AN IMPROVED LOWER BOUND OF THE HEILBRON TRIANGLE PROBLEM

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ABSTRACT. Using the method of compression we improve on the current lower bound of the Heilbron triangle problem. In particular, by letting $\Delta(s)$ denotes the minimal area of the triangle induced by s points in the plane $\mathbb{R}^2$. Then we have the lower bound

$$\Delta(s) \gg \frac{\log^2 s}{s^2}.$$

1. Introduction

The Heilbron triangle conjecture had long remained open until in 1982 when it was proven to be false by Szemeredi and Pitnz [1]. In particular they constructed a set of points in the plane whose minimal area of their induced triangles, denoted $\Delta(s)$ satisfies the lower bound (see [1])

$$\Delta(s) \gg \frac{\log s}{s^2}.$$

Indeed Erdős had shown earlier to the effect of the Heilbron conjecture that the upper bound cannot be reduced any further. What remains open now is the asymptotic growth rate of the minimal area of the triangle determined by a finite set of points in a plane $\mathbb{R}^2$. To that effect the quest for improved lower and upper bounds are of worthy pursuit. The first non-trivial upper bound was obtained by Roth [3] given

$$\Delta(s) \ll \frac{1}{s\sqrt{\log\log s}}$$

and eventually improved to (see [2])

$$\Delta(s) \ll \frac{e^{c\sqrt{\log s}}}{s^{\frac{8}{7}}}.$$

In this paper we obtain an improved lower bound of the minimal area of the triangle

induced by s points in the plane in the following result, by considering a particular type of configuration

Theorem 1.1. *Let $\Delta(s)$ denotes the minimal area of the triangle formed by s points in the plane. Then we have the lower bound*

$$\Delta(s) \gg \frac{\log^2 s}{s^2}.$$

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2. Preliminaries and background

Definition 2.1. By the compression of scale $m > 0$ on $\mathbb{R}^n$ we mean the map $\mathbb{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right)$$

for $n \geq 2$ and with $x_i \neq 0$ for all $i = 1, \ldots, n$.

Remark 2.2. The notion of compression is in some way the process of re scaling points in $\mathbb{R}^n$ for $n \geq 2$. Thus it is important to notice that a compression pushes points very close to the origin away from the origin by certain scale and similarly draws points away from the origin close to the origin.

Proposition 2.1. *A compression of scale $m > 0$ with $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map.*

Proof. Suppose $\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \mathbb{V}_m[(y_1, y_2, \ldots, y_n)]$, then it follows that

$$\left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right) = \left(\frac{m}{y_1}, \frac{m}{y_2}, \ldots, \frac{m}{y_n}\right).$$

It follows that $x_i = y_i$ for each $i = 1, 2, \ldots, n$. Surjectivity follows by definition of the map. Thus the map is bijective. □

2.1. The mass of compression.

In this section we recall the notion of the mass of compression on points in space and study the associated statistics.

Definition 2.3. By the mass of a compression of scale $m > 0$ we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

Lemma 2.4. *The estimate remain valid*

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$

where $\gamma = 0.5772\cdots$.

Remark 2.5. Next we prove upper and lower bounding the mass of the compression of scale $m \geq 1$.

Proposition 2.2. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$, then the estimates holds*

$$m \log\left(1 - \frac{n-1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \ll m \log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)}\right)$$

for $n \geq 2$.

Proof. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \geq 1$. Then it follows that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = m \sum_{j=1}^{n} \frac{1}{x_j}$$

$$\leq m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) + k}$$

and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound

$$\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) = m\sum_{j=1}^{n}\frac{1}{x_j} \geq m\sum_{k=0}^{n-1}\frac{1}{\sup(x_j)-k}.$$

□

Definition 2.6. Let $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ with $x_i\neq 0$ for all $i=1,2\ldots,n$. Then by the gap of compression of scale m $\mathbb{V}_m$, denoted $\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]$, we mean the expression

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)] = \left|\left|\left(x_1-\frac{m}{x_1},x_2-\frac{m}{x_2},\ldots,x_n-\frac{m}{x_n}\right)\right|\right|$$

Definition 2.7. Let $(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ with $x_i\neq x_j$ for all $1\leq i<j\leq n$. Then by the ball induced by $(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ under compression of scale m, denoted $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1,x_2,\ldots,x_n)]$ we mean the inequality

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},x_2+\frac{m}{x_2},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|\leq\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)].$$

A point $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1,x_2,\ldots,x_n)]$ if it satisfies the inequality. We call the ball the circle induced by points under compression if we take the dimension of the underlying space to be $n=2$.

Remark 2.8. The circle induced by points under compression is the ball induced on points when we take $n=2$.

Proposition 2.3. *Let $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ for $n\geq 2$ with $x_j\neq 0$ for $j=1,\ldots,n$, then we have*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2=\mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right]+m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)]-2mn.$$

In particular, we have the estimate

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2=\mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right]-2mn+O\left(m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)]\right)$$

for $\vec{x}\in\mathbb{N}^n$.

Lemma 2.9 (Compression estimate)**.** *Let $(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ for $n\geq 2$, then we have*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2\ll n\sup(x_j^2)+m^2\log\left(1+\frac{n-1}{\mathrm{Inf}(x_j)^2}\right)-2mn$$

and

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2\gg n\mathrm{Inf}(x_j^2)+m^2\log\left(1-\frac{n-1}{\sup(x_j^2)}\right)^{-1}-2mn.$$

Theorem 2.10. *Let* $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{N}^n$ *with* $z_i \neq z_j$ *for all* $1 \leq i < j \leq n$. *Then* $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ *if and only if*

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{y}].$$

Proof. Let $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ for $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{N}^n$ with $z_i \neq z_j$ for all $1 \leq i < j \leq n$, then it follows that $||\vec{y}|| > ||\vec{z}||$. Suppose on the contrary that

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] > \mathcal{G} \circ \mathbb{V}_m[\vec{y}],$$

then it follows that $||\vec{y}|| < ||\vec{z}||$, which is absurd. Conversely, suppose

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{y}]$$

then it follows from Proposition 2.3 that $||\vec{z}|| \leq ||\vec{y}||$ and $\sup(z_j) \leq \sup(y_j)$ by Lemma 2.9. It follows that

$$\begin{aligned}\left|\left|\vec{z} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| &\leq \left|\left|\vec{y} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| \\ &\leq \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{y}].\end{aligned}$$

This certainly implies $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ and the proof of the theorem is complete. □

Theorem 2.11. *Let* $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ *with* $x_i \neq x_j$ *for all* $1 \leq i < j \leq n$. *If* $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ *then*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \subseteq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Proof. First let $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ and suppose for the sake of contradiction that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \not\subseteq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there must exist some $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ such that $\vec{z} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. It follows from Theorem 2.10 that

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] > \mathcal{G} \circ \mathbb{V}_m[\vec{x}].$$

It follows that

$$\begin{aligned}\mathcal{G} \circ \mathbb{V}_m[\vec{y}] &\geq \mathcal{G} \circ \mathbb{V}_m[\vec{z}] \\ &> \mathcal{G} \circ \mathbb{V}_m[\vec{x}] \\ &\geq \mathcal{G} \circ \mathbb{V}_m[\vec{y}]\end{aligned}$$

which is absurd, thereby ending the proof. □

Remark 2.12. Theorem 2.11 tells us that points confined in certain balls induced under compression should by necessity have their induced ball under compression covered by these balls in which they are contained.

2.2. **Admissible points of balls induced under compression.** We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

Definition 2.13. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{N}^n$ with $y_i\neq y_j$ for all $1\leq i<j\leq n$. Then $\vec{y}$ is said to be an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ if

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

Remark 2.14. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball. Next we show that all balls can in principle be generated by their admissible points.

Theorem 2.15. *The point $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is admissible if and only if*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$.

Proof. First let $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ be admissible and suppose on the contrary that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\neq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there exist some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ such that

$$\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}].$$

Applying Theorem 2.10, we obtain the inequality

$$\mathcal{G}\circ\mathbb{V}_m[\vec{y}]<\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

It follows from Proposition 2.3 that $||\vec{x}||<||\vec{y}||$ or $||\vec{y}||<||\vec{x}||$. By joining this points to the origin by a straight line, this contradicts the fact that the point $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is an admissible point.. This contradicts the fact that the point $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is an admissible point. Now we notice that $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ certainly implies $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Conversely we notice as well that $\vec{x}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$, which certainly implies $\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{y}]$ by Theorem 2.10. Thus the conclusion follows. Conversely, suppose

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Then it follows that the point $\vec{y}$ must satisfy the inequality

$$\begin{aligned}\left|\left|\vec{z}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right|&=\left|\left|\vec{z}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|\\&\leq\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]&=\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|\\&\leq\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\end{aligned}$$

and $\vec{y}$ is indeed admissible, thereby ending the proof. ∎

Next we obtain an equivalent notion of the area of the circle induced by points under compression in the plane $\mathbb{R}^2$ in the following result.

Proposition 2.4. *Let $\vec{x} \in \mathbb{N}^2 \subset \mathbb{R}^2$. Then the area of the circle induced by point $\vec{x}$ under compression of scale m, denote by $\mathbb{V}_m[\vec{x}]$ is given by*

$$\delta(\mathbb{V}_m[\vec{x}]) = \frac{\pi(\mathcal{G} \circ \mathbb{V}_m[\vec{x}])^2}{4}.$$

Proof. This follows from the mere definition of the area of a circle and noting that the radius r of the circle induced by the point $\vec{x} \in \mathbb{R}^2$ under compression is given by

$$r = \frac{\mathcal{G} \circ \mathbb{V}_m[\vec{x}]}{2}.$$

□

3. Lower bound

Theorem 3.1. *Let $\Delta(s)$ denotes the minimal area of the triangle formed by s points in the plane. Then we have the lower bound*

$$\Delta(s) \gg \frac{\log^2 s}{s^2}.$$

Proof. First let $m \in \mathbb{R}^+$ be fixed and let $s \geq 4$. Pick arbitrarily a point $\vec{x} \in \mathbb{N}^2 \subset \mathbb{R}^2$ and apply the compression of scale m, given by $\mathbb{V}_m[\vec{x}]$. Next construct the circle induced by the compression given by

$$\mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}]}[\vec{x}]$$

with radius $\frac{(\mathcal{G} \circ \mathbb{V}_m[\vec{x}])}{2}$. On this circle locate $(s-3)$ admissible points so that the chord joining each pair of adjacent $(s-1)$ admissible points including $\vec{x}$ and $\mathbb{V}_m[\vec{x}]$ are equidistant. Let us now join each of the $(s-1)$ admissible point considered to the center of the circle given by

$$\vec{y} := \frac{1}{2}\bigg(x_1 + \frac{m}{x_1}, x_2 + \frac{m}{x_2}\bigg).$$

Invoking proposition 2.4, the area of the circle induced under compression is given by

$$\delta(\mathbb{V}_m[\vec{x}]) = \frac{\pi(\mathcal{G} \circ \mathbb{V}_m[\vec{x}])^2}{4}.$$

We note that we can use the area of each segment to approximate the area of each of the tringles inscribed in the segment. It follows that the area of each segment formed must be the same and given by

$$\mathcal{A} := \frac{\pi\mathcal{G} \circ \mathbb{V}_m[\vec{x}])^2}{4 \times (s-1)}$$
$$\gg \frac{2\mathrm{Inf}(x_j^2) + m^2 \log\bigg(1 - \frac{1}{\sup(x_j^2)}\bigg)^{-1} - 4m}{4 \times s}.$$

The lower bound follows by taking

$$m := \frac{\log^2 s}{s} \quad \text{and} \quad \mathrm{Inf}(x_j) := \left\lceil K \frac{\log s}{\sqrt{s}} \right\rceil$$

for $K \geq 4$. □

[1].

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