Affirmative resolve of the Riemann Hypothesis

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Abstract

Riemann Hypothesis has been the unsolved conjecture for 170 years. This conjecture is the last one of conjectures without proof in "Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse"(B. Riemann). The statement is the real part of the non-trivial zero points of the Riemann Zeta function is 1/2. Very famous and difficult this conjecture has not been solved by many mathematicians for many years. In this paper, I try to solve the proposition about the Mobius function equivalent to the Riemann Hypothesis. First, the non-trivial formula for Mobius function is proved in theorem1. In theorem3, I think this formula into 2 parts. By calculation for the latter part, I get upper bound for the sum of the mobius functions (for meaning of R.H. See theorem3).

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Handles propositions equivalent to the Riemann Hypothesis. I express the Riemann Hypothesis as R.H, and the Mobius function as $\mu(n)$.

Next theorem is well-known

Theorem .

$$\sum_{n=1}^{m} \mu(n) = O(m^{\frac{1}{2}+\epsilon}) \Leftrightarrow R.H$$

I will prove Left hand formula.

Lemma 1.

$$\sum_{n|m} \mu(n) = 1(m = 1), \sum_{n|m} \mu(n) = 0(m \neq 1)$$

Proof. First, if $m = 1$, it is $\sum_{n|m} \mu(n) = \mu(1) = 1$. Second case. There is a little explanation for this. Let m's prime factorization be $m = p_1^{n_1} p_2^{n_2} p_3^{n_3} \cdots p_k^{n_k}$. Then it becomes $\sum_{n|m} \mu(n) =_k C_0 -_k C_1 +_k C_2 -_k C_3 + \cdots _k C_k = (1-1)^k = 0$. □

Theorem 1.

$$\sum_{n \le m} \mu(n)[\frac{m}{n}] = 1$$

Proof. $\sum_{m'=1}^{m} \sum_{n|m'} \mu(n) = 1$ is from Lemma1

$$1 = \sum_{m'=1}^{m} \sum_{n|m'} \mu(n) = (\mu(1)) + (\mu(1) + \mu(2)) + (\mu(1) + \mu(3))$$

$$+(\mu(1) + \mu(2) + \mu(4)) + \cdots$$

See $\mu(n)$ in this expression as a character. $\mu(1)$ appears m times in the expression. $\mu(2)$ appears $[\frac{m}{2}]$ times that is a multiple of 2 less than m. In general, the number of occurrences of $\mu(n)(n < m)$ in this expression is the number $\left[\frac{m}{n}\right]$ that is a multiple of n below m. I get $\sum_{n \le m} \mu(n)[\frac{m}{n}] = 1$. □

example
$m = 10$ case, $10 - 5 - 3 - 2 + 1 - 1 + 1 = 1.m = 13$ case, $13 - 6 - 4 - 2 + 2 - 1 + 1 - 1 - 1 = 1$ etc..

Theorem 2.

$$\sum_{n=1}^{x} \mu(n) \textit{ changes sign at } n_0 \in [m^{\frac{1}{2}(1-\epsilon')}, m^{\frac{1}{2}}](m > \exists m_\epsilon$$

$$\sum_{n=1}^{x} \frac{m}{n}\mu(n) \textit{ changes sign at } n' \in [m'^{(1-\epsilon')}, m'](m > m' > \exists m_{\epsilon'})$$

Proof. $\sum_{n=1}^{x} \mu(n)$ changes sign in the interval $[m^{\frac{1}{2}(1-\epsilon')}, m^{\frac{1}{2}}], m > \exists m_{\epsilon'}$ ([11]). $\sum_{n=1}^{x} mn\mu(n)$ changes sign in the interval $[m'^{(1-\epsilon')}, m'], m > m' > \exists m_{\epsilon'}$ ([11]). □

Theorem 3.

$$|\sum_{n=1}^{m}\mu(n)| < Km^{\frac{1}{2}+\epsilon}$$

R.H. is got.

Proof. From theorem1

$$\sum_{n\le n_0}\mu(n)[\frac{m}{n}] + \sum_{n_0<n\le m}\mu(n)[\frac{m}{n}] = 1$$

By theorem2 $m^{\frac{1}{2}(1-\epsilon')} < n_0 < m^{\frac{1}{2}}$ is the point satisfies $\sum_{n\le n_0}\mu(n) = 0$. I take ϵ as $\epsilon \approx \frac{1}{2}\epsilon'$.

The following is obtained by calculation for $\sum_{n_0<n\le m}\mu(n)[\frac{m}{n}]$. This represents the terms coresponds to $[\sqrt{m}]$ are $\sqrt{m}$ to $m/(\sqrt{m}-1)$, the terms coresponds to $[\sqrt{m}]-1$ are $m/(\sqrt{m}-1)$ to $m/(\sqrt{m}-2)$ and the terms corresponds to 1 are $\frac{m}{2}$ to m.

$[\sqrt{m}]$ term is sum of all terms satisfy $[\frac{m}{n}] = [\sqrt{m}]-1$, $m/\sqrt{m} = \sqrt{m} \ge [\sqrt{m}]$ and $m/(m/(\sqrt{m}-1)) = \sqrt{m}-1 \ge [\sqrt{m}-1]$, $(m/(m/(\sqrt{m}-1)+1) = m(\sqrt{m}-1)/(m+\sqrt{m}-1) < \sqrt{m}-1,)$ so the range is $\sqrt{m}$ to $m/(\sqrt{m}-1)$. Next term is sum of all terms satisfy $[\frac{m}{n}] = [\sqrt{m}]-2$, $m/(m/(\sqrt{m}-2)) \ge [\sqrt{m}-2]$. The range is $m/(\sqrt{m}-1)$ to $m/(\sqrt{m}-2)$. The last term satisfy $[\frac{m}{n}] = 1$, that is $\frac{m}{2}$ to m.

$$\sum_{n_0<n\le m}\mu(n)[\frac{m}{n}] = ([m/(n_0)]-1)\times\sum_{m/([m/(n_0)])<n\le m/([m/(n_0)]-1)}\mu(n)+\cdots+$$

$$([\sqrt{m}])\times\sum_{m/(\sqrt{m}+1)<n\le m/\sqrt{m}}\mu(n)+([\sqrt{m}]-1)\times\sum_{\sqrt{m}<n\le m/(\sqrt{m}-1)}\mu(n)+([\sqrt{m}]-2)\times$$

$$\sum_{m/(\sqrt{m}-1)<n\le m/(\sqrt{m}-2)}\mu(n)+\cdots+1\times\sum_{m/2<n\le m}\mu(n)$$

By induction for Riemann Hypothesis, $|\sum_{n\le m/N}\mu(n)| < K(\frac{m}{N})^{\frac{1}{2}+\epsilon}$. I get $|\sum_{m/(N+1)<n\le m/N}\mu(n)| < K(\frac{m}{N+1})^{\frac{1}{2}+\epsilon} + K(\frac{m}{N})^{\frac{1}{2}+\epsilon}$,

I want to calculate some terms. $|\sum_{m/2\le n\le m}\mu(n)| < K(m-1)^{\frac{1}{2}+\epsilon}+1+K(\frac{m}{2})^{\frac{1}{2}+\epsilon}$, $|\sum_{m/3\le n\le m/2}\mu(n)|$is less than $K(\frac{m}{3})^{\frac{1}{2}+\epsilon}+K(\frac{m}{2})^{\frac{1}{2}+\epsilon}$.
$|\sum_{\sqrt{m}<n\le m/(\sqrt{m}-1)}\mu(n)|$ is less than $K(\frac{m}{[\sqrt{m}]})^{\frac{1}{2}+\epsilon}+K(\frac{m}{[\sqrt{m}]-1})^{\frac{1}{2}+\epsilon}$

Later, I calculate in the real examples.
example:$m = 100$ case.

$$-6 = 10 - 9 + 0 + 0 + 6 - 5 \times 2 - 4 - 3 + 2 + 1 \times 4$$

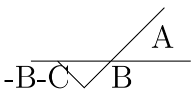
$1 \times 4 - 4 + 2 - 3$, $4 - 1 + 1 - 1 = 3$ give the almost value of $\sum_{[100/n_0]<n\leq 100} \mu(n)$. Actualy, $\sum_{9<n\leq 100} \mu(n) = 2$. This gives $|\sum_{[100/n_0]<n\leq 100} \mu(n)| < K[100/n_0] = K \times 10$,
example:$m = 10000$ case.

$$-95 = \cdots + 3 \times 18 - 2 \times 15 - 1 \times 25$$

$-1 \times 25 + 3 \times 8 - 2 \times 15 + 3 \times 10$, $-25 + 8 - 15 + 10 = -22$ gives the almost value of $\sum_{93<n\leq 10000} \mu(n) = -23$. This gives $|\sum_{[10000/n_0]<n\leq 10000} \mu(n)| < K[10000/n_0] = K \times 107$.

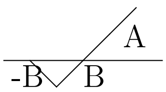
Why does the value coinside? By theorem2, $\sum_{n_0\leq n\leq n_1} \mu(n) = 0 \Leftrightarrow \sum_{n_0\leq n\leq n_1} [\frac{m}{n}]\mu(n) \approx 0, n_1 << m$.

3 terms case cover all patern of all possible case about. At first,



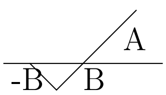
$(C > 0$,$A < K(m-1)^{\frac{1}{2}+\epsilon} + 1$,$B < (\frac{m}{2})^{\frac{1}{2}+\epsilon})$ case. I get $A < K(m-1)^{\frac{1}{2}+\epsilon} + 1 - C < Km^{\frac{1}{2}+\epsilon}$. $C \leq 0$case, I retake $B + C$ as B. Next 3 cases and 1 case can be occur.

For example, $26 \times 1 - 10 \times 2 - 2 \times 3$ case. $+ - - = 0$ case.



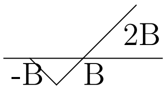
$(A < K(m-1)^{\frac{1}{2}+\epsilon} + 1$,$B < (\frac{m}{2})^{\frac{1}{2}+\epsilon})$
$A + B = \frac{(2\alpha+3\beta)}{\alpha+\beta} B$ holds. This is $A = \frac{(\alpha+2\beta)}{\alpha+\beta} B$. I supose $\alpha \neq 0$ and $A < \sqrt{3}\frac{\beta}{\alpha+\beta} B, (\frac{\alpha}{\alpha+\beta} < \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}})$. $A = |\sum_{n_0<n\leq m} \mu(n)| << Km^{\frac{1}{2}+\epsilon}$. Later, I see $\alpha = 0$ case.

For example, $17 \times 1 + 5 \times 2 - 9 \times 3$ case. $+ + - = 0$ case.

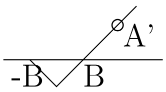


$(A + B)\frac{(\alpha+2\beta)}{\alpha+\beta} = 3B$ holds. This is $A = B(3\frac{(\alpha+\beta)}{\alpha+2\beta} - 1)$. I supose $\beta \neq 0$ and $A < \sqrt{3}B$. $A = |\sum_{n_0<n\leq m} \mu(n)| << Km^{\frac{1}{2}+\epsilon}$.

$\beta = 0$ or the former $\alpha = 0$ case.



$|\sum_{n\le m}\mu(n)| < 2K(\frac{m}{3})^{\frac{1}{2}+\epsilon}$ at best. $+--=0$ case is almost same as other cases. I treat 3 cases similarlly. In this case, for example, I prove temporaly $|\sum_{n\le\frac{3}{4}m}\mu(n)| < K(\frac{3}{4}m)^{\frac{1}{2}+\epsilon}$. $|\sum_{n\le\frac{3}{4}m}\mu(n)|$ is about less than $K\frac{3}{2}(\frac{m}{3})^{\frac{1}{2}+\epsilon}$. Next picture corresponds.



Generally, I think the case $|\sum_{n\le km}\mu(n)| < K(km)^{\frac{1}{2}+\epsilon}, (\frac{1}{2} << k << 1)$. $|\sum_{n\le km}\mu(n)|$ is about less than $K\sqrt{3k}(\frac{m}{3})^{\frac{1}{2}+\epsilon}$. This condition holds for enough small k. The proof for m (in the before 3 cases) is got by using large $M > m$. I can take $m < M < 2m$. I only use the term less than m. It does not contradivt to induction for m. For M, I think 3 terms' case. For $m < M$, I can lead $|\sum_{n\le m}\mu(n)| < Km^{\frac{1}{2}+\epsilon}$. I write down the step just in case.

For example, $12\times 1 - 12\times 2 + 4\times 3$ case. $+-+=0$ case.

$$max\{(K(\frac{m}{2})^{\frac{1}{2}+\epsilon}+K(m-1)^{\frac{1}{2}+\epsilon}+1)(1-\frac{1}{2}), -(-K(\frac{m}{3})^{\frac{1}{2}+\epsilon}-K(\frac{m}{2})^{\frac{1}{2}+\epsilon})(1-\frac{2}{3})\}$$

$$= (K(\frac{m}{2})^{\frac{1}{2}+\epsilon} + K(m-1)^{\frac{1}{2}+\epsilon}+1)(1-\frac{1}{2}) < Km^{\frac{1}{2}+\epsilon}$$

I can take 4 or more terms.
example:m=15000

$$-25 + 13\times 2 + 9\times 3 - 1\times 4 - 5\times 5 \approx 0$$

But,

$$-25 + 13\times 2 + 0\times 3$$

give almost value $\sum_{n\le m}\mu(n)$. And latter terms are opposite sign. If $+-+=0$'s first plus is 3 term, 1,2,3 terms' sum is minus. 1,2,3 terms' sum and rest of 3 and 4,5 terms' sum take opposite sign. If 6 terms case and in 1,2,3 terms $\sum_{n<x}[\frac{m}{n}]\mu(n)$ is 0 case, them the 4th term and the first term take opposite sign about. Because $\sum_{n<x}[\frac{m}{n}]\mu(n)$ is $+--=0$ case, there is change sign point, so the 4th term is minus. 1,2,3 terms' sum and 4,5,6 terms' sum take opposite sign. These cases cover all cases. So $|\sum_{n\le m}\mu(n)| < Km^{\frac{1}{2}+\epsilon}$ is got.

By calculation result, 4 or more latter terms' influence are gradually small.

First some (≥ 3) terms decide all value.
The calculation kee is $+-+=0$ or $-+-=0$. By theorem2, $\sum_{n=1}^{x} \frac{m}{n}\mu(n)$ takes 0 frequently. This suport the calculation.

example
m=10000 case.

$$-95 = 107+106+105+0-103+0+0+0-99-98-97+0-95+94-93+0-91$$
$$+0+0-88-87+86+0+0+84\times 2+0+0+81\times 2+-0+0-78+77$$
$$-76\times 2+75+74+0-72\times 2-71+70\times 2+69+68\times 2-67-66+0+0+0$$
$$+62\times 2-61+0-59-58-57\times 2+56\times 2-55\times 2+54+0-52\times 2-51-50\times 2$$
$$+49\times 3+48\times 2+47+46\times 4+45\times 2-44-43\times 3+0-41\times 2+40+0-38$$
$$+0-36\times 2-35\times 3-34+33\times 6-32-31+30\times 4+29\times 3-28\times 4-27+0$$
$$+25\times 3+0-23\times 6+22-21\times 2+0+19\times 4+18\times 7+17-16\times 9-15\times 9+14\times 9$$
$$+0+0+11\times 2+10\times 3-9\times 15+8\times 10+7\times 12-6\times 20+5\times 16-4\times 6+3\times 18-2\times 15$$
$$-1\times 25$$

By calculation, (mobius function's property, I vanish some terms.)

$$-1\times 25-2\times 15+3\times 18$$

I get $-25-15+18=-22$. $-+-=0$ is well taken as,

$$-4\times 6+5\times 16-6\times 10$$

This part's the sum of mobius function is $-6+16-10=0$.Next, $+-+=0$ is well taken as

$$7\times 2+8\times 10-9\times 15+10\times 4$$

This part's the sum of mobius function is $2+10-15+4=1$. Next $-+-=0$ is well taken as

$$-16\times 8+17+18\times 7+19\times 4-21\times 4$$

This part's the sum of mobius function is $-8+1+7+4-4=0$. Mobius function's partial sum is gradually small.
Mobius function's partial sum is $-22+1=-21$. It is the 3 terms' case.
I get enough result.

$$|\sum_{n\leq m}\mu(n)| < 0+Km^{\frac{1}{2}+\epsilon}$$

R.H. is got. □

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