# AN IMPROVED LOWER BOUND OF HEILBRONN'S TRIANGLE PROBLEM

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Abstract. Using the method of compression we improve on the current lower bound of Heilbronn's triangle problem. In particular, by letting $\Delta(s)$ denotes the minimal area of the triangle induced by $s$ points in a unit disc. Then we have the lower bound

$$\Delta(s) \gg \frac{\log s}{s\sqrt{s}}.$$

## 1. Introduction

Let $\mathcal{D}$ denotes any convex shape in the plane and $\Delta(S)$ denotes the minimal area of the triangle induced by a set of $s$ points in $\mathcal{D}$ so that $\Delta(s)$ denotes the supremum of all the $\Delta(S)$. Then Heilbronn conjectured what is now known as Heilbronn's triangle problem, which states

**Conjecture 1.1.** The minimal area of the triangle induced by $s$ points in $\mathcal{D}$ satisfies

$$\Delta(s) = O\left(\frac{1}{s^2}\right).$$

Indeed Erdős had shown earlier to the effect of Heilbronn's conjecture the lower bound

$$\Delta(s) \gg \frac{1}{s^2}.$$

This lower bound would have vindicated Heilbronn's conjectured upper bound as the sharpest if it had been proven to be true. Heilbronn's triangle problem had long remained open and it was indeed a breathrough in 1982 when the first chuck of it was solved by Komlos, Pintz and Szemeredi [1]. In particular, they constructed a set of points in $\mathcal{D}$ whose minimal area of their induced triangles, denoted $\Delta(s)$ satisfies the lower bound (see [1])

$$\Delta(s) \gg \frac{\log s}{s^2}.$$

What remains open now is the asymptotic growth rate of the minimal area of the triangle determined by a finite set of points in $\mathcal{D}$. To that effect the quest for improved lower and upper bounds are of worthy pursuit. The first non-trivial upper bound was obtained by Roth [4] given as

$$\Delta(s) \ll \frac{1}{s\sqrt{\log\log s}}.$$

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A refinement of a method in [3] eventually yields the best currently known upper bound (see [2])

$$\Delta(s) \ll \frac{e^{c\sqrt{\log s}}}{s^{\frac{8}{7}}}.$$

In this paper we obtain an improved lower bound of the minimal area of the triangle induced by $s$ points in a unit disc, by considering a particular type of configuration:

**Theorem 1.1.** *Let $\Delta(s)$ denotes the minimal area of the triangle formed by $s$ points in the unit disc. Then we have the lower bound*

$$\Delta(s) \gg \frac{\log s}{s\sqrt{s}}.$$

## 2. Preliminaries and background

**Definition 2.1.** By the compression of scale $m > 0$ ($m \in \mathbb{R}$) fixed on $\mathbb{R}^n$ we mean the map $\mathbb{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1, x_2, \dots, x_n)] = \left(\frac{m}{x_1}, \frac{m}{x_2}, \dots, \frac{m}{x_n}\right)$$

for $n \geq 2$ and with $x_i \neq x_j$ for $i \neq j$ and $x_i \neq 0$ for all $i = 1, \dots, n$.

*Remark* 2.2. The notion of compression is in some way the process of re scaling points in $\mathbb{R}^n$ for $n \geq 2$. Thus it is important to notice that a compression roughly speaking pushes points very close to the origin away from the origin by certain scale and similarly draws points away from the origin close to the origin.

**Proposition 2.1.** *A compression of scale $m > 0$ with $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map.*

*Proof.* Suppose $\mathbb{V}_m[(x_1, x_2, \dots, x_n)] = \mathbb{V}_m[(y_1, y_2, \dots, y_n)]$, then it follows that

$$\left(\frac{m}{x_1}, \frac{m}{x_2}, \dots, \frac{m}{x_n}\right) = \left(\frac{m}{y_1}, \frac{m}{y_2}, \dots, \frac{m}{y_n}\right).$$

It follows that $x_i = y_i$ for each $i = 1, 2, \dots, n$. Surjectivity follows by definition of the map. Thus the map is bijective. $\square$

### 2.1. The mass of compression.

In this section we recall the notion of the mass of compression on points in space and study the associated statistics.

**Definition 2.3.** By the mass of a compression of scale $m > 0$ ($m \in \mathbb{R}$) fixed, we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \dots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

It is important to notice that the condition $x_i \neq x_j$ for $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is not only a quantifier but a requirement; otherwise, the statement for the mass of compression will be flawed completely. To wit, suppose we take $x_1 = x_2 = \cdots = x_n$, then it will follows that $\mathrm{Inf}(x_j) = \mathrm{Sup}(x_j)$, in which case the mass of compression of scale $m$ satisfies

$$m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)-k} \leq \mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) \leq m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)+k}$$

and it is easy to notice that this inequality is absurd. By extension one could also try to equalize the sub-sequence on the bases of assigning the supremum and the Infimum and obtain an estimate but that would also contradict the mass of compression inequality after a slight reassignment of the sub-sequence. Thus it is important for the estimate to make any good sense to ensure that any tuple $(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$ must satisfy $x_i \neq x_j$ for all $1 \leq i,j \leq n$. Hence in this paper this condition will be highly extolled. In situations where it is not mentioned, it will be assumed that the tuple $(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$ is such that $x_i \leq x_j$ for $1 \leq i,j \leq n$.

**Lemma 2.4.** *The estimate remain valid*

$$\sum_{n\leq x}\frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$

*where* $\gamma = 0.5772\cdots$.

*Remark* 2.5. Next we prove upper and lower bounding the mass of the compression of scale $m \geq 1$.

**Proposition 2.2.** *Let* $(x_1,x_2,\ldots,x_n) \in \mathbb{N}^n$, *then the estimates holds*

$$m\log\left(1-\frac{n-1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) \ll m\log\left(1+\frac{n-1}{\mathrm{Inf}(x_j)}\right)$$

*for* $n \geq 2$.

*Proof.* Let $(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \geq 1$. Then it follows that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) &= m\sum_{j=1}^{n}\frac{1}{x_j}\\ &\leq m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)+k}\end{aligned}$$

and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) &= m\sum_{j=1}^{n}\frac{1}{x_j}\\ &\geq m\sum_{k=0}^{n-1}\frac{1}{\sup(x_j)-k}.\end{aligned}$$

□

**Definition 2.6.** Let $(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1,2\ldots,n$. Then by the gap of compression of scale $m$ $\mathbb{V}_m$, denoted $\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]$, we mean the expression

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)] = \left|\left|\left(x_1-\frac{m}{x_1},x_2-\frac{m}{x_2},\ldots,x_n-\frac{m}{x_n}\right)\right|\right|$$

**Definition 2.7.** Let $(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ with $x_i\neq x_j$ for all $1\leq i<j\leq n$. Then by the ball induced by $(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ under compression of scale $m$, denoted $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1,x_2,\ldots,x_n)]$ we mean the inequality

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},x_2+\frac{m}{x_2},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|\leq\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)].$$

A point $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1,x_2,\ldots,x_n)]$ if it satisfies the inequality. We call the ball the circle induced by points under compression if we take the dimension of the underlying space to be $n=2$.

*Remark* 2.8. The circle induced by points under compression is the ball induced on points when we take $n=2$.

**Proposition 2.3.** *Let $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ for $n\geq 2$ with $x_j\neq 0$ for $j=1,\ldots,n$, then we have*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2=\mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right]+m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)]-2mn.$$

*In particular, we have the estimate*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2=\mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right]-2mn+O\left(m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)]\right)$$

*for $\vec{x}\in\mathbb{N}^n$, where $m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)]$ is the error term in this case.*

**Lemma 2.9** (Compression estimate)**.** *Let $(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ for $n\geq 2$, then we have*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2\ll n\mathrm{sup}(x_j^2)+m^2\log\left(1+\frac{n-1}{\mathrm{Inf}(x_j)^2}\right)-2mn$$

*and*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2\gg n\mathrm{Inf}(x_j^2)+m^2\log\left(1-\frac{n-1}{\mathrm{sup}(x_j^2)}\right)^{-1}-2mn.$$

**Theorem 2.10.** *Let $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathbb{N}^n$ with $z_i\neq z_j$ for all $1\leq i<j\leq n$. Then $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ if and only if*

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{y}].$$

*Proof.* Let $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ for $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathbb{N}^n$ with $z_i\neq z_j$ for all $1\leq i<j\leq n$, then it follows that $||\vec{y}||>||\vec{z}||$. Suppose on the contrary that

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]>\mathcal{G}\circ\mathbb{V}_m[\vec{y}],$$

then it follows that $||\vec{y}||<||\vec{z}||$, which is absurd. Conversely, suppose

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{y}]$$

then it follows from Proposition 2.3 that $||\vec{z}|| \leq ||\vec{y}||$ and $\sup(z_j) \leq \sup(y_j)$ by Lemma 2.9. It follows that

$$\left|\left|\vec{z} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \dots, y_n + \frac{m}{y_n}\right)\right|\right| \leq \left|\left|\vec{y} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \dots, y_n + \frac{m}{y_n}\right)\right|\right| \leq \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{y}].$$

This certainly implies $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ and the proof of the theorem is complete. □

**Theorem 2.11.** *Let $\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$. If $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ then*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \subseteq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

*Proof.* First let $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ and suppose for the sake of contradiction that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \not\subseteq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there must exist some $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ such that $\vec{z} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. It follows from Theorem 2.10 that

$$\mathcal{G} \circ \mathbb{V}_m[\vec{z}] > \mathcal{G} \circ \mathbb{V}_m[\vec{x}].$$

It follows that

$$\begin{aligned}\mathcal{G} \circ \mathbb{V}_m[\vec{y}] &\geq \mathcal{G} \circ \mathbb{V}_m[\vec{z}] \\ &> \mathcal{G} \circ \mathbb{V}_m[\vec{x}] \\ &\geq \mathcal{G} \circ \mathbb{V}_m[\vec{y}]\end{aligned}$$

which is absurd, thereby ending the proof. □

*Remark* 2.12. Theorem 2.11 tells us that points confined in certain balls induced under compression should by necessity have their induced ball under compression covered by these balls in which they are contained.

2.2. **Admissible points of balls induced under compression.** We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

**Definition 2.13.** Let $\vec{y} = (y_1, y_2, \dots, y_n) \in \mathbb{N}^n$ with $y_i \neq y_j$ for all $1 \leq i < j \leq n$. Then $\vec{y}$ is said to be an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ if

$$\left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \dots, x_n + \frac{m}{x_n}\right)\right|\right| = \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}].$$

*Remark* 2.14. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball. Next we show that all balls can in principle be generated by their admissible points.

**Theorem 2.15.** *The point $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is admissible if and only if*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] = \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

*and $\mathcal{G} \circ \mathbb{V}_m[\vec{y}] = \mathcal{G} \circ \mathbb{V}_m[\vec{x}]$.*

*Proof.* First let $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{x}]}[\vec{x}]$ be admissible and suppose on the contrary that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{y}]}[\vec{y}] \neq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there exist some $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{x}]}[\vec{x}]$ such that

$$\vec{z} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{y}]}[\vec{y}].$$

Applying Theorem 2.10, we obtain the inequality

$$\mathcal{G}\circ \mathbb{V}_m[\vec{y}] < \mathcal{G}\circ \mathbb{V}_m[\vec{z}] \leq \mathcal{G}\circ \mathbb{V}_m[\vec{x}].$$

It follows from Proposition 2.3 that $||\vec{x}|| < ||\vec{y}||$ or $||\vec{y}|| < ||\vec{x}||$. By joining this points to the origin by a straight line, this contradicts the fact that the point $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{x}]}[\vec{x}]$ is an admissible point.. This contradicts the fact that the point $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{x}]}[\vec{x}]$ is an admissible point. Now we notice that $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{x}]}[\vec{x}]$ certainly implies $\mathcal{G}\circ \mathbb{V}_m[\vec{y}] \leq \mathcal{G}\circ \mathbb{V}_m[\vec{x}]$. Conversely we notice as well that $\vec{x} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{y}]}[\vec{y}]$, which certainly implies $\mathcal{G}\circ \mathbb{V}_m[\vec{x}] \leq \mathcal{G}\circ \mathbb{V}_m[\vec{y}]$ by Theorem 2.10. Thus the conclusion follows. Conversely, suppose

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{y}]}[\vec{y}] = \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ \mathbb{V}_m[\vec{y}] = \mathcal{G}\circ \mathbb{V}_m[\vec{x}]$. Then it follows that the point $\vec{y}$ must satisfy the inequality

$$\begin{aligned}\left|\left|\vec{z} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| &= \left|\left|\vec{z} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| \\ &\leq \frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{x}].\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{x}] &= \left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| \\ &\leq \frac{1}{2}\mathcal{G}\circ \mathbb{V}_m[\vec{x}]\end{aligned}$$

and $\vec{y}$ is indeed admissible, thereby ending the proof. □

Next we obtain an equivalent notion of the area of the circle induced by points under compression in the plane $\mathbb{R}^2$ in the following result.

**Proposition 2.4.** *Let $\vec{x} \in \mathbb{N}^2 \subset \mathbb{R}^2$. Then the area of the circle induced by point $\vec{x}$ under compression of scale $m$, denote by $\mathbb{V}_m[\vec{x}]$ is given by*

$$\delta(\mathbb{V}_m[\vec{x}]) = \frac{\pi(\mathcal{G}\circ \mathbb{V}_m[\vec{x}])^2}{4}.$$

*Proof.* This follows from the mere definition of the area of a circle and noting that the radius $r$ of the circle induced by the point $\vec{x} \in \mathbb{R}^2$ under compression is given by

$$r = \frac{\mathcal{G}\circ \mathbb{V}_m[\vec{x}]}{2}.$$

□

## 3. Lower bound

**Theorem 3.1.** *Let $\Delta(s)$ denotes the minimal area of the triangle formed by $s$ points in the unit disc. Then we have the lower bound*

$$\Delta(s) \gg \frac{\log s}{s\sqrt{s}}.$$

*Proof.* First let $s \geq 4$ and let $m := m(s) > 0$ be fixed. Pick arbitrarily a point $(x_1, x_2) = \vec{x} \in \mathbb{R}^2$ with $x_j > 1$ for $1 \leq j \leq 2$ such that $\mathcal{G} \circ \mathbb{V}_m[\vec{x}] < 1$. This ensures the circle induced under compression is contained in some unit disc. Next we apply the compression of scale $m > 0$, given by $\mathbb{V}_m[\vec{x}]$ and construct the circle induced by the compression given by

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

with radius $\frac{(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])}{2}$. On this circle locate $(s-3)$ admissible points so that the chord joining each pair of adjacent $(s-1)$ admissible points including $\vec{x}$ and $\mathbb{V}_m[\vec{x}]$ are equidistant. Let us now join each of the $(s-1)$ admissible point considered to the center of the circle given by

$$\vec{y} := \frac{1}{2}\left(x_1 + \frac{m}{x_1}, x_2 + \frac{m}{x_2}\right).$$

Invoking Proposition 2.4, the area of the circle induced under compression is given by

$$\delta(\mathbb{V}_m[\vec{x}]) = \frac{\pi(\mathcal{G} \circ \mathbb{V}_m[\vec{x}])^2}{4}.$$

We join all pairs of adjacent admissible points considered by a chord and produce $(s-1)$ triangles of equal area. We note that we can use the area of each sector formed from this construction to approximate the area of each of the triangles inscribed in the sector as we increase the number of such admissible points on the circle. It follows that the area of each sector formed must be the same and given by

$$\begin{aligned}\mathcal{A} &:= \frac{\pi(\mathcal{G} \circ \mathbb{V}_m[\vec{x}])^2}{4 \times (s-1)} \\ &\gg \frac{2\mathrm{Inf}(x_j^2) + m^2 \log\left(1 - \frac{1}{\sup(x_j^2)}\right)^{-1} - 4m}{4 \times s}.\end{aligned}$$

The lower bound follows by taking

$$m := \frac{\log^2 s}{4s} \quad \text{and} \quad \mathrm{Inf}(x_j) := 1 + \frac{\log s}{\sqrt{s}}$$

since points $\vec{x} = (x_1, x_2)$ can only have a compression gap $\mathcal{G} \circ \mathbb{V}_m[\vec{x}] < 1$ if $x_1 = x_2 = 1 + \epsilon$ for any small $\epsilon > 0$. □

Albeit Heilbronn's triangle problem is a max − min problem, the area of each triangle espoused in the construction is the same, to which the underlying condition has little relevance in this particular framework.

[1].

[1]

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