A. Ben Hadj Salem. (2020) “Assuming $c < rad^2(abc)$ Implies abc Conjecture True,”
International Mathematics Research Notices, Vol. 2020, Article ID rnn999, 3 pages.
doi:10.1093/imrn/rnn999

Assuming $c < rad^2(abc)$ Implies *abc* Conjecture True

Abdelmajid Ben Hadj Salem[1]

[1]Résidence Bousten 8, Mosquée Raoudha, Bloc B, 1181 Soukra Raoudha, Tunisia

Correspondence to be sent to: abenhadjsalem@gmail.com

In this paper about the abc conjecture, assuming the condition $c < rad^2(abc)$ holds, and the constant $K(\epsilon)$ is a smooth decreasing function and having a derivative for $\epsilon \in]0,1[$, then we give the proof of the abc conjecture.

To the memory of my Father who taught me arithmetic

*To my wife **Wahida**, my daughter **Sinda** and my son **Mohamed Mazen***

1 Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 1.1. (***abc* Conjecture**): Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3}$$

$K(\epsilon)$ depending only of ϵ. □

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine, in September 2018, about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the abc conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [1]. A conjecture was proposed that $c < rad^2(abc)$ [3]. It is the key to resolve the abc conjecture. In my paper, we assume that the last conjecture holds, and the constant $K(\epsilon)$ for $\epsilon \in]0,1[$ is a smooth function. The paper is organized as follows: in the second section, we begin by presenting some proprieties of the constant $K(\epsilon)$, then we give the proof of the abc conjecture.

Received 27 October 2020; Revised 11 Month 20XX; Accepted 21 Month 20XX
Communicated by A. Editor

2 The Proof of the *abc* Conjecture

Let a, b, c positive integers relatively prime with $c = a + b, a > b, b \geq 2$. We denote $R = rad(abc), I =]0, 1[$. For $c < R$, it is trivial that the *abc* conjecture holds. In the following, we consider the triples (a, b, c) with a, b, c relatively coprime and $c > R$. As we assume that $c < R^2$, it follows that $\forall\, \epsilon \geq 1$, it suffices to take $K(\epsilon) = 1$ and c satisfies $c < K(\epsilon)R^{1+\epsilon}$ and the *abc* conjecture is true.

2.1 Proprieties of the constant $K(\epsilon)$

- From the definition of the *abc* conjecture, above, the constant $K(\epsilon)$ is a positive real number, and for every $\epsilon > 0$, it exists a number $K(\epsilon)$ dependent only of ϵ.

- In the following, we consider that $\epsilon \in I$. We can say that K is a function $K : \epsilon \in I \longrightarrow K(\epsilon) \in]0, +\infty[$, so that $c < K(\epsilon)R^{1+\epsilon}$ holds, if the *abc* conjecture is true. Assuming that $c < R^2$ is satisfying, we can adopt that $K(\epsilon = 1) = 1$, because $c < K(1)R^{1+1}$.

- We obtain that $K(\epsilon) > 1$ if $\epsilon \in I$. If not, we consider the example $9 = 8 + 1$, we take $\epsilon = 0.2$, then $c < K(0.2)R^{1.02} < 1.R^{1.2}$. But $c = 9 > 6^{1.2} \cong 8.58$, then the contradiction.

- We take one value $\epsilon \in I$, let one triplet (a, b, c) so that $c > R$ and $c < K(\epsilon)R^{1+\epsilon}$. When $\epsilon \searrow \Longrightarrow R^{1+\epsilon} \searrow$, the last formula is satisfied if $K(\epsilon) \nearrow$. If not, in the case where $K(\epsilon)$ continues in $\searrow$, it exists $c' > c$ with $rad(abc) = rad(a'b'c') = R$ that verifies $c' > K(\epsilon')R^{1+\epsilon'}$ for some $0 < \epsilon' < \epsilon$. It follows the contradiction. Let the example:

$1 + 2^5 \times 7 = 3^2 \times 5^2 \Longleftrightarrow 1 + 224 = 225 = c$, $R = rad(abc) = 2 \times 3 \times 5 \times 7 = 210 < c$

$2^5 + 7^3 = 3 \times 5^3 \Longleftrightarrow 32 + 343 = 375 = c' > c$, $rad(a'b'c') = R$, $c' - c = 150 \approx 66\%.c$.

Fix one $\epsilon_0 \in I$, and if the *abc* conjecture holds, we have $225 < K(\epsilon_0)210^{1+\epsilon_0}$, if $K(\epsilon) \searrow$ for $\epsilon < \epsilon_0$, $\exists \epsilon' < \epsilon_0$, so that $375 < K(\epsilon')210^{1+\epsilon'}$ is not satisfied. Then the contradiction.

- In 1996, A. Nitaj had confirmed that the constant $K(\epsilon)$ verifies [4]:

$$lim_{\epsilon \longrightarrow 0} K(\epsilon) = +\infty \tag{4}$$

It follows that the function $K(\epsilon)$ is a decreasing function.

2.2 The proof of the *abc* conjecture

Proof. Let us suppose that $K(\epsilon)$ is a smooth decreasing function having a derivative in every point $\in]0, 1[$. We denote :

$$Y_c(\epsilon) = LogK(\epsilon) + (1 + \epsilon)LogR - Logc \tag{5}$$

We obtain $lim_{\epsilon \to 1} Y_c(\epsilon) = 2LogR - Logc = y_1 > 0$, assuming $c < R^2$, and $lim_{\epsilon \longrightarrow 0} Y_c(\epsilon) = +\infty$. The derivative of $Y_c(\epsilon)$ gives:

$$Y_c'(\epsilon) = \frac{K'(\epsilon)}{K(\epsilon)} + LogR \tag{6}$$

We have the following cases:

i)- If $Y_c'(\epsilon) > 0$ for all $\epsilon \in]0, 1[$, then Y is an increasing function of ϵ. It follows the contradiction because $lim_{\epsilon \longrightarrow 0} Y_c(\epsilon) = +\infty$.

ii) - If $Y_c'(\epsilon) < 0$ for all $\epsilon \in]0, 1[$, then Y is a decreasing function of ϵ. It follows $\forall\, \epsilon$, $Y_c(\epsilon) > 0 \Longrightarrow c < K(\epsilon)R^{1+\epsilon}$ is satisfied. As c is an arbitrary integer with the condition $c > R$, we deduce that the *abc* conjecture is true.

iii) - We suppose that $Y_c'(\epsilon) = 0$ for some $\epsilon_0 \in]0, 1[$. This is possible because as $K(\epsilon)$ is a decreasing function, $K'(\epsilon) < 0$, so that we obtain the equation:

$$-\frac{K'(\epsilon_0)}{K(\epsilon_0)} = LogR$$

* If $Y_c(\epsilon_0)$ is positive, then $Y_c(\epsilon) > 0$. As above, we deduce that the *abc* conjecture holds.

** If $Y_c(\epsilon_0)$ is negative, then it exists two values ϵ_1, ϵ_2 with $0 < \epsilon_1 < \epsilon_0 < \epsilon_2 < 1$, so that $Y_c(\epsilon_1) = Y_c(\epsilon_2) = 0$. It follows that $c = K(\epsilon_1)R^{\epsilon_1}.rad(abc)$. Suppose that $K(\epsilon_1)R^{\epsilon_1}$ is an integer, we obtain that a, b, c are not coprime. Then the contradiction.

Then, the *abc* conjecture holds for $\forall\epsilon \in I$. ■

Q.E.D

End of the mystery!

3 Conclusion

Finally, assuming $c < R^2$, and choosing the constant $K(\epsilon)$ as a smooth function, having a derivative, we have given an elementary proof that the *abc* conjecture is true. We can announce the important theorem:

Theorem 3.1. Let a, b, c positive integers relatively prime with $c = a + b$, assuming $c < rad^2(abc)$, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{7}$$

where $K(\epsilon)$ is a constant depending only of ϵ and varying smoothly. □

Acknowledgements

The author is very grateful to Professors Mihăilescu Preda, and Gérald Tenenbaum for their comments about errors found in previous manuscripts concerning proposed proofs of the *abc* conjecture.

References

[1] Waldschmidt, M. "On the abc Conjecture and some of its consequences", presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6-9, 2013. (2013)

[2] Klaus Kremmerz, K. for Quanta Magazine. "Titans of Mathematics Clash Over Epic Proof of ABC Conjecture". *The Quanta Newsletter*, 20 September 2018. www.quantamagazine.org. (2018).

[3] Mihăilescu, P. "Around ABC". *European Mathematical Society Newsletter N*° 93, September 2014, (2014): 29-34.

[4] Nitaj, A. "Aspects expérimentaux de la conjecture *abc*". Séminaire de Théorie des Nombres de Paris (1993-1994), *London Math. Soc. Lecture Note Ser.*, Vol n°235. Cambridge Univ. Press,(1996): 145-156.