Euler's derivation of rigid body equations

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Abstract

In this work, we give the modern version of Euler's derivation of equations governing rigid body rotations. This derivation helps understand the rigid body rotations.

Keywords: Rigid body, Euler equations, rotational motion

1 Introduction

Since Euler's derivation of Euler equations, we have been studying rigid body rotations by using them [1, 2, 3]. However, nearly all classical mechanics books use another derivation to get Euler equations [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Since Euler did his derivation a few centuries ago, his notation is not so easy to understand by students. One can find historical details of Euler's derivation in recent work [18].

In this work, we will present a derivation in modern fashion. This derivation can help understand rigid body rotations and inertia products. We will start by explaining the general structure and continue by giving derivations of Euler equations in the stationary reference frame and in the body reference frame.

Euler first derived equations in the stationary reference frame, and then he realized that moments of inertia and inertia products are time-dependent. Then, he derived equations in the body reference frame. We will also follow the same path.

2 Governing equations of rigid body rotations

In figure 1, one can see a rigid body, stationary reference frame (x', y', z'), body reference frame (x, y, z), applied force $\vec{F_a}$ and position vector $\vec{l}$ showing acting position of the force. In the figure, the rigid body is represented by a stone to emphasize that it can have any shape, and it should be solid. Euler defines a rigid body with the usage of solid instead of rigid by following statement "A body is called solid whose interior is not subject to any change, or which is such that all its parts constantly maintain the same relative position, whatever motion the whole body may have." [1].

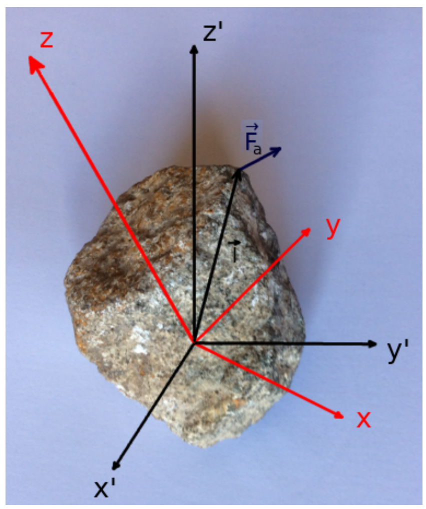

Figure 1: A rigid body (stone), stationary reference frame (x', y', z'), body reference frame (x, y, z), applied force $\vec{F_a}$ and position vector $\vec{l}$.

In general, a rigid body can make both translational and rotational motion. In this work, we will follow Euler and will consider only the rotational part of the motion. In figure 1, the origins of reference frames are fixed to the (approximate) center of mass.

We assume that the axes of the body reference frame lie on the principal axes of the rigid body. This assumption will provide simplicities in the calculations in the body reference frame by letting inertia products equal to zero.

An applied force $\vec{F_a}$ acting on the rigid body can cause changes in its angular momentum. We will follow Euler and consider an infinitesimal part of the rigid body to find how applied force changes the rigid body's angular momentum. This infinitesimal part should not lie on the rotation axis for the sake of generality.

2.1 Derivation in the stationary reference frame

In this section, we will follow the steps of Euler's derivation for the rotation of a rigid body in a stationary or inertial reference frame.

Like Euler, we will consider an infinitesimal part of the rotating rigid body. We can write its position in a stationary reference frame as

$$\vec{r} = x'\hat{x}' + y'\hat{y}' + z'\hat{z}'. \tag{1}$$

We can write the angular velocity of this infinitesimal part as

$$\vec{w} = w_{x'}\hat{x}' + w_{y'}\hat{y}' + w_{z'}\hat{z}'. \tag{2}$$

Then, we can obtain its velocity from the cross product $\vec{v} = \vec{w} \times \vec{r}$ as

$$\vec{v} = (z'w_{y'} - y'w_{z'})\hat{x}' - (z'w_{x'} - x'w_{z'})\hat{y}' + (y'w_{x'} - x'w_{y'})\hat{z}'. \tag{3}$$

As a result, we can write the components of velocity as

$$\begin{aligned} \frac{dx'}{dt} &= z'w_{y'} - y'w_{z'}, \\ \frac{dy'}{dt} &= x'w_{z'} - z'w_{x'}, \\ \frac{dz'}{dt} &= y'w_{x'} - x'w_{y'}. \end{aligned} \tag{4}$$

If we take the derivative of these with respect to time and use them in derivatives, we can get components of acceleration as

$$\begin{aligned} \frac{d^2x'}{dt^2} &= z'\dot{w}_{y'} - y'\dot{w}_{z'} + y'w_{x'}w_{y'} - x'w_{y'}^2 - x'w_{z'}^2 + z'w_{z'}w_{x'}, \\ \frac{d^2y'}{dt^2} &= x'\dot{w}_{z'} - z'\dot{w}_{x'} + z'w_{y'}w_{z'} - y'w_{z'}^2 - y'w_{x'}^2 + x'w_{x'}w_{y'}, \\ \frac{d^2z'}{dt^2} &= y'\dot{w}_{x'} - x'\dot{w}_{y'} + x'w_{z'}w_{x'} - z'w_{x'}^2 - z'w_{y'}^2 + y'w_{y'}w_{z'}. \end{aligned} \tag{5}$$

By multiplying components of acceleration with the mass of the infinitesimal part, ρ, we can get components of force $\vec{f} = f_{x'}\hat{x}' + f_{y'}\hat{y}' + f_{z'}\hat{z}'$ according to Newton's second law $\vec{F} = m\vec{a}$. Then by using the force $\vec{f}$ and equations (5), we can obtain moments of the force $n_{x'} = y'f_{z'} - z'f_{y'}$, $n_{y'} = z'f_{x'} - x'f_{z'}$ and $n_{z'} = x'f_{y'} - y'f_{x'}$ as

$$\begin{aligned} n_{x'} &= \rho[({y'}^2 + {z'}^2)\dot{w}_{x'} - x'y'\dot{w}_{y'} - x'z'\dot{w}_{z'} + x'y'w_{x'}w_{z'} - x'z'w_{x'}w_{y'} \\ &\quad -y'z'(w_{y'}^2 - w_{z'}^2) + w_{y'}w_{z'}({y'}^2 - {z'}^2)], \\ n_{y'} &= \rho[({x'}^2 + {z'}^2)\dot{w}_{y'} - x'y'\dot{w}_{x'} - y'z'\dot{w}_{z'} + y'z'w_{x'}w_{y'} - y'x'w_{y'}w_{z'} \\ &\quad -x'z'(w_{z'}^2 - w_{x'}^2) + w_{x'}w_{z'}({z'}^2 - {x'}^2)], \\ n_{z'} &= \rho[({x'}^2 + {y'}^2)\dot{w}_{z'} - x'z'\dot{w}_{x'} - y'z'\dot{w}_{y'} + x'z'w_{y'}w_{z'} - y'z'w_{x'}w_{z'} \\ &\quad -x'y'(w_{x'}^2 - w_{y'}^2) + w_{x'}w_{y'}({x'}^2 - {y'}^2)]. \end{aligned} \tag{6}$$

We will integrate equations (6) to find the effect of applied force on the rigid body. Integration of the left-hand side gives the moment of the applied force relevant to that axis, e.g., $N_{x'} = \int n'_x dv'$. By using integrals at the right-hand side, we can define components of moments of inertia tensor as

$$\begin{array}{ll} I_{x'x'} = \int \rho({y'}^2 + {z'}^2)dv', & I_{x'y'} = I_{y'x'} = -\int \rho x'y'dv', \\ I_{y'y'} = \int \rho({x'}^2 + {z'}^2)dv', & I_{x'z'} = I_{z'x'} = -\int \rho x'z'dv', \\ I_{z'z'} = \int \rho({x'}^2 + {y'}^2)dv', & I_{y'z'} = I_{z'y'} = -\int \rho y'z'dv'. \end{array} \tag{7}$$

Inertia products are not equal to zero since the principal axes of the rigid body does not lie on the axes of the stationary reference frame. Then from the integration of equations (6), we can obtain the following equations

$$\begin{aligned} N_{x'} &= I_{x'x'}\dot{w}_{x'} + I_{x'y'}\dot{w}_{y'} + I_{x'z'}\dot{w}_{z'} - I_{x'y'}w_{x'}w_{z'} + I_{x'z'}w_{x'}w_{y'} \\ &\quad +I_{y'z'}(w_{y'}^2 - w_{z'}^2) + w_{y'}w_{z'}(I_{z'z'} - I_{y'y'}), \\ N_{y'} &= I_{y'y'}\dot{w}_{y'} + I_{x'y'}\dot{w}_{x'} + I_{y'z'}\dot{w}_{z'} - I_{y'z'}w_{x'}w_{y'} + I_{x'y'}w_{y'}w_{z'} \\ &\quad +I_{x'z'}(w_{z'}^2 - w_{x'}^2) + w_{x'}w_{z'}(I_{x'x'} - I_{z'z'}), \\ N_{z'} &= I_{z'z'}\dot{w}_{z'} + I_{x'z'}\dot{w}_{x'} + I_{y'z'}\dot{w}_{y'} - I_{x'z'}w_{y'}w_{z'} + I_{y'z'}w_{x'}w_{z'} \\ &\quad +I_{x'y'}(w_{x'}^2 - w_{y'}^2) + w_{x'}w_{y'}(I_{y'y'} - I_{x'x'}). \end{aligned} \tag{8}$$

These equations describe the rotation of a rigid body in the stationary reference frame by the effect of the moment of force, $\vec{N} = \vec{l} \times \vec{F}_a$.

One should be careful while using these equations since moments of inertia and inertia products are time dependent.

2.2 Derivation in the body reference frame

In this section, we will follow Euler's method in the stationary reference frame to get equations of rigid body rotations in the body reference frame under the influence of torque. We will consider an infinitesimal part of the rigid body to see the effect of torque on the rotation of that infinitesimal part. The axes of the body reference frame are chosen as the principal axes of the body.

In the body reference frame, we can show the position of an infinitesimal part of the rigid body by

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}. \tag{9}$$

We can write the components of angular velocity as

$$\vec{w} = w_x\hat{x} + w_y\hat{y} + w_z\hat{z}. \tag{10}$$

Then, we can get the velocity of the infinitesimal part from cross-product $\vec{v} = \vec{w} \times \vec{r}$ as

$$\vec{v} = (zw_y - yw_z)\hat{x} - (zw_x - xw_z)\hat{y} + (yw_x - xw_y)\hat{z}. \tag{11}$$

As a result, we can write the components of the velocity as

$$\begin{aligned} \frac{dx}{dt} &= zw_y - yw_z, \\ \frac{dy}{dt} &= xw_z - zw_x, \\ \frac{dz}{dt} &= yw_x - xw_y. \end{aligned} \tag{12}$$

By taking the time derivative of equations (12) and using them, we can find the components of acceleration as

$$\begin{aligned} \frac{d^2x}{dt^2} &= z\dot{w}_y - y\dot{w}_z + yw_xw_y - xw_y^2 - xw_z^2 + zw_zw_x, \\ \frac{d^2y}{dt^2} &= x\dot{w}_z - z\dot{w}_x + zw_yw_z - yw_z^2 - yw_x^2 + xw_xw_y, \\ \frac{d^2z}{dt^2} &= y\dot{w}_x - x\dot{w}_y + xw_zw_x - zw_x^2 - zw_y^2 + yw_yw_z, \end{aligned} \tag{13}$$

where dots over w_i's represent time derivatives. According to Newton's second law, $\vec{F} = m\vec{a}$, by multiplying equations (13) with mass of the infinitesimal part, ρ, we can get components of the force $\vec{f} = \rho d^2x/dt^2\hat{x} + \rho d^2y/dt^2\hat{y} + \rho d^2z/dt^2\hat{z}$. This force is responsible for the acceleration of the rigid body's infinitesimal part and different than the applied force. It results from the combination of applied force and rigidity of the object.

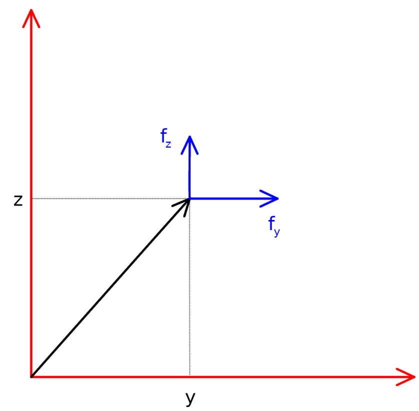

Figure 2: f_y and f_z components of force $\vec{f}$ acting on the infinitesimal part.

In figure 2, we see f_y and f_z components of the force acting on the infinitesimal part on yz-plane. y component of the force will cause a moment zf_y in the clockwise direction, and z component of the force will cause a moment yf_z in the counter-clockwise direction. The counter-clockwise direction is the positive direction according to the right-hand rule, then, the moment around x-axis will be equal to $yf_z - zf_y$, which is x component of the cross product $\vec{r} \times \vec{f}$. We can similarly obtain the other moments, $zf_x - xf_z$ and $xf_y - yf_x$, and they are also consistent with the cross product. Hence, by considering cross-product $\vec{r} \times \vec{f}$, we can write the moment of the force on the infinitesimal part of the rigid body as

$$\begin{aligned} \vec{n} &= n_x\hat{x} + n_y\hat{y} + n_z\hat{z} \\ &= (yf_z - zf_y)\hat{x} - (xf_z - zf_x)\hat{y} + (xf_y - yf_x)\hat{z}. \end{aligned} \tag{14}$$

Then by using the force $\vec{f}$, equations (13) and (14), we can obtain the effects of moments of the force on angular accelerations and angular velocities as

$$
\begin{aligned}
n_x &= \rho[(y^2+z^2)\dot{w}_x - xy\dot{w}_y - xz\dot{w}_z + xyw_xw_z - xzw_xw_y \\
&\quad -yz(w_y^2-w_z^2) + w_yw_z(y^2-z^2)], \\
n_y &= \rho[(x^2+z^2)\dot{w}_y - xy\dot{w}_x - yz\dot{w}_z + yzw_xw_y - yxw_yw_z \\
&\quad -xz(w_z^2-w_x^2) + w_xw_z(z^2-x^2)], \\
n_z &= \rho[(x^2+y^2)\dot{w}_z - xz\dot{w}_x - yz\dot{w}_y + xzw_yw_z - yzw_xw_z \\
&\quad -xy(w_x^2-w_y^2) + w_xw_y(x^2-y^2)].
\end{aligned} \tag{15}
$$

We can integrate equations (15) over the volume of the rigid body. If we consider the torque on different infinitesimal parts, it changes as the distance from the rotation axis changes, and the moments of force are distributed over the rigid body to satisfy the rotation of the rigid body as a whole with the help of atomic interactions. This is the result of the object's rigidity. Then, the integration of the left-hand side gives torque relevant to that axis, e.g., $N_x = \int n_x dv$. Integrals at the right-hand side are related to moments of inertia and inertia products, and they are defined as

$$
\begin{array}{ll}
I_{xx} = \int \rho(y^2+z^2)dv, & I_{xy} = I_{yx} = -\int \rho xy dv = 0, \\
I_{yy} = \int \rho(x^2+z^2)dv, & I_{xz} = I_{zx} = -\int \rho xz dv = 0, \\
I_{zz} = \int \rho(x^2+y^2)dv, & I_{yz} = I_{zy} = -\int \rho yz dv = 0.
\end{array} \tag{16}
$$

We choose the body axes as principal axes, then terms in the second column, inertia products, are equal to zero since the definition of principal axes is based on this.

Then the integration of equations (15) over the volume of the rigid body gives

$$
\begin{aligned}
N_x &= I_{xx}\dot{w}_x - w_yw_z(I_{yy}-I_{zz}), \\
N_y &= I_{yy}\dot{w}_y - w_zw_x(I_{zz}-I_{xx}), \\
N_z &= I_{zz}\dot{w}_z - w_xw_y(I_{xx}-I_{yy}).
\end{aligned} \tag{17}
$$

These are Euler equations for rigid bodies describing the change of the angular momentum under the influence of torque, $\vec{N} = \vec{l} \times \vec{F}_a$, in the body reference frame.

In these equations, moments of inertia and inertia products are independent of time.

3 Summary and Conclusion

By following Euler's ideas, we have obtained equations describing the rotation of a rigid body. We have used Euler's method to obtain equations in the stationary reference frame. While obtaining equations in the body reference frame, we have followed Euler's method in the stationary reference frame and have not use direction cosines though Euler used them. Our derivation gives the same equations as equations in the body reference frame, known as Euler equations. The equations in the stationary reference frame are more complicated than the ones in the body reference frame due to non-zero inertia products. In addition to this, it should be remembered that inertia products and moments of inertia change in the stationary reference frame during the rotation.

If we rotate a rigid body with an initial effect, it will rotate around one of its principal axes provided that there is not any disturbing effect. We have seen during the derivation that studying rotation of a rigid body by using principal axes provides simplifications since inertia products become zero. In formal definition, principal axes are defined by using inertia products: if we choose an axis passing through the center of mass and if inertia products related to that axis are zero then that axis is principal axes. As we mentioned this axis is physically exist and the rigid body naturally rotates around it, if an initial rotation is given. If we try to define the rotation of a rigid body by using axes other than principal axes, we need to consider inertia products. Moments of inertia and inertia products play a similar role to mass in translational motion, in which mass is related to inertia against the change in linear momentum. Moment of inertia corresponds to the inertia of the rigid body against the change in angular momentum around rotation axis, and inertia product is the inertia of the rigid body against the change in angular momentum around rotation axis due to effects of other rotations and asymmetries in mass distribution.

During the derivation, we considered that the fixed point is the center of mass of the rigid body. However, this derivation is also valid if the fixed point of the rigid body is different than the center of mass. In such cases, the origins should be set to that fixed point, and moments of inertia in the body reference frame should be calculated by considering rotations around the fixed point. This calculation can be done with the help of Steiner's parallel-axis theorem [5].

During the derivation, we considered an applied force. Any force that can change atomic interactions will deform the object, and it can not be considered as a rigid body anymore. Hence, the force that will be considered in Euler equations should not change the atomic structure. If it changes, other physical interactions should be considered, and such cases are not subject to rigid body rotations.

During the derivation, we have considered an infinitesimal part and the force $\vec{f}$, which provides acceleration to that infinitesimal part. This force $\vec{f}$ is the result of the combination of atomic forces or internal effects and applied force $\vec{F}_a$. After the integration, internal effects are simplified and the net torque $\vec{N} = \vec{l} \times \vec{F}_a$ remains. This torque is natural result of applied force $\vec{F}_a$.

The body reference frame is a non-inertial reference frame, and in non-inertial reference frames, there can be some fictitious effects. One can see that the second terms at the right-hand side of equations (17) are extra, and they are fictitious effects. We should note that the centrifugal force is simplified by the rigidity of the object.

One can also consider the situation that the applied force is equal to zero. In that case, external torque becomes zero, and Euler equations describe rotations without it.

References

[1] Euler L 1752 Decouverte d'un noveau principe de mecanique *The Euler Archive, E-177* http://eulerarchive.maa.org

[2] Euler L 1765a Du mouvement de rotation des corps solides autour d'un axe variable *The Euler Archive, E-292* http://eulerarchive.maa.org

[3] Lamb H 1920 *Higher Mechanics* (Cambridge: Cambridge University Press)

[4] Goldstein H 1980 *Classical Mechanics* 2nd Ed (Massachusetts: Addison-Wesley)

[5] Marion J B and Thornton S T 2004 *Classical Dynamics of Particles and Systems* 5th Ed (Belmont: Brooks/Cole)

[6] Sommerfeld A 1952 *Mechanics. Lectures on Theoretical Physics. Volume 1* (New York: Academic Press)

[7] Taylor J R 2005 *Classical Mechanics* (Dulles: University Science Books)

[8] Fowles G R and Cassiday G L 2005 *Analytical Mechanics* 7th Ed (Belmont: Brooks/Cole)

[9] Barger V and Olson M 1994 *Classical Mechanics: A Modern Perspective* (New York: McGraw-Hill)

[10] Jorge V J and Eugene J S 1998 *Classical dynamics. A contemporary approach* (Cambridge: Cambridge University Press)

[11] McCauley J L 1997 *Classical mechanics: Transformations, flows, integrable, and chaotic dynamics* (Cambridge: Cambridge University Press)

[12] Corinaldesi E 1999 *Classical Mechanics for Physics Graduate Students* (Singapore: World Scientific Publishing Company)

[13] Desloge E A 1982 *Classical Mechanics. Volume 1* (New York: John Wiley and Sons)

[14] Symon K R 1971 *Mechanics* 3rd Ed (Massachusetts: Addison-Wesley)

[15] Gregory R D 2006 *Classical Mechanics* (New York: Cambridge University Press)

[16] Arnold V I 1989 *Mathematical Methods of Classical Mechanics* 2nd Ed (New York: Springer-Verlag)

[17] Landau L D and Lifshitz E M 2000 *Mechanics* 3rd Ed (New Delhi: Butterworth-Heinenann)

[18] Marquina J E, Marquina M L, Marquina V and Hernández-Gómez J J 2017 Leonhard Euler and the mechanics of rigid bodies *Eur. J. Phys.* **38** 015001 https://doi.org/10.1088/0143-0807/38/1/015001