_k{}^{q_k}$

From the expression ③,

$a_k(p_k{}^{q_k}+\cdots+1)=nb_kp_k{}^{q_k}\ \ldots ④$

When $r=1$,

$p_1{}^{q_1}+\cdots+1=np_1{}^{q_1}$

Since $1\equiv 0\ (\mathrm{mod}\ p_1)$ holds, when $r=1$, odd n-multiperfect numbers do not exist.

When $r \geqq 2$,

From the equation ④,

$a_k(p_k^{q_k+1} - 1) = nb_k p_k^{q_k}(p_k - 1)$

$a_k p_k - nb_k(p_k - 1) = a_k / p_k^{q_k}$

Since the left side is an integer, let $c_k$ be an integer,

$c_k = a_k / p_k^{q_k} = a_k p_k - nb_k(p_k - 1)$ …⑤

holds.

$c_k p_k^{q_k+1} - nb_k(p_k - 1) = c_k$

$nb_k(p_k - 1) = c_k(p_k^{q_k+1} - 1)$

$nb_k = c_k(p_k^{q_k} + \cdots + 1)$ …⑥

When $p_k > 1$,

$p_k^{q_k} - 1 < p_k^{q_k}$

$(p_k^{q_k} - 1)/(p_k - 1) < p_k^{q_k}/(p_k - 1)$

$p_k^{q_k-1} + \cdots + 1 < p_k^{q_k}/(p_k - 1)$

Because $p_k$ is an odd prime and $p_k \geqq 3$ holds,

$p_k^{q_k-1} + \cdots + 1 < p_k^{q_k}/2$

From the equation ⑤ and the equation ⑥,

$nb_k - a_k = c_k(p_k^{q_k} + \cdots + 1) - c_k p_k^{q_k} = c_k(p_k^{q_k-1} + \cdots + 1)$

$nb_k - a_k < c_k p_k^{q_k}/2 = a_k/2$

$a_k/b_k > 2n/3$ …⑦

When $r = 2$,

$a_1 = p_2^{q_2} + \cdots + 1$

$b_1 = p_2^{q_2}$

$a_1/b_1 = (p_2^{q_2} + \cdots + 1)/p_2^{q_2} = (p_2^{q_2+1} - 1)/(p_2^{q_2}(p_2 - 1)) < p_2/(p_2 - 1)$

If $p_1 < p_2$, since $p_2 \geqq 5$ holds,

$a_1/b_1 < 5/4$

This inequality contradicts the inequality ⑦ when $n > 1$. Therefore, there are no odd n-multiperfect numbers when $r = 2$.

When $r \geqq 3$,
From the equation ④,
$a_k/b_k = np_k^{q_k}/(p_k^{q_k} + \cdots + 1)$ ...⑧

When n is divided by $p_k^{q_k} + \cdots + 1$, let $n'$ be an integer,
$n' = np_k^{q_k}/(p_k^{q_k} + \cdots + 1)$
$a_k = n'b_k$
hold. The equation ⑧ is an equation for obtaining $n'$-multiperfect numbers. When n is divisible by $p_k^{q_k} + \cdots + 1$ with respect to a plurality of $q_k$ with the same $p_k$, n is divided by the number of one of the $q_k$. By repeating this operation, it is possible to prevent n from being divisible by $p_k^{q_k} + \cdots + 1$ for all k.

A case where n cannot be divided by $p_k^{q_k} + \cdots + 1$ for all k is considered. At this time, the right side is not an integer. $p_k^{q_k} + \cdots + 1$ is the product of the prime factors $p_1$ to $p_r$ excluding $p_k$ and the divisor of n. Let $d_k$ be an integer and if $p_k^{q_k} + \cdots + 1 = d_k b_k$ holds, $c_k = n/d_k$ is established. $d_k$ must be the divisor of n since $c_k$ is an integer. Therefore, if $c_k = n/d_k$ does not hold, after the denominators of both sides are divided by its greatest common divisor C, at least one of the prime factors $p_1$ to $p_r$ excluding $p_k$ remains in the denominator on the left side.

When the denominators of both sides are divided by the GCD C, if $nC$ is a multiple of the denominator of the left side, let m be an integer,
$nC = m(p_k^{q_k} + \cdots + 1)$
holds. The value of the left side of the equation ⑧ is $mp_k^{q_k}/C$. If this value is assumed to be an integer, m is a multiple of C since $p_k$ does not exist as the prime factor of C. However, this contradicts the condition that n is not divided by $p_k^{q_k} + \cdots + 1$. Therefore, when the C is transposed from the denominator of the left side to the right side, the right side does not become an integer.

After this transposition, the product of the prime factors remaining in the denominator is assumed to be an odd integer P. If the numerator of the left side is a multiple of P, it becomes inconsistent since the left side is an integer and the right side is not an integer. Thereby, when the left side is reduced, at least one of the prime factors $p_s$ of P remains in the denominator. At this time, it becomes a contradiction since $p_s$ does not exist in the denominator of the right side. From the above, the equation ⑧ does not hold when $c_k = n/d_k$ does not hold.

When $c_k = n/d_k$ holds, since $c_k \leqq n$ is established,

$a_k = c_k p_k^{q_k} \leqq n p_k^{q_k}$

Since this inequality holds for all k,

$$\prod_{k=1}^{r} a_k \leqq \prod_{k=1}^{r} (n p_k^{q_k})$$

$a^{r-1} \leqq n^r b$

From the expression ③,

$(nb)^{r-1} \leqq n^r b$

$b^{r-2} \leqq n$ …⑨

From the inequality ⑦,

$$\prod_{k=1}^{r} (a_k/b_k) > (2n/3)^r$$

$(a/b)^{r-1} > (2n/3)^r$

$n^{r-1} > (2n/3)^r$

$n < (3/2)^r$

From the inequality ⑨,

$b < (3/2)^{r/(r-2)}$

This inequality does not hold since the right side is a monotonically decreasing function in the range of $r > 2$ and the maximum value is $27/8$ when $r \geqq 3$. Therefore, there are no odd n-multiperfect numbers when $r \geqq 3$.

From the above, there are no odd n-multiperfect numbers when $n > 1$.

## 3. Acknowledgement

For the proof about the existence of odd perfect number, we asked anonymous reviewers to point out several tens of mistakes. We would like to thank you for giving appropriate guidance and counter-arguments.

## 4. References

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