Solving Brocard's Problem

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Abstract: The article provides a solution to Brocard's problem by proving the impossibility of representing other factorials, with the exception of the known three, as a product of two natural numbers with a difference of 2.

Key words: factorial, natural number, square, Diophantine equation, proof.

1 INTRODUCTION

Brocard's problem is a mathematical problem in which you need to find the integer values *m* and *n* for which $n! + 1 = m^2$. This mathematical problem was formulated by Henri Brocard in two articles in 1876 and 1885 [1, 2], later in 1913 this problem was re-presented by Srinivasa Ramanujan [3, 4]. To date, only three solutions to Brocard's problem are known: $4! + 1 = 5^2$, $5! + 1 = 11^2$ и $7! + 1 = 71^2$.

2 BROCARD'S THEOREM

Based on the results of studying Brocard's problem, we formulated the following theorem, which is called Brocard's theorem.

Theorem 2.1. *The Diophantine equation $n! + 1 = m^2$ has only three solutions m and n: (5, 4), (11, 5), (71, 7).*

Theorem 2.1 is proved by proving that the factorial can be represented as a product of two natural numbers whose difference is 2, only for three cases: 4!, 5! and 7!

In view of the above, the features of representing the square of a natural number as a product of two natural numbers, the difference of which is 2, is described below.

3 SQUARE REPRESENTATION OF NATURAL NUMBERS

Theorem 3.1. *The square of any natural number greater than 1 is expressed as the product of two adjacent natural numbers by the formula*

(3.1) $$m^2 = (m-1)(m+1) + 1.$$

From Theorem 3.1, which does not require proof in view of the obviousness, it follows that a solution to Brocard's problem exists only in the case when

(3.2) $$n! = (m - 1)(m + 1).$$

Note that equation (3.2) is another representation of the equality $n! = m^2 - 1$.

Further, if we take the notation $m - 1 = b$, then equation (3.2) will have the form

(3.3) $$n! = b(b + 2).$$

Based on equation (3.3), we can state the following: To obtain equality $n! + 1 = m^2$ the factorial must be equal to the product of two natural numbers, the difference of which is 2.

Based on Theorem 3.1 and equality (3.3), we formulate the following theorem.

Theorem 3.2. *If there is a natural number b such that* $n! = b(b + 2)$*, then we will certainly get the equality* $n! + 1 = (b + 1)^2$.

Next, we will find out in what cases the factorial can be represented as the product of two natural numbers differing from each other by 2.

4 REPRESENTATION OF A FACTORIAL AS A PRODUCT OF TWO NATURAL NUMBERS

As a result of studying the product of natural numbers, the following theorem was formulated.

Theorem 4.1. *The product of any sequentially located four natural numbers a · (a + 1) · (a + 2) · (a + 3) can be represented as a product of two natural numbers, the difference of which is 2, i.e. a · (a + 1) · (a + 2) · (a + 3) = d (d + 2), while a · (a + 3) = d and (a + 1) · (a + 2) = (d + 2).*

By considering different versions of the product of the left-hand side of the equality

(4.1) $$a \cdot (a + 1) \cdot (a + 2) \cdot (a + 3) = d(d + 2).$$

two elements each (according to the 2x2 scheme), it is not difficult to establish that if equality (4.1) exists, then there must certainly be $a \cdot (a + 3) = d$ and $(a + 1) \cdot (a + 2) = (d + 2)$, since only in this case the product of four numbers on the left side of formula (4.1) can be represented as a product of two natural numbers differing by 2.

To show what has been said, we open the brackets and get: $a^2 + 3a \ = d;\quad a^2 + 3a + 2 \ = d + 2,$ i.e. the difference between the left sides of the equations is 2. This means that Theorem 4.1 is true and it is proved.

Thus, if it is possible to represent any factorial as a product of four consecutive natural numbers, then this factorial will be represented as a product of two natural numbers differing by 2. In accordance with Theorem 3.2, this means that such a factorial plus one will be equal to square of a natural number. Based on what has been said, we formulate the following theorem.

Theorem 4.2. *The Diophantine equation $n! + 1 = m^2$ has a natural solution only if n! can be represented as a product of any sequentially located four natural numbers.*

Note that only the above three factorials, which are solutions to Brocard's problem, have a representation in the form of equality (4.1):

I) $1 \cdot 2 \cdot 3 \cdot 4 = 4 \cdot (4 + 2);$ II) $2 \cdot 3 \cdot 4 \cdot 5 = 10 \cdot (10 + 2);$ III) $7 \cdot 8 \cdot 9 \cdot 10 = 70 \cdot (70 + 2).$

To prove Theorem 4.2, we first answer the following question:

Question 4.1. *Can a factorial represented in the form $n! = b(b + 2)$ not have a representation in the form $n! = a \cdot (a + 1) \cdot (a + 2) \cdot (a + 3)$?*

The answer to Question 4.1 is based on the following obvious factorial laws. Obviously, if the factorial argument is equal to or greater than 4, then such a factorial can be represented as a product of four natural numbers, three of which are consecutive numbers, and one of the three consecutive numbers must necessarily be *n*. This means that if $n! = b(b + 2)$, then $b(b + 2)$ must match one of the following:

i. $b(b + 2) = n \cdot (n + 1) \cdot (n + 2) \cdot c$, where $c = \frac{n!}{n(n+1)(n+2)}$;

ii. $b(b + 2) = (n - 1) \cdot n \cdot (n + 1) \cdot c$, where $c = \frac{n!}{(n-1) \cdot n \cdot (n+1)}$;

iii. $b(b + 2) = (n - 2) \cdot (n - 1) \cdot n \cdot c$, where $c = \frac{n!}{(n-2) \cdot (n-1) \cdot n}$.

Based on Theorem 3.2 and the above formulas, we can say the following.

a) First, we proved that for the Diophantine equation $n! + 1 = m^2$ to have a solution, the factorial must be the product of two natural numbers, the difference of which is 2, i.e. we know which factorial we need;

b) Second, we know that any factorial is greater than 4! can be represented as a product of four natural numbers, three of which are consecutive numbers, and one of them is equal to the factorial argument.

It turns out that we do not know only one of the four elements, so we will answer the following question:

Question 4.2: *If the product of two natural numbers of the form $b(b+2)$ can be represented as a product of four natural numbers, and three of them are consecutive natural numbers, then what should be the fourth factor?*

The answer to Question 4.2 is Theorem 4.1, i.e. the fourth factor must be one of four consecutive natural numbers, i.e. should be $b(b+2) = a \cdot (a+1) \cdot (a+2) \cdot (a+3)$.

Thus, if the factorial has a representation as a product of two natural numbers, the difference of which is 2, i.e. $n! = b(b+2)$ then this factorial certainly has a representation as a product of four consecutive natural numbers $n! = a \cdot (a+1) \cdot (a+2) \cdot (a+3)$, while one of factors must be equal to the factorial argument, i.e. Theorem 4.2 is true and it is proved.

As shown above, if

(4.2) $$n! = a \cdot (a+1) \cdot (a+2) \cdot (a+3),$$

then this equality is equivalent to the equality $n! = b(b+2)$, while the condition must be satisfied: $a \cdot (a+3) = b$ и $(a+1) \cdot (a+2) = (b+2)$.

If

(4.3) $$n(n-1)! = a \cdot (a+1) \cdot (a+2) \cdot (a+3),$$

then it is obvious that

(4.4) $$(n-1)! = \frac{a \cdot (a+1) \cdot (a+2) \cdot (a+3)}{n}.$$

Hence it follows that if $n! = b(b+2)$, then equality (4.4) must be satisfied.

Thus, in order to solve Brocard's problem, you need to answer the question:

Question 4.3: *Why is it impossible to represent other factorials (with the exception of the known three) as the product of two natural numbers, the difference of which is 2?*

The answer to this question is the following pattern:

If $n > 7$, then $\frac{a\cdot(a+1)\cdot(a+2)\cdot(a+3)}{n} < (n-1)!$

For example, if *n* = 8, then $\frac{8\cdot(8+1)\cdot(8+2)\cdot(8+3)}{8} = 990,\ \ 990 < (8-1)! = 5040.$

Obviously, with increasing *n*, the difference $(n-1)! - \frac{a\cdot(a+1)\cdot(a+2)\cdot(a+3)}{n} = S$ will also increase, and very strongly. This means that if $n > 7$, then the inequality $\frac{a\cdot(a+1)\cdot(a+2)\cdot(a+3)}{n} < (n-1)!$

Conclusion

It is proved that the Diophantine equation $n! + 1 = m^2$ can have a solution only if the factorial can be represented as a product of two natural numbers, the difference of which is 2. Then it is proved that the products of four consecutive natural numbers can be represented as the product of two natural numbers with a difference of 2. Further, it is proved that it is impossible to represent the factorial (except for the known three) as a product of four consecutive natural numbers. Thus, it has been proved that the above Diophantine equation can have only three solutions *m* and *n*: (5, 4), (11, 5), (71, 7), since only in these three cases the factorial can be represented as a product of four sequentially located natural numbers.

References

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