On the Riemann Hypothesis. Formulas explained - ψ(x) as equivalent RH. Mathematical connections with "Aurea" section and some sectors of String Theory

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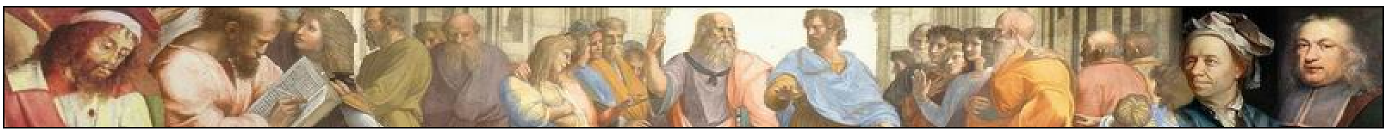
Abstract

In this work the authors will examine the themes of RH, equivalent RH and GRH already presented in [25]. The authors will explain some formulas and will show other special functions that are usually introduced with the PNT and useful to investigate other ways. In the **Sections 1** and **2**, we describe ψ(x), i.e. the 2nd Chebyshev's function as equivalent RH. In the **Section 3**, we describe a step function and a generalization of Polignac.

In the **Section 4**, we describe some equations concerning p-adic strings, p-adic and adelic zeta functions, zeta strings and zeta nonlocal scalar fields.

In conclusion, in the **Section 5**, we have described some possible mathematical connections between adelic strings and Lagrangians with Riemann zeta function with some equations in Number Theory above examined.



1. ψ(x) equivalent RH

In [25] we saw that Riemann defined ζ(s) as a function of complex variable s. The first step of Riemann was to extend (or to *analytically continue*) ζ(s) to all of X \ {1} This can be accomplished by noticing that s=σ+it and $n^{-s} = s\int_n^\infty x^{-s-1}dx$ then:

$$\begin{aligned}\varsigma(s) &= \sum_{n=1}^{\infty}\frac{1}{n^s} = \sum_{n=1}^{\infty}\left(s\int_n^\infty \frac{dx}{x^{s+1}}\right) = s\sum_{n=1}^{\infty}\int_n^\infty \frac{dx}{x^{s+1}} \\ &= s\int_1^\infty\left(\sum_{n\le x}1\right)\frac{dx}{x^{s+1}} = s\int_1^\infty \frac{[x]}{x^{s+1}}dx = s\int_1^\infty \frac{x-\{x\}}{x^{s+1}}dx \qquad (1.1) \qquad (^1) \\ &= \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx,\ \ \sigma > 1\end{aligned}$$

Since {x}∈[0,1), it follows that the last integral converges for σ>0 and defines a continuation of . ζ(s) to the half-plane σ=Re(s)>0. We can extend ζ(s) to a holomorphic function on all X \ {1}, in fact from the last integral s=1 is a simple pole with residue 1. We note that for s real and s>0 the integral in (1.1) is always positive real. Then from (1.1) ζ(s)<0, s∈(0.1) and ζ(s)>0, s∈(1.∞).

A popular expression of Euler is:

$$\varsigma(s) = \prod_{p=prime}(1-p^{-s})^{-1}$$

$$\ln\varsigma(s) = -\sum_{p=prime}\ln(1-p^{-s}) = \sum_{p=primes}\sum_{k=1}^{\infty}\frac{p^{-ks}}{k} \qquad (1.2)$$

In (1.2) we have applied the integration of Newton linked to expression:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

If the previous expression is integrated for x and you change the sign to bring the term 1-x to numerator, then we obtain:

$$-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \ldots$$

Now, we introduce the ***von Mangoldt's function*** *(also called lambda function)*:

$$\Lambda(n) = \begin{cases}\log p, & if\ n=p^k,\ \ p\ \ prime,\ k\ge 1 \\ 0, & \text{otherwise}\end{cases} \qquad (1.3)$$

[1] [x] is the greatest integer ≤ x or floor of x; {x}=x-[x] is the fractional part of x.

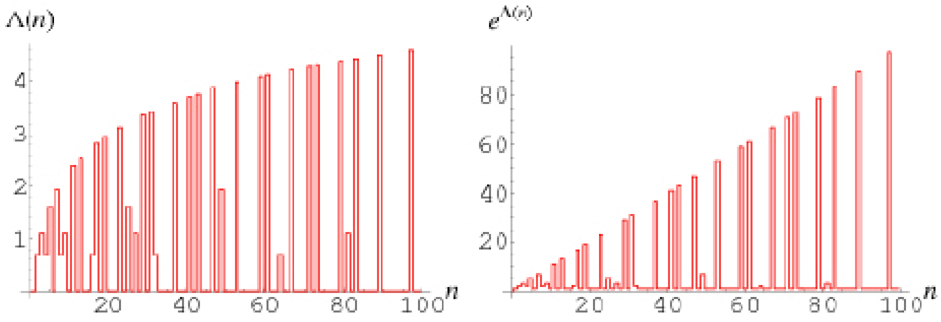


Figure 1 – von Mangoldt's function

From (1.2) we have:

$$p^{-ks} = \begin{cases} n^{-s}, & if\ n=p^k \\ 0, & \text{otherwise} \end{cases}$$

and if we use the rules of logarithm: $n = p^k$, $k = \log_p n = \log n / \log p$ then:

$$\frac{1}{k} = \begin{cases} \frac{\log p}{\log n} = \frac{\Lambda(n)}{\log n}, & when\ \ n=p^k \\ 0, & \text{otherwise} \end{cases}$$

Further the (1.2) becomes:

$$\ln \varsigma(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n \ \ n^s} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s} \qquad (1.4)$$

The (1.4) consents to pass from "a multiplicative problem" to "an additive problem", even if we are started from the Euler's product.

Consequently if we do the derivative of (1.4) then we obtain:

$$\frac{\varsigma'(s)}{\varsigma(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \qquad (1.5)$$

The von Mongoldt's function isn't a multiplicative function nor an additive function. Moreover it's:

$$\log n = \sum_{d|n} \Lambda(d) \qquad \text{where d | n are divisor of n}$$

<u>Example</u>

n=12

We remember $12=2^2*3$ and that the divisors of 12 are: 1, 2, 3, 4, 6, 12, then:
$\log 12=\Lambda(1)+ \Lambda(2)+ \Lambda(3)+ \Lambda(2^2)+ \Lambda(2*3)+ \Lambda(2^2*3)$

From (1.3) it is:
log 12= 0 + log 2 + log 3 + log 2 + 0 + 0= log(2*3*2) = log 12

Pafnuty Lvovich Chebyshev introduced two functions:

$$\theta(x) = \sum_{p \le x} \log p \qquad \text{1st Chebyshev's function}$$

$$\psi(x) = \sum_{n \le x} \Lambda(n) \qquad \text{2nd Chebyshev's function}$$

This functions are very important in the proofs linked to prime numbers, because they are simple to use.

Another formulas equivalent for θ(x) is:

$$\theta(x) = \sum_{p \le x} \log p = \sum_{k=1}^{\pi(x)} \log p_k = \ln \left| \prod_{k=1}^{\pi(x)} p_k \right|$$

Hardy and Wright [23] showed that:

$$\lim_{x \to \infty} \frac{x}{\theta(x)} = 1$$

Or:

$$\theta(x) \sim x$$

From here we have the figure 1.

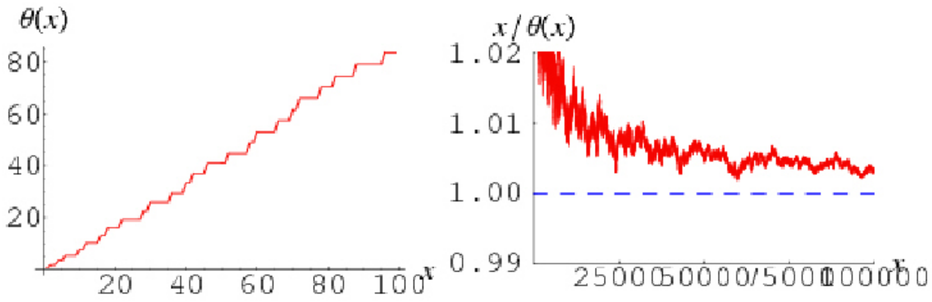


Figure 2 - θ(x) Chebyshev's function

We also can write ψ(x):

$$\psi(x) = \sum_{\substack{p \le x \\ p^k \le x}} \log p = \sum_{k=1}^{\pi(x)} \Lambda(k) \qquad (1.6)$$

In the previous formula the sum runs over all prime numbers p and positive integers k such that $p^k \le x$ and therefore potentially includes some primes multiple times.

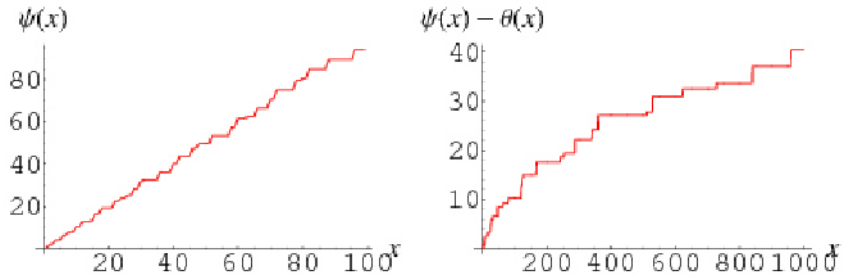


Figure 3 - ψ(x) Chebyshev's function

A simple and nice formula for ψ(x) is:

$$\psi(x) = \ln|lcm(1,2,3,4,...x)| \quad ([2])$$

or

$$lcm(1,2,3,4,...x) = e^{\psi(x)}$$

Example
x=10
lcm(1,2,3,4,5,6,7,8,9,10)=$5*7*2^3*3^2$=2520

ψ(10)=ln 2520 = ln $5*7*2^3*3^2$= ln 5 + ln 7 + 3ln 2 + 2 ln 3

Now an *equivalent PNT* or *an equivalent RH* is:

$$\psi(x) \sim x \qquad (1.7)$$

Finally ψ(x) and θ(x) are linked:

$$\psi(x) = \sum_{k=1}^{\infty} \theta(x^{\frac{1}{k}})$$

The previous formula has got a finite number of terms, because $\theta(x^{1/2})=0$ for $n>\log_2 x$.

The functions ψ(x) and θ(x) are in some ways more natural then prime counting function π(x), because they deal with multiplication of primes. In a multiplicative problem they are better.

It can be obtained a link from ζ(s) and ψ(x) by inverting (1.5); in fact, starting from (1.5), the Fourier inversion formula implies, for each a>1:

$$\psi(x) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) x^s \frac{ds}{s}, \; x > 0 \qquad (1.8)$$

A link between ψ(x) and the nontrivial zeros (with multiplicities) of Riemann zeta function is the so-called *explicit formula (Riemann-von Mangoldt)*:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - (\ln 2\pi) - \frac{1}{2}\ln(1-x^{-2}) \qquad (1.9)$$

[2] For italian readers the term *lcm* (least common multiple) is equivalent to the term *mcm*

For x>1 and x not prime number or prime power and ρ a nontrivial zero. The (1.9) gives a very precise description of the error in the approssimations (1.7), and, more important, it relations the estimation of this error to the location of the nontrivial zeros.

We note that de la Vallèe-Poussin showed that the term-by-term integration of both sides of (1.9) is a valid operation for x>1:

$$\psi_1(x) = \int_0^x \psi(t)dt = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - \sum_n \frac{x^{-2n+1}}{2n(2n\text{-}1)} - x\log(2\pi) + const \quad (1.10)$$

It is clear that, as $x\rightarrow\infty$, the last three terms on the right hand side of the (1.10) are all $o(x^2)$.

2. Why ψ(x) is an equivalent RH

Now we can show that *ζ(1+it)≠ 0* or that *there aren't nontrivial zeros on the line σ=1*. If we remember that s=σ+it, taking the real part, from (1.4) is:

$$\mathrm{Re}(\ln\varsigma(s)) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n\ n^\sigma}\cdot\cos(t\log n)$$

By trigonometric identity $3 + 4\cos t + \cos 2t = 2(1 + \cos t)^2 \geq 0$ then:

$$3\,\mathrm{Re}(\ln\varsigma(s)) + 4\,\mathrm{Re}(\ln\varsigma(\sigma+\mathrm{it})) + \mathrm{Re}(\ln\varsigma(\sigma+2\mathrm{it})) \geq 0$$

So exponentiating, this gives:

$$|\varsigma(\sigma)|^3|\varsigma(\sigma+\mathrm{it})|^4|\varsigma(\sigma+2\mathrm{it})| \geq 1 \qquad (2.1)$$

As we have seen in (1.1), ζ(s) has got a simple pole in s=1 with residue 1. This is equivalent to says:

$$\lim_{s\to1}(s-1)\varsigma(s) = 1 \quad (2.2)$$

Now we suppose that ζ(s) has got a zero of order m≥1 at $s_0 = 1 + it_0$ then it is equivalent to:

$$\lim_{s\to s_0}(s-s_0)^{-m}\varsigma(s) = c \quad (2.3)$$

For some c ∈ X \{0}. Taking $s = \sigma + it_0$ and σ>1 then we can rewrite (2.1) as:

$$\begin{aligned}&|\varsigma(\sigma)|^3|\sigma-1|^3\cdot\frac{|\varsigma(\sigma+\mathrm{it}_0)|^4}{|s-s_0|^{4m}}\cdot|\varsigma(\sigma+2\mathrm{it}_0)| \geq \frac{|\sigma-1|^3}{|s-s_0|^{4m}} \\ &= \frac{|\sigma-1|^3}{|\sigma-1|^{4\mathrm{m}}} = \frac{1}{|\sigma-1|^{4\mathrm{m}\text{-}1}}\end{aligned} \qquad (2.4)$$

Letting $\sigma \to 1^+$, and taking account the two limits above, we obtain that there is a pole of order $4m-3 \geq 1$ at $s=1+2it_0$. This is impossible, then $\zeta(1+it) \neq 0$ for $t \in R \setminus \{0\}$ is true. Therefore if ρ is a nontrivial zero of $\zeta(s)$, then $Re(\rho)<1$, $|x^{\rho}-1|<1$ and the infinite sum $\sum_{\rho} \frac{1}{\rho(\rho+1)}$ in (1.10) converges absolutely. This implies that $\sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho+1)}$ converges uniformly in x and:

$$\lim_{x\to\infty} \sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho+1)} = \sum_{\rho} \lim_{x\to\infty} \frac{x^{\rho-1}}{\rho(\rho+1)} = \sum_{\rho} 0 = 0 \qquad (2.5)$$

So also the second term of (1.10) is bounded by $o(x^2)$. Therefore we can conclude that $\psi_1(x) \sim x^2/2$.

In general if two functions are asymptotic one can't conclude their derivative are asymptotic; but we know that the derivative $\psi=\psi_1$' is a monotonic non-decreasing function, then we can conclude that (1.7) is true or $\psi(x) \sim x$.

3. A step function and a generalization of Polignac

Now, we introduce the ***von Mangoldt's function*** *(also called lambda function)*:

$$\Lambda(n) = \begin{cases} \log p, & if\ n=p^k,\ \ p\ \ prime,\ k\geq 1 \\ 0, & \text{otherwise} \end{cases} \qquad (3.1)$$

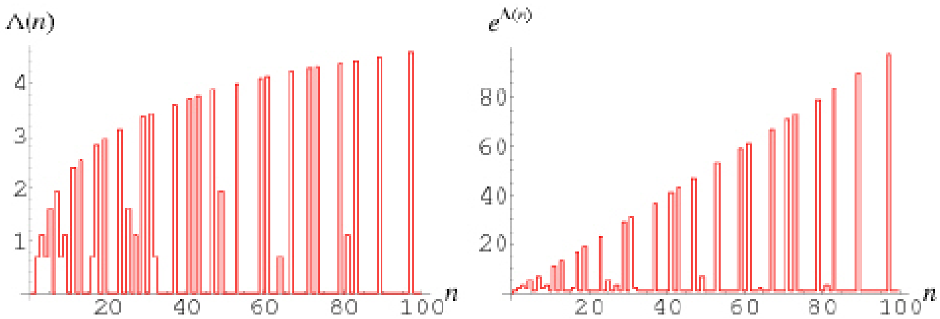


Figure 4 – von Mangoldt's function

The von Mongoldt's function isn't a multiplicative function nor an additive function. Moreover it's:

$$\log n = \sum_{d|n} \Lambda(d) \qquad \text{where d | n are divisor of n}$$

<u>Example</u>

n=12

We remember $12=2^2*3$ and that the divisors of 12 are: 1, 2, 3, 4, 6, 12, then:
$\log 12=\Lambda(1)+ \Lambda(2)+ \Lambda(3)+ \Lambda(2^2)+ \Lambda(2*3)+ \Lambda(2^2*3)$

From (3.1) it is:
$\log 12= 0 + \log 2 + \log 3 + \log 2 + 0 + 0= \log(2*3*2) = \log 12$

We know π(N) as a counting prime function (a step function):

$$\pi(N) = \sum_{p \le x} 1$$

If we introduce the von Mangoldt's function Λ(N) then *we propose* a "*step function* $\upsilon(N)$":

$$\nu(N) = \frac{\pi(N)}{N\,\Lambda(N)}$$

$$\frac{G(N)}{N} \sim \nu(N) \quad \text{or} \quad \lim_{x\to\infty} \frac{G(x)}{x} \Big/ \nu(x) = 1$$

For example

π(N)	Λ(N)	υ(N)	G(N)/N
π(10) = 4	Λ(10) = ln 10 = 2,3	υ(10) = 4/(10*2,3)=0,17	G(10)/10 = 0,1
π(30) = 10	Λ(30) = ln 30 = 3,401	υ(30) = 10/(30*3,401)=0,098≈0,1	G(30)/30 = 3/30=0,1
π(100) = 25	Λ(100) = ln 100 = 4,605	υ(100) = 25/(100*4,605)=0,054	G(100)/100 = 6/100=0,06
π(1000) = 168	Λ(1000) = ln 1000 = 6,907	υ(1000) = 168/(1000*6,907)=0,0243	G(1000)/1000 = 28/1000=0,028

The approssimations of the step function v(N) improve when N grows (see figure 5).

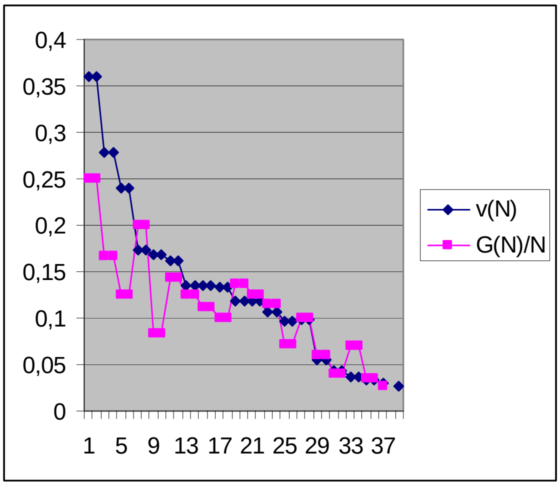


Figure 5 – step function v(N)

Can we generalize this result as *a generalization of Polignac*? Yes. If we call *P(x , d)* the number of primes ≤x and which are far d, if we remember the GRH [see 25], since it is:

$$\pi(x,a,d) = \frac{1}{\varphi(d)} \int_2^x \frac{1}{\ln t} dt + O(x^{\frac{1}{2}+\varepsilon}), \quad x \to \infty \quad (3.2)$$

where a, d are such that gcd(a, d)=1 and φ(d) is the *totient function* of Euler.

Then it is:

$$\left| \frac{P(x,d)}{x} - \frac{\pi(x,a,d)}{x \ln x} \right| < kC3(x), k = 1, C3(x) = \frac{1}{\ln x}$$

or (3.3)

$$\left| \frac{P(x,d)}{x} - \frac{1}{\varphi(d)} \int_2^x \frac{1}{t(\ln t)^2} dt \right| = O(x^{\frac{1}{2}+\varepsilon})$$

Example

x=127, a=2,d=9

gcd(a , d) =1

a + d: 11, 20, 29, 38, 47, 56, 65, 74, 83, 92, 101, 110, 119, 128 …

We have underlined the prime numbers above.

φ(9) = 6, in fact 1,2,4,5,7,8 are the numbers without nothing in common with 9,

Li ~ x/ln x

π(127-2,9) ~ [1/φ(9)]*(127/ln 127) ≈ 4,36 about 5.

But this result is also P(x, 9). In fact if we consider a=1,2,4,5,7,8 or the numbers less than 9 and without nothing in common with 9, we have six *arithmetic progressions*:

a=1, a + d: 10, 19, 28, 37, 46, 55, 64, 73, 82, 91, 100, 109, 118, 127 …
a=2, a + d: 11, 20, 29, 38, 47, 56, 65, 74, 83, 92, 101, 110, 119, 128 …
a=4, a + d: 13, 22, 31, 40, 49, 58, 67, 76, 85, 94, 103, 112, 121, 130 …
a=5, a + d: 14, 23, 32, 41, 50, 59, 68, 77, 86, 95, 104, 113, 122, 131 …
a=7, a + d: 16, 25, 34, 43, 52, 61, 70, 79, 88, 97, 106, 115, 124, 133 …
a=8, a + d: 17, 26, 35, 44, 53, 62, 71, 80, 89, 98, 107, 116, 125, 134 …

In all arithmetic progression we have prime numbers. How many are the prime numbers in each arithmetic progression?

About:

$$[1/\varphi(d)] * (x/\ln x).$$

Then the absolute value of difference [P(x, d)/x] - [π(x, a, d)/x ln x] is very little.

In fact for a = 2 is:

P(127,9)/127 = 5/127 = 0,00031
π(127, 2, 9)/(127 ln 127) = 0,007102
|[P(127, 9)/127] - [π(127, 2, 9)/127 ln 127]| = 0,00679 < 1/ln 127 = 0,206433.

We can obtain, with the integral, a value better than x/ln x; in fact (see Appendix) it is:

$$\int_2^x \frac{dt}{t \cdot \ln^2 t} = \frac{1}{\ln 2} - \frac{1}{\ln x}$$

Then it is:

π(127, 2, 9) = 1/6 * (1/ln 2 - 1/ln 127) = 0,20604
π(127, 2, 9) / 127 ln 127 = 0,000334 a similar result of P(x, d)/x

|[P(127, 9)/127] - [π(127, 2, 9)/127 ln 127]| = 0,000024 < 1/ln 127 = 0,206433.

So, also (3.3) is an equivalent RH.

With regard the values 4,36 and 0,206433 we have the following mathematical connections:

$(\Phi)^{21/7}+(\Phi)^{-30/7}=4{,}236067977+0{,}127156535=4{,}36322$;
$(\Phi)^{-28/7}+(\Phi)^{-41/7}=0{,}145898034+0{,}059693843=0{,}205591877\cong 0{,}2056$.

Note that $\Phi=\frac{\sqrt{5}+1}{2}\cong 1{,}6180339$ **is the Aurea ratio and that with regard the index n/7, n = 1, 2,…,** $+\infty$ **, n = -1, -2,…,** $-\infty$**, while 7 is the number of the compactified dimensions of M-Theory**.

4. On some equations concerning p-adic strings, p-adic and adelic zeta functions, zeta strings and zeta nonlocal scalar fields. [27] [28] [29] [30] [31]

Like in the ordinary string theory, the starting point of p-adic strings is a construction of the corresponding scattering amplitudes. Recall that the ordinary crossing symmetric Veneziano amplitude can be presented in the following forms:

$$A_\infty(a,b)=g^2\int_R |x|_\infty^{a-1}|1-x|_\infty^{b-1}dx=g^2\left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}+\frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)}+\frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)}\right]=g^2\frac{\zeta(1-a)}{\zeta(a)}\frac{\zeta(1-b)}{\zeta(b)}\frac{\zeta(1-c)}{\zeta(c)}=$$
$$=g^2\int DX\exp\left(-\frac{i}{2\pi}\int d^2\sigma\partial^\alpha X_\mu\partial_\alpha X^\mu\right)\prod_{j=1}^4\int d^2\sigma_j\exp\left(ik_\mu^{(j)}X^\mu\right),\quad (4.1-4.4)$$

where $\hbar=1$, $T=1/\pi$, and $a=-\alpha(s)=-1-\frac{s}{2}$, $b=-\alpha(t)$, $c=-\alpha(u)$ with the condition $s+t+u=-8$, i.e. $a+b+c=1$.
The p-adic generalization of the above expression

$$A_\infty(a,b)=g^2\int_R |x|_\infty^{a-1}|1-x|_\infty^{b-1}dx,$$

is:

$$A_p(a,b)=g_p^2\int_{Q_p}|x|_p^{a-1}|1-x|_p^{b-1}dx,\quad (4.5)$$

where $|\ldots|_p$ denotes p-adic absolute value. In this case only string world-sheet parameter x is treated as p-adic variable, and all other quantities have their usual (real) valuation.
Now, we remember that the Gauss integrals satisfy adelic product formula

$$\int_R\chi_\infty(ax^2+bx)d_\infty x\prod_{p\in P}\int_{Q_p}\chi_p(ax^2+bx)d_px=1,\quad a\in Q^\times,\quad b\in Q,\quad (4.6)$$

what follows from

$$\int_{Q_v} \chi_v\left(ax^2 + bx\right) d_v x = \lambda_v(a)|2a|_v^{-\frac{1}{2}} \chi_v\left(-\frac{b^2}{4a}\right), \quad v = \infty, 2, ..., p.... \quad (4.7)$$

These Gauss integrals apply in evaluation of the Feynman path integrals

$$K_v(x'', t''; x', t') = \int_{x', t'}^{x'', t''} \chi_v\left(-\frac{1}{h}\int_{t'}^{t''} L(\dot{q}, q, t)dt\right) D_v q, \quad (4.8)$$

for kernels $K_v(x'', t''; x', t')$ of the evolution operator in adelic quantum mechanics for quadratic Lagrangians. In the case of Lagrangian

$$L(\dot{q}, q) = \frac{1}{2}\left(-\frac{\dot{q}^2}{4} - \lambda q + 1\right),$$

for the de Sitter cosmological model one obtains

$$K_\infty(x'', T; x', 0) \prod_{p \in P} K_p(x'', T; x', 0) = 1, \quad x'', x', \lambda \in Q, \; T \in Q^\times, \quad (4.9)$$

where

$$K_v(x'', T; x', 0) = \lambda_v(-8T)|4T|_v^{-\frac{1}{2}} \chi_v\left(-\frac{\lambda^2 T^3}{24} + [\lambda(x'' + x') - 2]\frac{T}{4} + \frac{(x'' - x')^2}{8T}\right). \quad (4.10)$$

Also here we have the number 24 that correspond to the Ramanujan function that has 24 “modes”, i.e., the physical vibrations of a bosonic string. Hence, we obtain the following mathematical connection:

$$K_v(x'', T; x', 0) = \lambda_v(-8T)|4T|_v^{-\frac{1}{2}} \chi_v\left(-\frac{\lambda^2 T^3}{24} + [\lambda(x'' + x') - 2]\frac{T}{4} + \frac{(x'' - x')^2}{8T}\right) \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi t x w'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4}\right)}\right]}. \quad (4.10b)$$

The adelic wave function for the simplest ground state has the form

$$\psi_A(x) = \psi_\infty(x) \prod_{p \in P} \Omega\left(|x|_p\right) = \begin{cases} \psi_\infty(x), x \in Z \\ 0, x \in Q \setminus Z \end{cases}, \quad (4.11)$$

where $\Omega(|x|_p)=1$ if $|x|_p \le 1$ and $\Omega(|x|_p)=0$ if $|x|_p > 1$. Since this wave function is non-zero only in integer points it can be interpreted as discreteness of the space due to p-adic effects in adelic approach. The Gel'fand-Graev-Tate gamma and beta functions are:

$$\Gamma_\infty(a)=\int_R |x|_\infty^{a-1}\chi_\infty(x)d_\infty x=\frac{\zeta(1-a)}{\zeta(a)}, \quad \Gamma_p(a)=\int_{Q_p}|x|_p^{a-1}\chi_p(x)d_p x=\frac{1-p^{a-1}}{1-p^{-a}}, \quad (4.12)$$

$$B_\infty(a,b)=\int_R |x|_\infty^{a-1}|1-x|_\infty^{b-1}d_\infty x=\Gamma_\infty(a)\Gamma_\infty(b)\Gamma_\infty(c), \quad (4.13)$$

$$B_p(a,b)=\int_{Q_p}|x|_p^{a-1}|1-x|_p^{b-1}d_p x=\Gamma_p(a)\Gamma_p(b)\Gamma_p(c), \quad (4.14)$$

where $a,b,c \in C$ with condition $a+b+c=1$ and $\zeta(a)$ is the Riemann zeta function. With a regularization of the product of p-adic gamma functions one has adelic products:

$$\Gamma_\infty(u)\prod_{p\in P}\Gamma_p(u)=1, \quad B_\infty(a,b)\prod_{p\in P}B_p(a,b)=1, \quad u\neq 0,1, \quad u=a,b,c, \quad (4.15)$$

where $a+b+c=1$. We note that $B_\infty(a,b)$ and $B_p(a,b)$ are the crossing symmetric standard and p-adic Veneziano amplitudes for scattering of two open tachyon strings. Introducing real, p-adic and adelic zeta functions as

$$\zeta_\infty(a)=\int_R \exp(-\pi x^2)|x|_\infty^{a-1}d_\infty x=\pi^{-\frac{a}{2}}\Gamma\left(\frac{a}{2}\right), \quad (4.16)$$

$$\zeta_p(a)=\frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1}d_p x=\frac{1}{1-p^{-a}}, \quad \mathrm{Re}\,a>1, \quad (4.17)$$

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a), \quad (4.18)$$

one obtains

$$\zeta_A(1-a)=\zeta_A(a), \quad (4.19)$$

where $\zeta_A(a)$ can be called adelic zeta function. We have also that

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a)=\int_R \exp(-\pi x^2)|x|_\infty^{a-1}d_\infty x\cdot\frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1}d_p x. \quad (4.19b)$$

Let us note that $\exp(-\pi x^2)$ and $\Omega(|x|_p)$ are analogous functions in real and p-adic cases. Adelic harmonic oscillator has connection with the Riemann zeta function. The simplest vacuum state of the adelic harmonic oscillator is the following Schwartz-Bruhat function:

$$\psi_A(x)=2^{\frac{1}{4}}e^{-\pi x_\infty^2}\prod_{p\in P}\Omega(|x_p|_p), \quad (4.20)$$

whose the Fourier transform

$$\psi_A(k)=\int \chi_A(kx)\psi_A(x)=2^{\frac{1}{4}}e^{-\pi k_\infty^2}\prod_{p\in P}\Omega\left(\left|k_p\right|_p\right) \quad (4.21)$$

has the same form as $\psi_A(x)$. The Mellin transform of $\psi_A(x)$ is

$$\Phi_A(a)=\int \psi_A(x)|x|^a d_A^\times x=\int_R \psi_\infty(x)|x|^{a-1}d_\infty x\prod_{p\in P}\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|^{a-1}d_p x=\sqrt{2}\Gamma\left(\frac{a}{2}\right)\pi^{-\frac{a}{2}}\zeta(a) \quad (4.22)$$

and the same for $\psi_A(k)$. Then according to the Tate formula one obtains (4.19).
The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p=\frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi+\frac{1}{p+1}\varphi^{p+1}\right], \quad (4.23)$$

where p is any prime number, $\Box=-\partial_t^2+\nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+...+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L=\sum_{n\geq 1}C_n\mathcal{L}_n=\sum_{n\geq 1}\frac{n-1}{n^2}\mathcal{L}_n=\frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq 1}n^{-\frac{\Box}{2}}\phi+\sum_{n\geq 1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (4.24)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}=\prod_p\frac{1}{1-p^{-s}}, \quad s=\sigma+i\tau, \quad \sigma>1. \quad (4.25)$$

Employing usual expansion for the logarithmic function and definition (4.25) we can rewrite (4.24) in the form

$$L=-\frac{1}{g^2}\left[\frac{1}{2}\phi\zeta\left(\frac{\Box}{2}\right)\phi+\phi+\ln(1-\phi)\right], \quad (4.26)$$

where $|\phi|<1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x)=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk, \quad -k^2=k_0^2-\vec{k}^2>2+\varepsilon, \quad (4.27)$$

where $\tilde{\phi}(k)=\int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.
Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. **When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string".** Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi} \quad (4.28)$$

which has an evident solution $\phi=0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t)=\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}. \quad (4.29)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n, \quad (4.30)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk=\sum_{n\geq 1}\left[\theta^{n^2}+\frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right], \quad (4.31)$$

and one can easily see trivial solution $\phi=\theta=0$.

The exact tree-level Lagrangian of effective scalar field φ, which describes open p-adic string tachyon, is:

$$\mathcal{L}_p=\frac{m_p^D}{g_p^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2m_p^2}}\varphi+\frac{1}{p+1}\varphi^{p+1}\right], \quad (4.32)$$

where p is any prime number, $\Box=-\partial_t^2+\nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+\ldots+)$, as above. Now, we want to introduce a model which incorporates all the above string Lagrangians (4.32) with p replaced by $n\in N$. Thence, we take the sum of all Lagrangians $\mathcal{L}_n$ in the form

$$L=\sum_{n=1}^{+\infty}C_n\mathcal{L}_n=\sum_{n=1}^{+\infty}C_n\frac{m_n^D}{g_n^2}\frac{n^2}{n-1}\left[-\frac{1}{2}\phi n^{-\frac{\Box}{2m_n^2}}\phi+\frac{1}{n+1}\phi^{n+1}\right], \quad (4.33)$$

whose explicit realization depends on particular choice of coefficients C_n, masses m_n and coupling constants g_n.

Now, we consider the following case

$$C_n=\frac{n-1}{n^{2+h}}, \quad (4.34)$$

where h is a real number. The corresponding Lagrangian reads

$$L_h=\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\sum_{n=1}^{+\infty}n^{-\frac{\Box}{2m^2}-h}\phi+\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right] \quad (4.35)$$

and it depends on parameter h. According to the Euler product formula one can write

$$\sum_{n=1}^{+\infty} n^{-\frac{\Box}{2m^2}-h} = \prod_p \frac{1}{1-p^{-\frac{\Box}{2m^2}-h}}. \quad (4.36)$$

Recall that standard definition of the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1, \quad (4.37)$$

which has analytic continuation to the entire complex s plane, excluding the point $s = 1$, where it has a simple pole with residue 1. Employing definition (4.37) we can rewrite (4.35) in the form

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\zeta\left(\frac{\Box}{2m^2}+h\right)\phi + \sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right]. \quad (4.38)$$

Here $\zeta\left(\frac{\Box}{2m^2}+h\right)$ acts as a pseudodifferential operator

$$\zeta\left(\frac{\Box}{2m^2}+h\right)\phi(x) = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk, \quad (4.39)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

We consider Lagrangian (4.38) with analytic continuations of the zeta function and the power series $\sum\frac{n^{-h}}{n+1}\phi^{n+1}$, i.e.

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\zeta\left(\frac{\Box}{2m^2}+h\right)\phi + AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right], \quad (4.40)$$

where AC denotes analytic continuation.

Potential of the above zeta scalar field (4.40) is equal to $-L_h$ at $\Box = 0$, i.e.

$$V_h(\phi) = \frac{m^D}{g^2}\left(\frac{\phi^2}{2}\zeta(h) - AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right), \quad (4.41)$$

where $h \neq 1$ since $\zeta(1) = \infty$. The term with ζ-function vanishes at $h = -2, -4, -6, \ldots$. The equation of motion in differential and integral form is

$$\zeta\left(\frac{\Box}{2m^2}+h\right)\phi = AC\sum_{n=1}^{+\infty} n^{-h}\varphi^n, \quad (4.42)$$

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk = AC\sum_{n=1}^{+\infty} n^{-h}\phi^n, \quad (4.43)$$

respectively.
Now, we consider five values of h, which seem to be the most interesting, regarding the Lagrangian (4.40): $h=0$, $h=\pm 1$, and $h=\pm 2$. For $h=-2$, the corresponding equation of motion now read:

$$\zeta\left(\frac{\Box}{2m^2}-2\right)\phi=\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-2\right)\tilde{\phi}(k)dk=\frac{\phi(\phi+1)}{(1-\phi)^3}. \quad (4.44)$$

This equation has two trivial solutions: $\phi(x)=0$ and $\phi(x)=-1$. Solution $\phi(x)=-1$ can be also shown taking $\tilde{\phi}(k)=-\delta(k)(2\pi)^D$ and $\zeta(-2)=0$ in (4.44).
For $h=-1$, the corresponding equation of motion is:

$$\zeta\left(\frac{\Box}{2m^2}-1\right)\phi=\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-1\right)\tilde{\phi}(k)dk=\frac{\phi}{(1-\phi)^2}. \quad (4.45)$$

where $\zeta(-1)=-\frac{1}{12}$.
The equation of motion (4.45) has a constant trivial solution only for $\phi(x)=0$.
For $h=0$, the equation of motion is

$$\zeta\left(\frac{\Box}{2m^2}\right)\phi=\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}. \quad (4.46)$$

It has two solutions: $\phi=0$ and $\phi=3$. The solution $\phi=3$ follows from the Taylor expansion of the Riemann zeta function operator

$$\zeta\left(\frac{\Box}{2m^2}\right)=\zeta(0)+\sum_{n\geq 1}\frac{\zeta^{(n)}(0)}{n!}\left(\frac{\Box}{2m^2}\right)^n, \quad (4.47)$$

as well as from $\tilde{\phi}(k)=(2\pi)^D 3\delta(k)$.
For $h=1$, the equation of motion is:

$$\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+1\right)\tilde{\phi}(k)dk=-\frac{1}{2}\ln(1-\phi)^2, \quad (4.48)$$

where $\zeta(1)=\infty$ gives $V_1(\phi)=\infty$.
In conclusion, for $h=2$, we have the following equation of motion:

$$\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+2\right)\tilde{\phi}(k)dk=-\int_0^{\phi}\frac{\ln(1-w)^2}{2w}dw. \quad (4.49)$$

Since holds equality

$$-\int_0^1\frac{\ln(1-w)}{w}dw=\sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta(2)$$

one has trivial solution $\phi = 1$ in (4.49).

Now, we want to analyze the following case: $C_n = \frac{n^2-1}{n^2}$. In this case, from the Lagrangian (4.33), we obtain:

$$L = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\left\{\zeta\left(\frac{\square}{2m^2}-1\right)+\zeta\left(\frac{\square}{2m^2}\right)\right\}\phi+\frac{\phi^2}{1-\phi}\right]. \quad (4.50)$$

The corresponding potential is:

$$V(\phi) = -\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2. \quad (4.51)$$

We note that 7 and 31 are prime natural numbers, i.e. $6n \pm 1$ with $n=1$ and 5, with 1 and 5 that are Fibonacci's numbers. Furthermore, the number 24 is related to the Ramanujan function that has 24 "modes" that correspond to the physical vibrations of a bosonic string. Thence, we obtain:

$$V(\phi) = -\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2 \Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (4.51b)$$

The equation of motion is:

$$\left[\zeta\left(\frac{\square}{2m^2}-1\right)+\zeta\left(\frac{\square}{2m^2}\right)\right]\phi = \frac{\phi\left[(\phi-1)^2+1\right]}{(\phi-1)^2}. \quad (4.52)$$

Its weak field approximation is:

$$\left[\zeta\left(\frac{\square}{2m^2}-1\right)+\zeta\left(\frac{\square}{2m^2}\right)-2\right]\phi = 0, \quad (4.53)$$

which implies condition on the mass spectrum

$$\zeta\left(\frac{M^2}{2m^2}-1\right)+\zeta\left(\frac{M^2}{2m^2}\right) = 2. \quad (4.54)$$

From (4.54) it follows one solution for $M^2 > 0$ at $M^2 \approx 2.79m^2$ and many tachyon solutions when $M^2 < -38m^2$.

We note that the number 2.79 is connected with $\phi = \frac{\sqrt{5}-1}{2}$ and $\Phi = \frac{\sqrt{5}+1}{2}$, i.e. the "aurea" section and the "aurea" ratio. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^2+\frac{1}{2^2}\left(\frac{\sqrt{5}-1}{2}\right)=2{,}772542\cong 2{,}78.$$

Furthermore, we have also that:

$$(\Phi)^{14/7}+(\Phi)^{-25/7}=2{,}618033989+0{,}179314566=2{,}79734$$

With regard the extension by ordinary Lagrangian, we have the Lagrangian, potential, equation of motion and mass spectrum condition that, when $C_n=\frac{n^2-1}{n^2}$, are:

$$L=\frac{m^D}{g^2}\left[\frac{\phi}{2}\left\{\frac{\Box}{m^2}-\zeta\left(\frac{\Box}{2m^2}-1\right)-\zeta\left(\frac{\Box}{2m^2}\right)-1\right\}\phi+\frac{\phi^2}{2}\ln\phi^2+\frac{\phi^2}{1-\phi}\right],\quad (4.55)$$

$$V(\phi)=\frac{m^D}{g^2}\frac{\phi^2}{2}\left[\zeta(-1)+\zeta(0)+1-\ln\phi^2-\frac{1}{1-\phi}\right],\quad (4.56)$$

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)-\frac{\Box}{m^2}+1\right]\phi=\phi\ln\phi^2+\phi+\frac{2\phi-\phi^2}{(1-\phi)^2},\quad (4.57)$$

$$\zeta\left(\frac{M^2}{2m^2}-1\right)+\zeta\left(\frac{M^2}{2m^2}\right)=\frac{M^2}{m^2}.\quad (4.58)$$

In addition to many tachyon solutions, equation (4.58) has two solutions with positive mass: $M^2\approx 2.67m^2$ and $M^2\approx 4.66m^2$.
We note also here, that the numbers 2.67 and 4.66 are related to the “aureo” numbers. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^2+\frac{1}{2\cdot 5}\left(\frac{\sqrt{5}-1}{2}\right)\cong 2.6798,$$

$$\left(\frac{\sqrt{5}+1}{2}\right)^2+\left(\frac{\sqrt{5}+1}{2}\right)+\frac{1}{2^2}\left(\frac{\sqrt{5}+1}{2}\right)\cong 4.64057.$$

Furthermore, we have also that:

$$(\Phi)^{14/7}+(\Phi)^{-41/7}=2{,}618033989+0{,}059693843=2{,}6777278;$$
$$(\Phi)^{22/7}+(\Phi)^{-30/7}=4{,}537517342+0{,}1271565635=4{,}6646738.$$

Now, we describe the case of $C_n=\mu(n)\frac{n-1}{n^2}$. Here $\mu(n)$ is the Mobius function, which is defined for all positive integers and has values 1, 0, – 1 depending on factorization of n into prime numbers p. It is defined as follows:

$$\mu(n)=\begin{cases}0, & n=p^2m\\ (-1)^k, & n=p_1p_2...p_k,\ p_i\neq p_j\\ 1, & n=1,(k=0).\end{cases}\quad (4.59)$$

The corresponding Lagrangian is

$$L_\mu = C_0 \mathcal{L}_0 + \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^{\frac{\Box}{2m^2}}} \phi + \sum_{n=1}^{+\infty} \frac{\mu(n)}{n+1} \phi^{n+1} \right] \quad (4.60)$$

Recall that the inverse Riemann zeta function can be defined by

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}, \quad s = \sigma + it, \quad \sigma > 1. \quad (4.61)$$

Now (4.60) can be rewritten as

$$L_\mu = C_0 \mathcal{L}_0 + \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)} \phi + \int_0^\infty \mathcal{M}(\phi) d\phi \right], \quad (4.62)$$

where $\mathcal{M}(\phi) = \sum_{n=1}^{+\infty} \mu(n) \phi^n = \phi - \phi^2 - \phi^3 - \phi^5 + \phi^6 - \phi^7 + \phi^{10} - \phi^{11} - \ldots$ The corresponding potential, equation of motion and mass spectrum formula, respectively, are:

$$V_\mu(\phi) = -L_\mu(\Box = 0) = \frac{m^D}{g^2}\left[\frac{C_0}{2}\phi^2 \left(1 - \ln \phi^2\right) - \phi^2 - \int_0^\phi \mathcal{M}(\phi) d\phi \right], \quad (4.63)$$

$$\frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)} \phi - \mathcal{M}(\phi) - C_0 \frac{\Box}{m^2} \phi - 2C_0 \phi \ln \phi = 0, \quad (4.64)$$

$$\frac{1}{\zeta\left(\frac{M^2}{2m^2}\right)} - C_0 \frac{M^2}{m^2} + 2C_0 - 1 = 0, \quad |\phi| << 1, \quad (4.65)$$

where usual relativistic kinematic relation $k^2 = -k_0^2 + \vec{k}^2 = -M^2$ is used.
Now, we take the pure numbers concerning the eqs. (4.54) and (4.58). They are: 2.79, 2.67 and 4.66. We note that all the numbers are related with $\Phi = \frac{\sqrt{5}+1}{2}$, thence with the aurea ratio, by the following expressions:

$$2{,}79 \cong (\Phi)^{15/7}; \qquad 2{,}67 \cong (\Phi)^{13/7} + (\Phi)^{-21/7}; \qquad 4{,}66 \cong (\Phi)^{22/7} + (\Phi)^{-30/7}. \quad (4.66)$$

5. Mathematical connections.

With regard the **Section 1** and **3**, we have the following possible interesting mathematical connections between the eqs. (1.1), (1.8), (1.10) and (3.3) and the equations (4.28), (4.30), (4.31), (4.43), (4.45), (4.46), (4.48), (4.49) and (4.62) of the **Section 4**. Indeed, with the eqs. (4.28), (4.30) and (4.31), for example, we obtain that:

$$\varsigma(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\sum_{n=1}^{\infty}\left(s\int_n^{\infty}\frac{dx}{x^{s+1}}\right)=s\sum_{n=1}^{\infty}\int_n^{\infty}\frac{dx}{x^{s+1}}$$

$$=s\int_1^{\infty}\left(\sum_{n\le x}1\right)\frac{dx}{x^{s+1}}=s\int_1^{\infty}\frac{[x]}{x^{s+1}}dx=s\int_1^{\infty}\frac{x-\{x\}}{x^{s+1}}dx$$

$$=\frac{s}{s-1}-s\int_1^{\infty}\frac{\{x\}}{x^{s+1}}dx,\ \sigma>1$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\sum_{n\ge1}\theta^{\frac{n(n-1)}{2}}\phi^n\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk=\sum_{n\ge1}\left[\theta^{n^2}+\frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right].\quad(5.1)$$

$$\psi(x)=\lim_{T\to\infty}\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)x^s\frac{ds}{s},\ x>0\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\sum_{n\ge1}\theta^{\frac{n(n-1)}{2}}\phi^n\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk=\sum_{n\ge1}\left[\theta^{n^2}+\frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right].\quad(5.2)$$

$$\psi_1(x)=\int_0^x\psi(t)dt=\frac{x^2}{2}-\sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)}-\sum_{n}\frac{x^{-2n+1}}{2n(2n-1)}-x\log(2\pi)+const\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\sum_{n\ge1}\theta^{\frac{n(n-1)}{2}}\phi^n\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk=\sum_{n\ge1}\left[\theta^{n^2}+\frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right].\quad(5.3)$$

$$| \frac{P(x,d)}{x} - \frac{\pi(x,a,d)}{x \ln x} | < kC3(x), k = 1, C3(x) = \frac{1}{\ln x}$$

or

$$| \frac{P(x,d)}{x} - \frac{1}{\varphi(d)} \int_2^x \frac{1}{t(\ln t)^2} dt | = O(x^{\frac{1}{2}+\varepsilon})$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2}\right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int e^{ixk} \zeta\left(-\frac{k^2}{2}\right) \tilde{\phi}(k) dk = \sum_{n \geq 1} \theta^{\frac{n(n-1)}{2}} \phi^n \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int e^{ixk} \zeta\left(-\frac{k^2}{4}\right) \tilde{\theta}(k) dk = \sum_{n \geq 1} \left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)} \theta^{\frac{n(n-1)}{2}-1} \left(\phi^{n+1} - 1\right) \right]. \quad (5.4)$$

In conclusion, with eqs. (4.46) and (4.62), for example, we have the following mathematical connections:

$$\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \left(s \int_n^{\infty} \frac{dx}{x^{s+1}} \right) = s \sum_{n=1}^{\infty} \int_n^{\infty} \frac{dx}{x^{s+1}}$$

$$= s \int_1^{\infty} \left(\sum_{n \leq x} 1 \right) \frac{dx}{x^{s+1}} = s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx = s \int_1^{\infty} \frac{x - \{x\}}{x^{s+1}} dx$$

$$= \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx, \ \sigma > 1$$

$$\Rightarrow \zeta\left(\frac{\Box}{2m^2}\right) \phi = \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta\left(-\frac{k^2}{2m^2}\right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi}. \quad (5.5)$$

$$\psi(x) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) x^s \frac{ds}{s}, \ x > 0 \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{\Box}{2m^2}\right) \phi = \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta\left(-\frac{k^2}{2m^2}\right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi}. \quad (5.6)$$

$$\psi_1(x) = \int_0^x \psi(t) dt = \frac{x^2}{2} - \sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)} - \sum_{n} \frac{x^{-2n+1}}{2n(2n-1)} - x \log(2\pi) + const \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{\Box}{2m^2}\right) \phi = \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta\left(-\frac{k^2}{2m^2}\right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi}. \quad (5.7)$$

$$|\frac{P(x,d)}{x}-\frac{\pi(x,a,d)}{x\ln x}|<kC3(x), k=1, C3(x)=\frac{1}{\ln x}$$

or

$$|\frac{P(x,d)}{x}-\frac{1}{\varphi(d)}\int_2^x\frac{1}{t(\ln t)^2}dt|=O(x^{\frac{1}{2}+\varepsilon})$$

$$\Rightarrow \zeta\left(\frac{\Box}{2m^2}\right)\phi=\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}. \quad (5.8)$$

$$\varsigma(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\sum_{n=1}^{\infty}\left(s\int_n^{\infty}\frac{dx}{x^{s+1}}\right)=s\sum_{n=1}^{\infty}\int_n^{\infty}\frac{dx}{x^{s+1}}$$

$$=s\int_1^{\infty}\left(\sum_{n\le x}1\right)\frac{dx}{x^{s+1}}=s\int_1^{\infty}\frac{[x]}{x^{s+1}}dx=s\int_1^{\infty}\frac{x-\{x\}}{x^{s+1}}dx$$

$$=\frac{s}{s-1}-s\int_1^{\infty}\frac{\{x\}}{x^{s+1}}dx,\ \sigma>1$$

$$\Rightarrow C_0\mathcal{L}_0+\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi+\int_0^{\infty}\mathcal{M}(\phi)d\phi\right]. \quad (5.9)$$

$$\psi(x)=\lim_{T\to\infty}\frac{1}{2\pi i}\int_{a-iT}^{a+iT}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)x^s\frac{ds}{s},\ x>0\Rightarrow$$

$$\Rightarrow C_0\mathcal{L}_0+\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi+\int_0^{\infty}\mathcal{M}(\phi)d\phi\right]. \quad (5.10)$$

$$\psi_1(x)=\int_0^x\psi(t)dt=\frac{x^2}{2}-\sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)}-\sum_{n}\frac{x^{-2n+1}}{2n(2n-1)}-x\log(2\pi)+const\Rightarrow$$

$$\Rightarrow C_0\mathcal{L}_0+\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi+\int_0^{\infty}\mathcal{M}(\phi)d\phi\right]. \quad (5.11)$$

$$| \frac{P(x,d)}{x} - \frac{\pi(x,a,d)}{x \ln x} | < kC3(x), k = 1, C3(x) = \frac{1}{\ln x}$$

or

$$| \frac{P(x,d)}{x} - \frac{1}{\varphi(d)} \int_2^x \frac{1}{t(\ln t)^2} dt | = O(x^{\frac{1}{2}+\varepsilon})$$

$$\Rightarrow C_0 \mathcal{L}_0 + \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)} \phi + \int_0^\infty \mathcal{M}(\phi) d\phi \right]. \quad (5.12)$$

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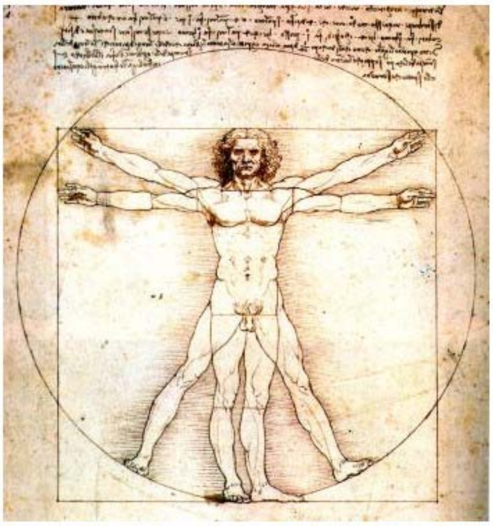

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