Angular momentum emission by a rotating dipole

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A new calculation confirms the presence of spin radiation along the axis of rotation of a dipole. This is further proof of the need to introduce the spin tensor into classical electrodynamics, along with the energy-momentum tensor.

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1. Introduction

Circularly polarized electromagnetic radiation contains angular momentum in the form of the angular momentum density [1,2].

> J. H. Poynting: “If we put E for the energy in unit volume and G for the torque per unit area, we have $G = E\lambda/2\pi$” [2, p. 565].

This means that such radiation is Weyssenhoff’s spin-fluid [3].

> J. Weyssenhoff: “By spin-fluid we mean a fluid each element of which possesses besides energy and linear momentum also a certain amount of angular momentum, proportional – just as energy and the linear momentum – to the volume of the element”.

This is recorded in textbooks [4,5]. Since Emma Noether, this angular momentum has been described by the spin tensor density [6-8]

$$\underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda}\delta^{\mu]}_{\alpha}\frac{\partial \mathsf{L}}{\partial(\partial_{\nu}A_{\alpha})} = -2A^{[\lambda}F^{\mu]\nu}, \tag{1}$$

where $\mathsf{L} = -F_{\mu\nu}F^{\mu\nu}/4$ is the free electromagnetic field Lagrangian, A^{λ} is the vector potential, and $F_{\mu\nu}$ is the field-strength tensor. The local sense of a spin tensor is as follows. Υ^{xyt} [J∗s/m^3] is spin volume density, Υ^{xyl} [J/m^2] is spin flux density, i.e. torque per unit area (cf. J. H. Poynting). The spin tensor is used in the publications [9-20]. However, the spin tensor is ignored in works expressing the common point of view, e.g. [21-25].

Besides spin, any electromagnetic field contains mass-energy and momentum, which are described by the energy-momentum tensor [26,27]

$$T^{\mu\nu} = -g^{\mu\lambda}F_{\lambda\alpha}F^{\nu\alpha} + g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}/4. \tag{2}$$

The local sense of the energy-momentum tensor is as follows. T^{xt} [N∗s/m^3] is momentum volume density, T^{tx} [kg/m^2∗s] is mass-energy flux density. It means, e.g., $dp^x = T^{xt}dV$ is the momentum in the volume dV.

Moment of momentum, e.g., $dL^{xy} = (xT^{yt} - yT^{xt})dV$ is the orbital angular momentum of the momentum contained in the volume dV. So, the total angular momentum possessed by the volume dV is

$$dJ^{ik} = dS^{ik} + dL^{ik} = (\Upsilon^{ikt} + 2r^{[i}T^{k]t})dV. \tag{3}$$

The total torque per the area da_l, i.e. angular momentum flux, is

$$d\tau^{ik} = d\underset{S}{\tau}^{ik} + dL^{ik}/dt = (\Upsilon^{ikl} + 2r^{[i}T^{k]l})da_l. \tag{4}$$

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It is important that spin is not associated with a moment of a linear momentum, or even with a motion of matter. **Hehl** writes about spin of an electron [28]:

> “The current density in Dirac’s theory can be split into a convective part and a polarization part. The polarization part is determined by the spin distribution of the electron field. It should lead to *no* energy flux in the rest system of the electron because the genuine spin ‘motion’ take place only within a region of the order of the Compton wavelength of the electron”.

2. Electromagnetic field of a rotating dipole

Electromagnetic field of a rotating dipole **p** is well known [27,29,30]

$$\mathbf{E}=\left[\frac{\omega^2(\mathbf{p}r^2-(\mathbf{pr})\mathbf{r})}{4\pi\varepsilon_0c^2r^3}+\frac{i\omega(\mathbf{p}r^2-3(\mathbf{pr})\mathbf{r}}{4\pi\varepsilon_0cr^4}-\frac{(\mathbf{p}r^2-3(\mathbf{pr})\mathbf{r}}{4\pi\varepsilon_0r^5}\right]\exp(ikr-i\omega t)\,, \tag{5}$$

$$\mathbf{H}=\left[\frac{\omega^2\mathbf{r}\times\mathbf{p}}{4\pi cr^2}+\frac{i\omega\mathbf{r}\times\mathbf{p}}{4\pi r^3}\right]\exp(ikr-i\omega t)\,. \tag{6}$$

The first terms of (5), (6) are proportional to $1/r$ and so represent radiation. This radiation is of circular polarization in the direction of the rotational axis, z-axis (see Fig. 1 from [31]). Therefore this field contains the spin flux Υ^{xyl}. We calculate this spin flux per sphere $r=Const$ in Section 3.

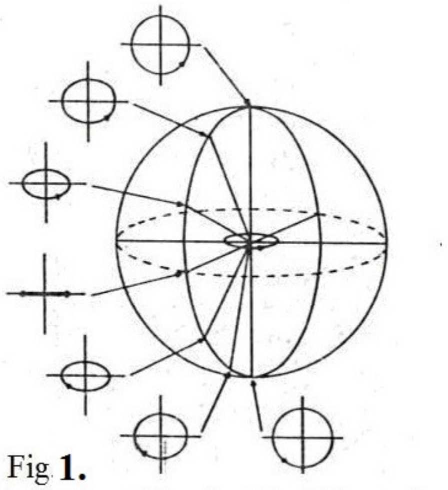

Fig. **1.**
Polarization of the electric field seen by looking from different directions at a circular oscillator

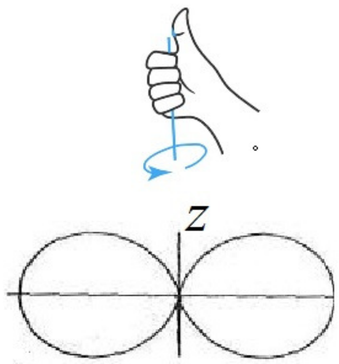

Fig. **2.**
sin²θ
Torque distribution

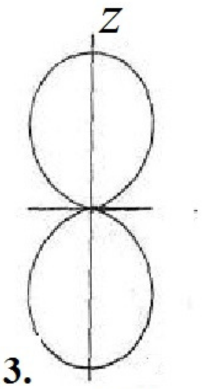

Fig. **3.**
cos²θ
Spin flux distribution

At the same time this radiation contains no orbital angular momentum flux per elements da_l of the sphere $r=Const$. $dL^{ik}/dt=2r^{[i}T^{k]l}da_l=0$. Really, the first terms fields **E** & **H** are orthogonal to each other and to the vector **r**. So, in any point, we can enter local Cartesian coordinates such that $da_l=\{0,0,da_z\}$, $\mathbf{E}=\{E_x,0,0\}$, $\mathbf{H}=\{0,H_y,0\}$, $\mathbf{r}=\{0,0,z\}$, i.e. F_{tx}, F^{tx}, F_{xz}, F^{xz} are not equal to zero only. Using this coordinates we find according to (2): $T^{xz}=-g^{xx}F_{x\alpha}F^{z\alpha}=0,\ T^{yz}=-g^{yy}F_{y\alpha}F^{z\alpha}=0$. So the orbital angular momentum is not radiated.

The second terms field of (5), (6) contains the orbital angular momentum flux, or torque, per the sphere $r=Const$. In Refs [32-37], spherical coordinates were used, and the angular distribution of the torque was obtained (see Fig. 2):

$$dL^{ik}/dtd\Omega=\omega^3p^2\sin^2\theta/16\pi^2\varepsilon_0c^3 \tag{7}$$

where $d\Omega=\sin\theta d\theta d\varphi$. This torque is located in the neighborhood of the plane of rotation where the polarization is near linear. This torque is not radiated. This torque is like a static torque that someone can apply (Fig. 2).

3. Spin radiation by a rotating dipole

Spin radiated by the first terms field was calculated in [15] using the spin volume density Υ^{xyt} on the assumption that this density is moving at the speed of light. Here the spin flux density Υ^{xyl} is used. This is more naturally.

Using

$$\mathbf{E} = \frac{\omega^2(\mathbf{p}r^2 - (\mathbf{pr})\mathbf{r})}{4\pi\varepsilon_0 c^2 r^3}\exp(ikr - i\omega t), \quad \mathbf{H} = \frac{\omega^2 \mathbf{r}\times\mathbf{p}}{4\pi c r^2}\exp(ikr - i\omega t), \quad p_x = p, \quad p_y = ip \tag{8}$$

yields

$$E_x = F_{tx} = \frac{\omega^2 p(r^2 - x^2 - ixy)}{4\pi\varepsilon_o c^2 r^3}, \; E_y = F_{ty} = \frac{\omega^2 p(ir^2 - xy - iy^2)}{4\pi\varepsilon_o c^2 r^3}, \; E_z = F_{tz} = \frac{-\omega^2 p(zx + izy)}{4\pi\varepsilon_o c^2 r^3}, \tag{9}$$

$$H_x = F^{zy} = \frac{-i\omega^2 pz}{4\pi c r^2}, \; H_y = F^{xz} = \frac{\omega^2 pz}{4\pi c r^2}, \; H_z = F^{yx} = \frac{\omega^2 p(ix - y)}{4\pi c r^2}. \tag{10}$$

Using $\mathbf{A} = -\int \mathbf{E}dt = -i\mathbf{E}/\omega$ yields

$$A_x = \frac{\omega p(-ir^2 + ix^2 - xy)}{4\pi\varepsilon_o c^2 r^3}, \; A_y = \frac{\omega p(r^2 + ixy - y^2)}{4\pi\varepsilon_o c^2 r^3}, \; A_z = \frac{\omega p(izx - zy)}{4\pi\varepsilon_o c^2 r^3}. \tag{11}$$

Accordingly to $\Upsilon^{\lambda\mu\nu} = -2A^{[\lambda}F^{\mu]\nu}$, we have

$$\Upsilon^{xyx} = -\frac{\Re}{2}\{\overline{A}^x F^{yx}\} = \frac{\omega^3 z^2 x}{32\pi^2\varepsilon_0 c^3 r^5}, \; \Upsilon^{xyy} = \frac{\Re}{2}\{\overline{A}^y F^{xy}\} = \frac{\omega^3 z^2 y}{32\pi^2\varepsilon_0 c^3 r^5},$$

$$\Upsilon^{xyz} = -\frac{\Re}{2}\{\overline{A}^x F^{yz} - \overline{A}^y F^{xz}\} = \frac{\omega^3(r^2 + z^2)z}{32\pi^2\varepsilon_0 c^3 r^5} \tag{12}$$

Because of $d\underset{S}{\tau}^{ik} = \Upsilon^{ikl}da_l$, we need the Cartesian coordinates of elements of the sphere $r = Const$, which spherical coordinates are $da_v = \{da_r = d\theta d\varphi, da_\theta = 0, da_\varphi = 0\}$. The transformation coefficients are; $\frac{\partial r}{\partial x} = \frac{x}{r}, \; \frac{\partial r}{\partial y} = \frac{y}{r}, \; \frac{\partial r}{\partial z} = \frac{z}{r}$, and $\sqrt{g} = r^2\sin\theta$. So we have $da_l = \{da_x = x\sin\theta d\theta d\varphi r, da_y = yr\sin\theta d\theta d\varphi, da_z = zr\sin\theta d\theta d\varphi\}$, and

$$d\underset{S}{\tau}^{xy} = \Upsilon^{xyl}da_l = \Upsilon^{xyx}da_x + \Upsilon^{xyy}da_y + \Upsilon^{xyz}da_z$$

$$= \frac{\omega^3 p^2(z^2x^2 + z^2y^2 + r^2z^2 + z^4)}{32\pi^2\varepsilon_0 c^3 r^4}\sin\theta d\theta d\varphi = \frac{\omega^3 p^2}{16\pi^2\varepsilon_0 c^3}\cos^2\theta\sin\theta d\theta d\varphi. \tag{13}$$

This result, $d\underset{S}{\tau}^{xy}/d\Omega = \frac{\omega^3 p^2}{16\pi^2\varepsilon_0 c^3}\cos^2\theta$, is coincided with Ref. [15]. The angular distribution of the spin radiation is represent in Fig. 3.

4. Conclusion

A rotating electric dipole emits angular momentum flux of two types: (i) spin flux, which is directed mainly along the axis of rotation and determined by the spin tensor, and (ii) orbital angular momentum flux determined by the energy-momentum tensor. The spin flux is not recognized by nowadays electrodynamics.

I am eternally grateful to Professor Robert Romer for the courageous publication of my question: "Does a plane wave really not carry spin?" [38] (was submitted on 07 October, 1999).

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The article was rejected without peer review many times.

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