# ON THE POINTWISE PERIODICITY OF MULTIPLICATIVE AND ADDITIVE FUNCTIONS

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Abstract. We study the problem of estimating the number of points of coincidences of an idealized gap on the set of integers under a given multiplicative function $g : \mathbb{N} \longrightarrow \mathbb{C}$ respectively additive function $f : \mathbb{N} \longrightarrow \mathbb{C}$. We obtain various lower bounds depending on the length of the period.

## 1. Introduction and problem statement

Let $f : \mathbb{N} \longrightarrow \mathbb{C}$, then we say $f$ is periodic on the set $[1, x] \subset \mathbb{N}$ with period $l$ if $f(n-l) = f(n) = f(n+l)$ for all $n \in [1, x]$. We say $f$ is pointwise left-periodic if for any $l > 0$ there exist some $n \in [1, x] \subset \mathbb{N}$ such that $f(n-l) = f(n)$. Similarly we say it is pointwise right-periodic if for any $l > 0$ there exist some $n \in [1, x] \subset \mathbb{N}$ such that $f(n) = f(n+l)$. We call $l := l(n) > 0$ the pointwise period. We say it is fully-pointwise periodic with period $l := l(n) > 0$ if $f(n-l) = f(n) = f(n+1)$. In this paper we study the problem of estimating the size of the quantity

$$\mathcal{G}(x,l)_f := \#\{n \leq x : f(n-l) = f(n) = f(n+l), \text{ for fixed } l > 0\}.$$

First we obtain a general theorem for problems of this flavour and narrow it down to specific examples by varying our arithmetic functions. We study the problem under the context of pointwise left-periodicity and pointwise right-periodicity. In particular, under a given multiplicative function $g$ or an additive function $f$, we study the size of the following sets

$$\mathcal{G}(x,l)_f^+ := \#\{n \leq x : f(n) = f(n+l), \text{ for fixed } l > 0\}$$

and

$$\mathcal{G}(x,l)_f^- := \#\{n \leq x : f(n-l) = f(n), \text{ for fixed } l > 0\}$$

respectively

$$\mathcal{G}(x,l)_g^+ := \#\{n \leq x : g(n) = g(n+l), \text{ for fixed } l > 0\}$$

and

$$\mathcal{G}(x,l)_g^- := \#\{n \leq x : g(n-l) = g(n), \text{ for fixed } l > 0\}.$$

In particular we obtain the following results

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**Theorem 1.1.** *Let $g : \mathbb{N} \longrightarrow \mathbb{C}$ be a completely multiplicative function with $g(t) \neq 0$ for $t \in \mathbb{N}$ such that*

$$\frac{g(n+1)}{g(n)} \ll C \log\log n$$

*for $C > 0$. Then we have*

$$\mathcal{G}(x,l)_g^+ \gg \frac{1}{C}\frac{\log(\frac{x}{l})}{\log\log(\frac{x}{l})}.$$

**Theorem 1.2.** *Let $f : \mathbb{N} \longrightarrow \mathbb{C}$ be a completely additive function with $f(n) \neq 0$ for $n \geq 2$ such that*

$$\frac{f(n+1)}{f(n)} \ll C \log n.$$

*Then we have the lower bound*

$$\mathcal{G}(x,l)_f^+ \gg \frac{1}{C}\log\log(\frac{x}{l}).$$

## 2. Notations

Through out this paper we consider arithmetic functions $f, g : \mathbb{N} \longrightarrow \mathbb{C}$, where by convention $g$ denotes multiplicative (resp. completely multiplicative) functions and $f$ denotes additive (resp. completely additive) function. We keep the usual standard notation: $f \ll h \Leftrightarrow |f(n)| \leq Ch(n)$ for all $n \geq n_0$ for some $C > 0$, and similarly $f \gg h \Leftrightarrow |f(n)| \geq Kh(n)$ for all $n \geq n_0$ for some $n_0 > 0$ and some $K > 0$. We denote the quantity $\mathcal{G}(x,l)_f^+ := \#\{n \leq x : g(n) = g(n+l),\ \text{for fixed } l > 0\}$ and similarly $\mathcal{G}(x,l)_f^- := \#\{n \leq x : g(n-l) = g(n),\ \text{for fixed } l > 0\}$.

## 3. Preliminary results

In this section we review the theory of extremal orders for arithmetic functions. We revisit, as is essential in our studies, the notion of the maximal and the minimal orders of various arithmetic functions. We then leverage this concepts in the sequel to establish particular examples to the main results of this paper.

**Theorem 3.1.** *Let $\varphi(n) := \sum\limits_{\substack{m \leq n \\ (m,n)=1}} 1$, then we have*

$$\varphi(n) < n$$

*and*

$$\varphi(n) \gg e^{-\gamma}\frac{n}{\log\log n}$$

*where $\gamma$ is the Euler-Macheroni constant.*

*Proof.* For a proof see [1]. □

**Theorem 3.2.** *Let $\tau(n) := \sum\limits_{d|n} 1$. Then for $n \geq 1$ we have*

$$\tau(n) \ll n^{\frac{\log 2}{\log\log n}}$$

*and $\tau(n) \geq 2$.*

*Proof.* For a proof see [1]. □

**Theorem 3.3.** *Let* $\Omega(n) = \sum\limits_{p||n} 1$*. Then for* $n \geq 1$ *we have* $\Omega(n) \geq 1$ *and*

$$\Omega(n) \ll \frac{\log n}{\log 2}.$$

*Proof.* For a proof see [1]. □

**Theorem 3.4.** *Let* $\sigma(n) := \sum\limits_{d|n} d$*. Then we have* $\sigma(n) \geq n$ *and*

$$\sigma(n) \ll e^{\gamma} n \log\log n.$$

*Proof.* For a proof see [1]. □

## 4. **Lower bound**

In this section we study the underlying problem in the setting of functions $f : \mathbb{N} \longrightarrow \mathbb{C}$ with the property $f(mn) = f(n) + f(m)$ and those of the form $g : \mathbb{N} \longrightarrow \mathbb{C}$ with the property $g(mn) = g(m)g(n)$. We estimate from below the size of each of these sets under a given arithmetic function.

**Theorem 4.1.** *Let* $g : \mathbb{N} \longrightarrow \mathbb{C}$ *be a completely multiplicative function with* $g(t) \neq 0$ *for* $t \in \mathbb{N}$ *such that*

$$\frac{g(n+1)}{g(n)} \ll C \log\log n$$

*for* $C > 0$*. Then we have*

$$\mathcal{G}(x,l)_g^+ \gg \frac{1}{C} \frac{\log(\frac{x}{l})}{\log\log(\frac{x}{l})}.$$

*Proof.* Clearly

$$\begin{aligned}
\mathcal{G}(x,l)_g^+ &= \#\{n \leq x : g(n) = g(n+l), \text{ for fixed } l > 0\} \\
&\geq \#\left\{n \leq x : g(n) = g(n+l), \ l|n, \text{ for fixed } l > 0\right\} \\
&= \#\left\{m \leq \frac{x}{l} : g(m) = g(m+1), \ l|n, \text{ for fixed } l > 0\right\} \\
&= \sum_{\substack{m \leq \frac{x}{l} \\ g(m)=g(m+1)}} 1 \\
&= \sum_{m \leq \frac{x}{l}} \frac{g(m)}{g(m+1)} \\
&\gg \frac{1}{C} \sum_{m \leq \frac{x}{l}} \frac{1}{\log\log m}
\end{aligned}$$

and the lower bound follows immediately. □

*Remark* 4.2. We particularize the result in Theorem 4.1 by varying our arithmetic functions in the following sequel.

**Corollary 4.1.** Let $\varphi(n) = \sum\limits_{\substack{m \leq n \\ (m,n)=1}} 1$. Then we have the lower bound

$$\mathcal{G}(x,l)_\varphi^+ \gg e^{-\gamma} \frac{\log(\frac{x}{l})}{\log\log(\frac{x}{l})}.$$

*Proof.* It follows from Theorem 3.1 the ratio

$$\frac{\varphi(n+1)}{\varphi(n)} \ll e^{\gamma}\log\log n$$

is satisfied and the lower bound follows immediately by virtue of Theorem 4.1. □

**Corollary 4.2.** Let $\sigma(n) := \sum_{d|n} d$. Then we have the lower bound

$$\mathcal{G}(x,l)^{+}_{\sigma} \gg e^{-\gamma}\frac{\log(\frac{x}{l})}{\log\log(\frac{x}{l})}.$$

*Proof.* It follows from Theorem 3.4

$$\frac{\sigma(n+1)}{\sigma(n)} \ll e^{\gamma}\log\log n$$

and the lower bound follows by appealing to Theorem 4.1. □

**Theorem 4.3.** *Let $g : \mathbb{N} \longrightarrow \mathbb{C}$ be a multiplicative function with $g(t) \neq 0$ for all $t \in \mathbb{N}$ and suppose*

$$\frac{g(n+1)}{g(n)} \ll 1$$

*with $g(st) \leq g(s)g(t)$. Then*

$$\mathcal{G}(x,l)^{+}_{g} \gg \frac{x}{l}.$$

*Proof.* Clearly

$$\begin{aligned}\mathcal{G}(x,l)^{+}_{g} &= \#\{n \leq x : g(n) = g(n+l),\ \text{for fixed } l > 0\}\\ &\geq \#\left\{n \leq x : g(n) = g(n+l),\ l|n,\ \text{for fixed } l > 0\right\}\\ &\geq \#\left\{m \leq \frac{x}{l} : g(ml) = g(l)g(m+1),\ (m+1,l) = 1,\ \text{for fixed } l > 0\right\}\\ &= \sum_{\substack{m \leq \frac{x}{l}\\ g(ml) = g(m+1)g(l)}} 1\\ &= \sum_{m \leq \frac{x}{l}} \frac{g(m+1)g(l)}{g(ml)}\\ &\gg \sum_{m \leq \frac{x}{l}} \frac{g(l)g(m+1)}{g(m)g(l)}\\ &= \sum_{m \leq \frac{x}{l}} \frac{g(m+1)}{g(m)}\\ &\gg \sum_{m \leq \frac{x}{l}} 1\end{aligned}$$

and the lower bound follows immediately. □

Theorem 4.3 can be useful in practice. It tells us that multiplicative functions obeying the underlying conditions with the correlation

$$\sum_{n \leq x} g(n)g(n+l)$$

for a fixed $l > 0$ can be well approximated by the partial sum of the corresponding square function

$$\sum_{n\leq x} g(n)^2.$$

**Corollary 4.3.** Let $\tau(n) := \sum\limits_{d|n} 1$. Then we have the lower bound

$$\mathcal{G}(x,l)_{\tau}^{+} \gg \frac{x}{l}.$$

*Proof.* By appealing to Theorem 3.2, we obtain

$$\begin{aligned}\frac{\tau(n+1)}{\tau(n)} &\ll \frac{1}{2} n^{\frac{\log 2}{\log\log n}}\\ &\ll 1\end{aligned}$$

and the result follows by applying Theorem 4.3. □

*Remark* 4.4. Next, we extend our result to multiplicative functions whose consecutive ratio grow by a poly-logarithmic power saving of a logarithm. We make these statement more precise in the following result. It needs to be said these result also holds if we replace our multiplicative function with an additive function.

**Theorem 4.5.** *Let $g : \mathbb{N} \longrightarrow \mathbb{C}$ be a completely multiplicative function with $g(n) \neq 0$ for $n \geq 2$ such that*

$$\frac{g(n+1)}{g(n)} \ll \frac{\log n}{(\log\log n)^c}$$

*for some $c > 0$. Then we have*

$$\mathcal{G}(x,l)_{g}^{+} \gg \frac{1}{c+1}\left(\log\log\left(\frac{x}{l}\right)\right)^{c+1}.$$

*Proof.* Clearly we can write

$$\begin{aligned}\mathcal{G}(x,l)_{g}^{+} &= \#\{n \leq x : g(n) = g(n+l),\ \text{for fixed } l > 0\}\\ &\geq \#\left\{n \leq x : g(n) = g(n+l),\ l|n,\ \text{for fixed } l > 0\right\}\\ &= \#\left\{m \leq \frac{x}{l} : g(m) = g(m+1),\ \text{for fixed } l > 0\right\}\\ &= \sum_{\substack{m\leq \frac{x}{l}\\ g(m)=g(m+1)}} 1\\ &= \sum_{m\leq \frac{x}{l}} \frac{g(m)}{g(m+1)}\\ &\gg \sum_{m\leq \frac{x}{l}} \frac{(\log\log n)^c}{\log n}\end{aligned}$$

and the lower bound follows by applying partial summation. □

*Remark* 4.6. Keeping in mind the different possible growth rate of the ratios of consecutive values of an additive function, we examine the situation where the ratio grow logarithmically is size in the following result.

**Theorem 4.7.** *Let $f:\mathbb{N}\longrightarrow\mathbb{C}$ be a completely additive function with $f(n)\neq 0$ for $n\geq 2$ such that*

$$\frac{f(n+1)}{f(n)}\ll C\log n.$$

*Then we have the lower bound*

$$\mathcal{G}(x,l)_f^+\gg \frac{1}{C}\log\log(\frac{x}{l}).$$

*Proof.* We observe that we can write

$$\begin{aligned}\mathcal{G}(x,l)_f^+ &= \#\{n\leq x: f(n)=f(n+l),\ \text{for fixed } l>0\}\\ &\geq \#\left\{n\leq x: f(n)=f(n+l),\ l|n,\ \text{for fixed } l>0\right\}\\ &= \#\left\{m\leq \frac{x}{l}: f(m)=f(m+1),\ \text{for fixed } l>0\right\}\\ &= \sum_{\substack{m\leq \frac{x}{l}\\ f(m)=f(m+1)}} 1\\ &= \sum_{m\leq \frac{x}{l}}\frac{f(m)}{f(m+1)}\\ &\gg \frac{1}{C}\sum_{m\leq \frac{x}{l}}\frac{1}{\log m}\end{aligned}$$

and the lower bound follows by applying partial summation. □

*Remark* 4.8. We provide a particular instance where these result might be useful, by considering the number of prime counting with multiplicity function $\Omega(n)$.

**Corollary 4.4.** Let $\Omega(n):=\sum_{p||n}1$. Then we have the lower bound

$$\mathcal{G}(x,l)_\Omega^+\gg(\log 2)\log\log(\frac{x}{l}).$$

*Proof.* By appealing to Theorem 3.3, we obtain the upper bound

$$\frac{\Omega(n+1)}{\Omega(n)}\ll\frac{\log n}{\log 2}$$

and the lower bound follows by applying Theorem 4.5. □

[1].

## References

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