Evaluating the Alignment of the Polarized Starlight from 99 Stars in a Region off the Disk of the Milky Way

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Abstract

Detecting polarized starlight projects an intriguing pattern of polarization directions on the Galaxy. Polarized starlight is a well known tracer of Galactic Magnetic fields and acts as a tool for understanding the electrodynamics of the dust that contaminates the view of more distant objects. Here, the alignment of the polarization directions of a sample of stars well off the Galactic Disk is investigated with a recently devised test. The Hub Test offers numerical metrics based on the geometry of spherical geodesics, *i.e.* great circles, to judge alignment. By comparing the directions of two vectors at a single point, the test avoids the issues related to parallel transport. The sample of 99 stars, located from longitude 15° to 35° and latitude 23° to +35°, is among the most highly aligned regions. The alignment function provides a full-sphere depiction of the collective alignment. The metrics include the likelihood that random polarization directions would produce equal or better alignments. For the sample considered here, the alignment occurs at the 20σ level, far away from random alignments. The source of the polarization data is the Heiles 2000 agglomeration catalog appended with data from the Berdyugin 2014 catalog. This article is a Mathematica notebook which can be accessed and run via a link in the References.

Keywords: Polarized Starlight; Alignment; Computer Program; Uncertainties; Hub Test; Galactic Structure; Galactic Magnetic Field

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In[1]:=
```
Print["The date and time that this statement was evaluated: ", Now]
```

```
The date and time that this statement was evaluated:  Sun 11 Apr 2021 16:07:50 GMT-4.
```

0. Preface

The pdf version of this notebook is available online from the viXra archive. To find the ready-to-run notebook follow the link in Ref. 1.

Notes:

(1) The pdf version of this notebook reflects a large number of uncertainty runs that consumed considerable computer time. The ready-to run notebook is set up to generate fewer uncertainty runs. [Experimental uncertainties produce uncertainties in the results. The “uncertainty runs” follow the process of alignment evaluation but with polarization directions allowed to vary in a way that reflects the uncertainties in measurement.]

(2) The pdf version quotes some numerical values that are associated with the particular settings and uncertainty runs that were current when the pdf version was created. Other sets of uncertainty runs should alter those numerical values only slightly.

(3) A template for performing calculations similar to those in this notebook, but with other data, can be found online, Ref. 2. Or alter this notebook.

(4) These notebooks were created using Wolfram Mathematica, Version Number: 12.1, Ref. 3.

(5) The formulas for creating Aitoff plots were found on Wikipedia, Ref. 4.

The Hub Test

This notebook presents an application of the Hub Test, which is discussed more fully in Ref. 5. The basic idea is that polarization directions are well-aligned with each other when they are well-aligned with some point on the Celestial Sphere.

Consider the well-known prescription for finding Polaris, the North Star, based on the alignment of the direction from the Merak to Dubhe with Polaris. Guided by Fig. 1, let the source S be the star Merak, take the interval from Merak to Dubhe in place of the direction of polarization $\hat{v}_{\psi}$, and let Polaris be the point H. Then the alignment of the Merak to Dubhe direction $\hat{v}_{\psi}$ with Polaris, the point H, illustrates the concept of alignment in the Hub Test. With Merak as S, Merak-Dubhe as $\hat{v}_{\psi}$, and Polaris as H, the angle η would be about $\eta = 3.47°$. In that case, the blue great circle and the purple great circle in Fig. 1 would almost coincide.

Out[∘]=

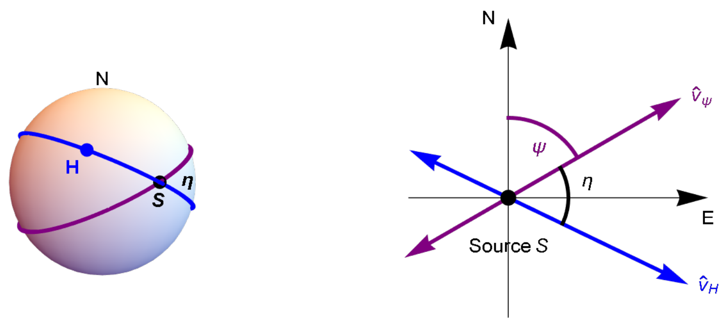

Figure 1: The Celestial sphere is pictured on the left and on the right is the plane tangent to the sphere at the source S. The linear polarization direction $\hat{v}_{\psi}$ lies in the tangent plane and determines the purple great circle on the sphere. A point H on the sphere and the point S determine a second great circle, the blue circle drawn on the sphere at the left. Clearly, H and S must be distinct in order to determine a great circle.

In Fig. 1, we select the acute angle η between the great circles at S, $0° \leq \eta \leq 90°$. This "alignment angle" η measures the alignment of the polarization direction $\hat{v}_{\psi}$ with the point H. Perfect alignment occurs when $\eta = 0°$ and the two great circles overlap. Perpendicular great circles, $\eta = 90°$, indicates maximum "avoidance" of the polarization direction $\hat{v}_{\psi}$ with the point H on the sphere. The halfway value, $\eta = 45°$, favors neither alignment nor avoidance.

With N sources S_i, $i = 1, ..., N$, there are N alignment angles η_{iH} for the point H and an average alignment angle $\overline{\eta}$ at H,

$$\overline{\eta}(\mathrm{H}) = \frac{1}{N} \sum_{i=1}^{N} \eta_{\mathrm{iH}} \, . \qquad (1)$$

The alignment angle $\overline{\eta}$(H) is a function of position H on the sphere. It is symmetric across diameters, $\overline{\eta}(\mathrm{H}) = \overline{\eta}(\mathrm{-H})$, because great circles are symmetric across diameters.

The function $\overline{\eta}$(H) measures convergence and divergence of the great circles determined by the polarization directions. For random polarization directions, the average $\overline{\eta}$(H) should be near 45°, since each alignment angle η_{iH} is acute, $0° \leq \eta_{\mathrm{iH}} \leq 90°$, and random polarization directions should not favor any one value. Points H where the alignment angle $\overline{\eta}$(H) is smaller than 45°, the great circles tend to converge, where $\overline{\eta}$(H) is larger than 45°, the great circles can be said to diverge.

Thus the basic concept includes "avoidance", as well as alignment. Avoidance is high when the two directions $\hat{v}_{\psi}$ and $\hat{v}_{H}$ differ by a large angle, $\eta \to 90°$. Perpendicular great circles at S , $\eta = 90°$, would indicate the maximum avoidance of the polarization direction and the point on the sphere. The N sources' polarization directions most avoid the points $H_{\max}$ and $-H_{\max}$ where the function $\overline{\eta}$(H) takes its maximum value $\overline{\eta}_{\max}$. The locations of the most extreme divergence are called "avoidance hubs".

The N sources' polarization directions are best aligned with the points $H_{\min}$ and $-H_{\min}$ where the alignment angle is a minimum $\overline{\eta}_{\min}$. The locations $H_{\min}$ and $-H_{\min}$ of their most extreme convergence are called "alignment hubs". Alignment and avoidance are equally viable, complementary concepts with the Hub Test.

The Hub test provides many calculated results to describe the collective behavior of the polarization directions in a sample. The alignment angle function $\overline{\eta}$(H), Eq. (1), can be mapped on the Celestial Sphere to give a visual display. The smallest alignment angle $\overline{\eta}_{\mathrm{min}}$ and the largest avoidance angle $\overline{\eta}_{\mathrm{max}}$ quantify the agreement of the directions. Known formulas, see Sec. 4 below, are available to calculate the significance of the alignment, *i.e.* the likelihood that random polarization directions would yield better results. The locations of the convergence hubs H_{min} and the divergence hubs H_{max} may provide clues to magnetic field direction and such quantities.

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References

1. Introduction

For those interested in the structure of the Milky Way, polarized starlight infers the direction of the Galactic magnetic field, see for example, Refs. 6 & 7. For those interested in deep space objects on the far side of the Galaxy, polarized starlight helps uncover the physics of the contaminating dust that obscures the objects of interest, see, for example, Refs. 8 & 9.

The Hub Test, described briefly in the Preface, supplies several quantitative measures that may be helpful in understanding the implications of the polarization directions of a given sample.

This work looks at a very significantly aligned sample of 99 stars occupying a region about 30° off the Galactic Disk. The stars' polarization directions are aligned at the 20σ level, with the chance that the alignment is random being nil. The alignment is quite well known and not surprising. Yet evaluating the correlations numerically requires suitable tests. Analyzing this sample with the

Hub Test illustrates its numerical metrics.

Certainly, alignment is an important characteristic. However, one aspect of collective behavior that is often overlooked is the concept of avoidance. It may be useful to know where the polarization directions do-not-point, *i.e.* avoidance, as well as where the polarization directions do-point, *i.e.* alignment.

The sample is chosen because it is off the Galactic Disk and has extreme alignment behavior. Also, an accident of nature has put the alignment direction closely coincident with the Celestial Equator.

The known distances to many of the stars is information that is not utilized. All stars, no matter how distant, are plotted, planetarium-like, on the 2D Celestial Sphere. Some preliminary formulas and the construction of the grid are presented in Sec. 2. The grid is a 2°x2° mesh of 10518 grid points that maintains equal spacings at high latitudes. The needed stellar data is taken from the catalogs Heiles 2000 and Berdyugin 2014, Ref. 10 - 13. Cuts were made for the % polarization, %p $\geq$ 0.1%, the polarization direction uncertainty, $|\sigma\psi| \leq 7°$, and the fractional uncertainty in %polarization, σp/p $\leq$ 0.25 . To keep this notebook self-contained, the needed data is included in Sec. 3. A list of identifying numbers for the stars is provided for those who wish to check the stellar data. Sec. 4 has the probability and significance formulas imported from previous work with the Hub Test.

Sec. 5 presents the analysis of the “best” polarization directions, where “best” indicates the values listed in the catalog as the observed polarization directions. One finds values for the smallest alignment angle $\overline{\eta}_{\text{min}}$, the largest avoidance angle $\overline{\eta}_{\text{max}}$, and the locations hubs on the sphere where these extreme alignment angles are found. The uncertainty in the statistics formulas give the significance of these results some uncertainty.

Inevitably, but importantly, one must estimate the uncertainties in the numerical results. The uncertainty for each measured polarization direction is supplied in the catalogs. The task of estimating uncertainties occupies Sec. 6 and Sec. 7 finishes the article with some concluding remarks.

2. Coordinates, grid, and sundry basic formulas

2a. Coordinates

Consider the “Celestial Sphere”, a sphere in 3 dimensional Euclidean space. See Fig. 1 in the Preface. The sphere is also called the “sphere” or sometimes “the sky”. The center of the sphere is the origin of a 3D Cartesian coordinate system with coordinates (x, y, z). The direction of the positive z -axis is due “North”. Galactic longitude, gLON and latitude, gLAT, are measured as in the Heiles 2000 catalog with the direction of the positive x-axis along (gLON,gLAT) = (0°, 0°). The similar appearance of the letter “*l*” and the number “1” when typed keep us from using the (l,b) notation in this computer program.

The view of the Galaxy is generally from inside the sphere, let us say from the origin to be specific. Then the direction of increasing gLON, *i.e.* local East, is to the left with up toward North. Latitude gLAT = 90° indicates the North Galactic Pole, the direction from the origin (0,0,0) to (0,0,1) . We do not use the conventional *UVW* notation.

Somewhat contrarily, from a point-of-view located outside the sphere, as in the sketch in Fig. 1, one pictures a source S plotted on the sphere and, in the 2D tangent plane at S, local North is upward and local East is to the right. A “position angle” at the point S on the sphere, such as the angle ψ in Fig. 1, is measured in the 2D plane tangent to the sphere at S. In the tangent plane as drawn in Fig.1, the position angle ψ is measured clockwise from local North with East to the right.

It is important to note that from a point of view inside the sphere, position angles are measured counterclockwise from North, since increasing gLON, *i.e.* East, is to the left when viewed from inside the sphere. But it is much easier to draw a sphere from the outside looking inward, as with Fig. 1.

Definitions:

er, eN, eE are unit vectors in a 3D Cartesian coordinate system

(gLON,gLAT) = galactic longitude and latitude
er(gLON,gLAT) = radial unit vectors from Origin
eN(gLON,gLAT) = local North at a point on the Celestial Sphere
eE(gLON,gLAT) = local East at a point on the Celestial Sphere
gLONFROMr(er) = gLON determined by radial unit vector er
gLATFROMr(er) = gLAT determined by radial unit vector er

Aitoff Plot Functions
αH(gLON,gLAT) , xH(gLON,gLAT) , yH(gLON,gLAT), where xH is centered on gLON = 0 and gLON increases from left-to-right.
xH180(gLON,gLAT) , yH180(gLON,gLAT), where xH is centered on gLON = 180° and gLON increases from left-to-right.
xHGal(gLON,gLAT) , yHGal(gLON,gLAT), where xH is centered on gLON = 0 and gLON increases from right-to-left, so gLON = +180° is on the left and gLON = -180° is to the right.

```
In[2]:= (* For a Source at (gLON,gLAT) = (gLON,gLAT): er, eN,
        eE are unit vectors from Origin to Source, local North, local East, resp.  *)
        er[gLON_, gLAT_] := er[gLON, gLAT] = {Cos[gLON] Cos[gLAT], Sin[gLON] Cos[gLAT], Sin[gLAT]}
        eN[gLON_, gLAT_] := eN[gLON, gLAT] = {-Cos[gLON] Sin[gLAT], -Sin[gLON] Sin[gLAT], Cos[gLAT]}
        eE[gLON_, gLAT_] := eE[gLON, gLAT] = {-Sin[gLON], Cos[gLON], 0}
        {"Check er.er = 1, er.eN = 0, er.eE = 0, eN.eN
           = 1, eN.eE = 0,eE.eE = 1, erXeE = eN, eEXeN = er, eNXer = eE: ",
         {0} == Union[Flatten[Simplify[{er[gLON, gLAT].er[gLON, gLAT] - 1, er[gLON, gLAT].eN[gLON, gLAT],
              er[gLON, gLAT].eE[gLON, gLAT], eN[gLON, gLAT].eN[gLON, gLAT] - 1, eN[gLON, gLAT].
                eE[gLON, gLAT], eE[gLON, gLAT].eE[gLON, gLAT] - 1, Cross[er[gLON, gLAT], eE[gLON, gLAT]] -
                eN[gLON, gLAT], Cross[eE[gLON, gLAT], eN[gLON, gLAT]] - er[gLON, gLAT],
              Cross[eN[gLON, gLAT], er[gLON, gLAT]] - eE[gLON, gLAT]}]]]}

Out[5]= {Check er.er = 1, er.eN = 0, er.eE = 0, eN.eN = 1,
          eN.eE = 0,eE.eE = 1, erXeE = eN, eEXeN = er, eNXer = eE: , True}
```

Get (gLON,gLAT) in radians from a radial vector r:

```
In[6]:= gLONFROMr[r_] := N[ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] ≥ 0 && r[[1]] > 0)
        gLONFROMr[r_] := N[π - ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] ≥ 0 && r[[1]] < 0)
        gLONFROMr[r_] := N[-π + ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] < 0 && r[[1]] < 0)
        gLONFROMr[r_] := N[-ArcTan[Abs[r[[2]] / r[[1]]]]] /; (r[[2]] < 0 && r[[1]] > 0)
        gLONFROMr[r_] := π / 2. /; (r[[2]] ≥ 0 && r[[1]] == 0)
        gLONFROMr[r_] := -π / 2. /; (r[[2]] < 0 && r[[1]] == 0)

In[12]:= gLATFROMr[r_] := N[ArcTan[r[[3]] / (√ (r[[1]]^2 + r[[2]]^2))]] /; (√ (r[[1]]^2 + r[[2]]^2) > 0)
         gLATFROMr[r_] := Sign[r[[3]]] (π / 2.) /; (√ (r[[1]]^2 + r[[2]]^2) == 0)
```

The following Aitoff Plot formulas can be found in Wikipedia, Ref. 4.
For these formulas the angles gLON and gLAT should be in degrees.
They give an Aitoff Plot that is centered on (0°,0°)

```
In[14]:= αH[gLON_, gLAT_] := αH[gLON, gLAT] = ArcCos[Cos[((2. π) / 360.) gLAT] Cos[((2. π) / 360.) gLON / 2.]]
         xH[gLON_, gLAT_] :=
          xH[gLON, gLAT] = (2. Cos[((2. π) / 360.) gLAT] Sin[((2. π) / 360.) gLON / 2.]) / Sinc[αH[gLON, gLAT]]
         yH[gLON_, gLAT_] := yH[gLON, gLAT] = Sin[((2. π) / 360.) gLAT] / Sinc[αH[gLON, gLAT]]
```

Using the following functions produces an Aitoff Plot that is centered on (180°,0°)

In[17]:=

```
xH180[gLON_, gLAT_] := xH180[gLON, gLAT] =
  (2. Cos[((2. π) / 360.) gLAT] Sin[((2. π) / 360.) (gLON - 180.) / 2.]) / Sinc[αH[(gLON - 180.), gLAT]]
yH180[gLON_, gLAT_] := yH180[gLON, gLAT] = Sin[((2. π) / 360.) gLAT] / Sinc[αH[(gLON - 180.), gLAT]]
```

For Galactic Coordinates, the following functions produces an Aitoff Plot that is centered on gLON = 0° and the gLON axis runs from +180° on the left to 0° at the center to -180° on the right. The viewpoint is inside the Celestial Sphere, looking out.

In[19]:=

```
(*The plots of the sky in Galactic coordinates have the gLON axis running from +
 180° on the left to -180° on the right. Angles gLON and gLAT are in degrees*)
xHGal[gLON_, gLAT_] := xHGal[gLON, gLAT] =
  (2. Cos[((2. π) / 360.) gLAT] Sin[-(2. π / 360.) gLON / 2.]) / Sinc[αH[-gLON, gLAT]]
yHGal[gLON_, gLAT_] := yHGal[gLON, gLAT] = Sin[((2. π) / 360.) gLAT] / Sinc[αH[-gLON, gLAT]]
```

2b. Grid

We avoid bunching at the poles by taking into account the diminishing radii of constant latitude circles as the latitude approaches the poles. Successive grid points along any latitude or along any longitude make an arc that subtends the same central angle dθ.

We grid one hemisphere at a time, then the grids are combined.

Definitions:

gridSpacing	separation in degrees between grid points on and between constant latitude circles
dθ1	grid spacing in radians
idN, ai, ji	dummy indices, ID #s for grid points, longitude, latitude
gLONpointH,gLATpointH	gLON and gLAT of the grid points H_j
grid, gridN, gridS	tables data associated with grid points, listings are below
nGrid	number of grid points
gLONGrid	longitudes at the grid points ($-\pi \leq$ gLON $\leq +\pi$)
gLATGrid	latitudes at the grid points ($-\pi/2 \leq$ gLON $\leq \pi/2$)
rGrid	radial unit vectors from origin to grid points, in 3D Cartesian coordinates

Tables: ***grid, gridN*** **and** ***gridS***

1. sequential point # 2. gLON index 3. gLAT index 4. gLON (rad) 5. gLAT (rad) 6. Cartesian coordinates of the grid point

In[21]:=

```
gridSpacing = 2.(*, in degrees.*);
```

In[22]:=
```
(*KEEP this cell - DO NOT DELETE*)
(*The Northern Grid "gridN". *)
dθ1 = ((2. π) / 360.) gridSpacing;
(*Convert gridSpacing to radians*)gridN = {};
idN = 1;
For[gLATj = 0., gLATj < π / (2. dθ1), gLATj++, gLATpointH = gLATj dθ1;
 For[ ai = 0., ai < Ceiling[((2. π) / dθ1) (Cos[gLATpointH] + 0.01)],
  ai++, gLONpointH = ai dθ1 / (Cos[gLATpointH] + 0.01);
  AppendTo[gridN, {idN, ai, gLATj, gLONpointH, gLATpointH, er[gLONpointH, gLATpointH]}];
  idN = idN + 1
 ]]
```

In[24]:=
```
(*KEEP this cell - DO NOT DELETE*)
(*The Southern Grid "gridS". *)
dθ1 = ((2. π) / 360.) gridSpacing;(*Convert gridSpacing to radians*)
gridS = {}; idN = 1;
For[gLATj = 1., gLATj < π / (2. dθ1), gLATj++, gLATpointH = -gLATj dθ1;
 For[ ai = 0., ai < Ceiling[((2. π) / dθ1) (Cos[gLATpointH] + 0.01)],
  ai++, gLONpointH = ai dθ1 / (Cos[gLATpointH] + 0.01);
  AppendTo[gridS, {idN, ai, gLATj, gLONpointH, gLATpointH, er[gLONpointH, gLATpointH]}];
  idN = idN + 1
 ]]
```

In[27]:=
```
(*KEEP this cell - DO NOT DELETE*)
grid = {}; j = 1;
For[jN = 1, jN ≤ Length[gridN], jN++, AppendTo[grid, {j, gridN[[jN, 2]], gridN[[jN, 3]],
   gLONFROMr[gridN[[jN, 6]] ], gLATFROMr[gridN[[jN, 6]] ], gridN[[jN, 6]]}];
 j = j + 1]
For[jS = 1, jS ≤ Length[gridS], jS++, AppendTo[grid, {j, gridS[[jS, 2]], gridS[[jS, 3]],
   gLONFROMr[gridS[[jS, 6]] ], gLATFROMr[gridS[[jS, 6]] ], gridS[[jS, 6]]}];
 j = j + 1]
```

In[30]:=
```
nGrid = Length[grid];
```

In[31]:=
```
gLONGrid = Table[grid[[j, 4]] , {j, nGrid}];
gLATGrid = Table[grid[[j, 5]] , {j, nGrid}];
rGrid  = Table[grid[[j, 6]] , {j, nGrid}];
```

2c. The mean and standard deviation are convenient functions. And we identify directories for getting and putting data.

Definitions

mean   the arithmetic average of a set of numbers, $\frac{1}{N}\sum_{i=1}^{N} n_i$

stanDev   the standard deviation. Given a set of *N* numbers n_i with mean value *m*, the standard deviation is $\left(\frac{1}{N}\sum_{i=1}^{N}(n_i - m)^2\right)^{1/2}$, the square root of the average of the squares of the differences of the numbers with the mean. Note that we divide by *N* to get the average of the deviations squared.

catalogDirectory   directory containing the catalog files
homeDirectory   directory containing the notebook and data files

In[34]:=
```
mean[data_] := (1 / Length[data]) Sum[data[[i4]], {i4, Length[data]}];
(* arithmetic average *)
stanDev[data_] :=
 ((1 / Length[data]) Sum[(data[[i5]] - mean[data])^2, {i5, Length[data]}])^(1/2)
 (*standard deviation*)
```

In[36]:=
```
catalogDirectory =
  "C:\\Users\\shurt\\Dropbox\\HOME_DESKTOP-0MRE5OJ\\SendXXX_CJP_CEJPetc\\SendViXra\\
    20210221StellarPolarization\\20210221Catalog";
 (* location of the catalog data file on my computer*)
```

In[37]:=
```
homeDirectory =
  "C:\\Users\\shurt\\Dropbox\\HOME_DESKTOP-0MRE5OJ\\SendXXX_CJP_CEJPetc\\SendViXra\\
    20210221StellarPolarization\\20210221Notebooks\\20210228GalacticCoordsNotebooks\\
    20210320Lon30Lat30offDisk";
 (*The notebook file and data files for this notebook are put in this directory. *)
```

Section Summary

In[38]:=
```
Print["The grid points are separated by gridSpacing = ",
 gridSpacing, "° arcs along latitude and longitude."]
Print["The number of grid points is ", nGrid, " ."]
```

```
The grid points are separated by gridSpacing = 2.° arcs along latitude and longitude.

The number of grid points is 10518 .
```

3. Polarization and Position Data

Definitions:

cat the catalog data, Heiles 2000 Ref. 10 combined with Berdyugin 2014 Ref. 12

allClumpsofStarsIDsInCatalog record numbers of the stars in the catalog for all clumps

clumpOfStarsIDinCatalog record numbers of the sample’s stars in the catalog (we treat this clump)

nSrc number of stars

gLONSrc galactic longitude (radians)

gLATSrc galactic latitude (radians)

ψn PPA, polarization position angle: counterclockwise from North with East to the left, as seen from inside the Celestial Sphere.

$\sigma\psi$n uncertainty in PPA

percentPol percentage of linear polarization

rSrc unit vector from Origin to Sources on Celestial Sphere

eNSrc Local North at the ith Source

eESrc Local East at the ith Source

sourceCenter unit radial vector to the arithmetic center of the sources

angleSourceToCenter arc from Source to Center

showClump1 map of significance for alignments in the catalog, needed to discuss sample selection

Catalog data

The HD or BD numbers for the stars in the sample are given in the following cell.
Most records can be found by searching the Heiles 2000 or the Berdyugin 2014 catalogs for the HD or BD number.
Just one star had neither an HD nor a BD number. The exceptional star is record # 5361 in the Heiles 2000 catalog with (gLON,gLAT) = (16.726632 hours, 6.0061°). To find record # 5361, one can search the Heiles 2000 catalog for the dec.RA entry which is “60061.16726632” .

In[40]:=
```
starIDnumbers = {{"HD", 151061.`}, {"HD", 154445.`}, {"HD", 155195.`}, {"HD", 156247.`},
   {"HD", 152126.`}, {"HD", 150764.`}, {"HD", 145085.`}, {"HD", 152974.`}, {"HD", 151219.`},
   {"HD", 157999.`}, {"HD", 152310.`}, {"HD", 152087.`}, {"HD", 153115.`}, {"HD", 145892.`},
   {"HD", 150752.`}, {"HD", 151026.`}, {"HD", 151812.`}, {"HD", 153147.`}, {"HD", 152067.`},
   {"HD", 152466.`}, {"HD", 152897.`}, {"Heiles 2000", "Record # 5361"}, {"HD", 146815.`},
   {"HD", 151828.`}, {"HD", 158836.`}, {"HD", 160140.`}, {"HD", 153033.`}, {"HD", 157278.`},
   {"HD", 151556.`}, {"HD", 150873.`}, {"HD", 155593.`}, {"HD", 151494.`},
   {"HD", 153303.`}, {"HD", 152532.`}, {"HD", 156130.`}, {"HD", 153272.`},
   {"HD", 160311.`}, {"HD", 156655.`}, {"HD", 151291.`}, {"HD", 155500.`},
   {"HD", 156404.`}, {"HD", 154762.`}, {"HD", 153797.`}, {"HD", 156732.`},
   {"HD", 154619.`}, {"HD", 154302.`}, {"HD", 155644.`}, {"HD", 156681.`},
   {"HD", 153540.`}, {"HD", 155422.`}, {"HD", 153835.`}, {"HD", 152447.`},
   {"HD", 159082.`}, {"HD", 159005.`}, {"HD", 157606.`}, {"BD", 13.328`}, {"HD", 151627.`},
   {"HD", 151072.`}, {"HD", 151545.`}, {"HD", 159119.`}, {"HD", 155581.`},
   {"HD", 154512.`}, {"HD", 152308.`}, {"HD", 153898.`}, {"BD", 15.3104`},
   {"BD", 15.3101`}, {"HD", 157741.`}, {"HD", 151203.`}, {"HD", 148035}, {"HD", 148512},
   {"HD", 146047}, {"HD", 147510}, {"HD", 149755}, {"HD", 149755}, {"HD", 147189},
   {"HD", 149413}, {"HD", 146026}, {"HD", 145568}, {"HD", 146561}, {"HD", 148622},
   {"HD", 147548}, {"HD", 148229}, {"BD", {"BD+10", 3004}}, {"HD", 150305},
   {"HD", 147252}, {"HD", 147836}, {"HD", 150123}, {"HD", 151059}, {"HD", 150268},
   {"HD", 151879}, {"HD", 147868}, {"HD", 150905}, {"HD", 150257}, {"HD", 148765},
   {"HD", 153225}, {"HD", 150830}, {"HD", 150568}, {"HD", 152155}, {"HD", 153301}};
```

For example, the Heiles 2000 catalog listing for the first star in the sample, Record # 4698, HD151061. :
{-30849.16753181 151061.0 -2.424200-999.900000-999.900000 2.390 0.035 87.4 0.4 144.8 14.3393 26.1434 0.60 -0.1 1 7.2 199.5 M6III 0000000000001000000000 10}

The combined Heiles 2000 and Berdyugin 2014 data file that we use has the Heiles data first followed by Berdyugin 2014 data. The Heiles 2000 part of the file contains the original unaltered catalog entries, except that the declination and Right Ascension have been separated and the object’s record number is appended to each record.
The Berdyugin 2014 catalog data requires some work to get it into the same form as the Heiles 2000 catalog. Some 39 stars appear in both catalogs and are deleted from the Berdyugin 2014 catalog.
We kept the 399 stars in the Berdyugin catalog that do not have polarization directions.
Also, the polarization direction in the Berdyugin 2014 catalog need to be converted from Equatorial to Galactic coordinates.
Once determined, the data was rearranged to conform to the Heiles 2000 catalog format. Any unknown quantities were flagged as “-999”, as in the Heiles 2000 catalog. The Bredyugin 2014 data is appended to the Heiles 2000 catalog, increasing the star count from 9286 to 11647 stars.

----

1. Declination (deg) 2 RA (hr) 3. HD number 4. Bonner DM number 5. Cordoba DM number 6. Cape DM number 7. Percentage polarization (%) 8. rms uncertainty on Pol (%) 9. Position angle, equatorial (deg.) 10. rms uncertainty on PA (deg.) 11. Position angle, Galactic (deg.) 12. Galactic longitude (deg.) 13. Galactic latitude (deg.) 14. Reddening

(mag.) 15. Discrepancy between PA and PAgal (deg.) 16. Primary stellar database 17. Visual magnitude (mag.) 18. Distance (pc) 19. Spectral type 20. Polarization catalog numbers 21. Distance catalog 22. Object # in the catalog

----

See the ReadMe files in Refs. 11 & 13 for details.

In[41]:=
```
(*galactic longitude in radians, rounded to six places*)
(*gLONSrc = Table[cat[[i,12]] (2.π/360.),{i,clumpOfStarsIDinCatalog}];*)
gLONSrc = 10^-6. {250 268, 336 777, 349 877, 396 797, 371 553, 355 747, 264 845, 392 120, 373 212,
   467 158, 393 484, 395 614, 416 793, 306 326, 383 883, 390 549, 404 983, 424 726, 410 660,
   422 532, 429 238, 402 380, 339 266, 419 410, 513 891, 530 055, 442 034, 497 609, 423 139,
   413 503, 480 505, 426 869, 452 972, 443 167, 493 094, 456 224, 544 635, 502 568, 433 076,
   499 134, 520 949, 508 835, 505 271, 558 219, 532 561, 530 339, 547 761, 565 138, 527 496,
   551 098, 535 399, 520 166, 610 464, 609 671, 619 517, 574 278, 553 250, 546 288, 553 193,
   653 334, 616 906, 611 382, 587 635, 608 045, 618 923, 616 325, 660 516, 589 513, 270 352,
   285 536, 290 074, 291 470, 314 683, 314 683, 338 245, 342 259, 345 575, 348 193, 366 694,
   396 015, 405 091, 406 662, 436 856, 445 059, 460 243, 460 941, 464 083, 468 970, 471 762,
   479 093, 487 121, 521 155, 524 297, 528 660, 580 147, 582 940, 588 874, 601 790, 602 662};
```

In[42]:=
```
nSrc = Length[gLONSrc];
```

In[43]:=
```
(*galactic latitude in radians, rounded to six places*)
(*gLATSrc=  Table[cat[[i,13]] (2.π/360.),{i,clumpOfStarsIDinCatalog}];*)
gLATSrc =
  10^-6. {456 288, 400 235, 383 845, 376 501, 488 809, 523 040, 650 730, 473 078, 516 980, 362 236,
   494 295, 502 109, 481 468, 648 983, 537 656, 531 592, 514 558, 484 264, 509 903, 503 908,
   493 726, 542 429, 640 304, 520 890, 361 623, 336 287, 495 586, 398 525, 531 022, 547 522,
   437 666, 534 664, 493 298, 511 650, 428 199, 495 468, 338 716, 417 793, 543 225, 449 026,
   433 278, 475 457, 502 053, 440 561, 489 956, 498 113, 466 692, 444 870, 518 293, 474 419,
   513 689, 547 677, 399 743, 401 410, 443 712, 534 818, 582 830, 596 885, 586 129, 416 671,
   496 506, 524 756, 577 118, 540 722, 531 568, 538 403, 455 449, 608 338, 569 675, 560 949,
   637 045, 596 204, 534 769, 534 769, 629 017, 559 378, 663 923, 677 188, 659 211, 611 040,
   647 866, 629 366, 635 998, 579 449, 678 235, 662 702, 593 063, 566 010, 592 016, 545 066,
   671 254, 592 365, 613 134, 658 338, 549 779, 617 148, 626 224, 586 082, 555 015};
```

In[44]:=
```
(* galactic position angle in radians, rounded to six places*)
(*ψn=  Table[cat[[i,11]] (2.π/360.),{i,clumpOfStarsIDinCatalog}];*)
ψn = 10^-6. {2 527 237, 2 614 503, 2 677 335, 2 590 069, 2 513 274, 2 665 118, 3 071 779, 2 492 330,
   2 617 994, 2 584 833, 2 830 924, 2 602 286, 2 638 938, 2 569 125, 2 513 274, 2 672 099, 2 635 447,
   2 752 384, 2 604 031, 2 631 957, 2 522 001, 2 523 746, 2 624 975, 2 644 174, 2 528 982, 2 328 269,
   2 614 503, 2 616 249, 2 445 206, 2 457 424, 2 745 403, 2 460 914, 2 406 809, 2 588 323,
   2 708 751, 2 501 057, 2 705 260, 2 352 704, 2 642 428, 2 389 356, 2 513 274, 2 972 296,
   2 227 040, 2 560 398, 2 993 240, 2 792 527, 2 912 955, 2 858 849, 2 595 305, 2 918 191,
   2 537 709, 2 523 746, 2 834 415, 2 647 664, 2 888 520, 2 724 459, 2 499 311, 2 703 515,
   2 560 398, 2 412 045, 2 679 080, 205 949, 2 483 604, 2 771 583, 2 935 644, 2 911 209, 2 645 919,
   2 471 386, 2 585 376, 2 610 617, 2 581 367, 2 422 949, 2 485 061, 2 554 874, 2 607 604,
   2 604 617, 2 440 555, 2 548 246, 2 661 401, 2 566 889, 2 593 929, 2 731 882, 2 801 263,
   2 482 942, 2 681 701, 2 783 884, 2 424 506, 2 579 781, 2 480 488, 2 494 087, 2 764 966,
   2 522 077, 2 597 706, 2 732 176, 2 784 442, 2 749 822, 2 824 960, 2 454 031, 2 725 736};
```

In[45]:=
```
(*uncertainty in ψ in radians, rounded to six places*)
(*σψn=  Table[cat[[i,10]] (2.π/360.),{i,clumpOfStarsIDinCatalog}];*)
σψn =
  10^-6. {6981, 3491, 8727, 8727, 43633, 15708, 83776, 15708, 10472, 17453, 34907, 3491, 8727,
    50615, 5236, 6981, 10472, 10472, 6981, 6981, 3491, 20944, 50615, 41888, 104720, 38397,
    26180, 19199, 41888, 61087, 41888, 12217, 40143, 47124, 24435, 3491, 69813, 29671,
    15708, 116937, 59341, 50615, 106465, 36652, 116937, 66323, 38397, 26180, 26180,
    54105, 43633, 71558, 45379, 80285, 52360, 83776, 33161, 41888, 54105,
    97738, 24435, 90757, 92502, 102974, 111701, 55851, 97738, 62832, 34907,
    34907, 27925, 34907, 34907, 34907, 26180, 52360, 34907, 52360, 52360,
    34907, 52360, 104720, 69813, 34907, 61087, 34907, 52360, 34907, 34907,
    52360, 69813, 69813, 52360, 69813, 69813, 34907, 34907, 69813, 52360};
```

In[46]:=
```
(* % polarization, rounded to six places*)
(*percentPol=Table[cat[[i,7]],{i,clumpOfStarsIDinCatalog}]; *)
percentPol =
  10^-6. {2390000, 3420000, 2300000, 2002000, 718000, 154000, 210000, 493000, 660000,
    1010000, 199000, 621000, 1706000, 340000, 728000, 645000, 931000, 1193000, 563000,
    1009000, 896000, 623000, 360000, 585000, 330000, 590000, 660000, 1150000, 545000,
    583000, 700000, 548000, 698000, 602000, 1460000, 811000, 410000, 1490000, 554000,
    150000, 610000, 420000, 340000, 920000, 150000, 550000, 460000, 660000, 680000,
    560000, 490000, 290000, 380000, 260000, 330000, 820000, 540000, 510000, 460000,
    180000, 710000, 460000, 190000, 290000, 390000, 790000, 180000, 280000, 443000,
    336000, 484000, 368000, 489000, 551000, 575000, 540000, 469000, 305000, 272000,
    507000, 306000, 219000, 375000, 285000, 211000, 305000, 481000, 408000, 390000,
    346000, 200000, 400000, 212000, 198000, 359000, 327000, 346000, 299000, 382000};
```

In[47]:=
```
σpercentPol =
  10^-6. {35000, 24000, 42000, 34000, 63000, 5000, 35000, 16000, 13000, 35000, 14000, 5000,
    29000, 35000, 8000, 10000, 20000, 26000, 7000, 15000, 5000, 27000, 36000, 50000,
    69000, 46000, 35000, 46000, 45000, 72000, 58000, 14000, 57000, 57000, 73000,
    5000, 58000, 87000, 17000, 35000, 73000, 42000, 73000, 69000, 35000, 73000,
    35000, 35000, 35000, 60000, 42000, 42000, 35000, 42000, 35000, 138000,
    35000, 42000, 50000, 35000, 35000, 83000, 35000, 60000, 87000, 87000, 35000,
    35000, 32000, 23000, 27000, 25000, 35000, 40000, 30000, 49000, 30000, 26000,
    32000, 36000, 29000, 43000, 48000, 24000, 26000, 26000, 50000, 31000, 29000,
    34000, 26000, 61000, 24000, 27000, 51000, 25000, 25000, 38000, 37000};
```

In[48]:=
```
Print["There are ", nSrc, " stars in the sample."]
Print["Check that the Sample obeys the data cuts:"]
Print[
 "Check that the smallest % polarization p in the sample is 0.1% or more. Smallest: ",
 Sort[percentPol][[1]], "% ."]
Print["Check that the largest fractional uncertainty in % polarization, σp/p,
   is less than 0.25. Largest: ", Sort[σpercentPol / percentPol][[-1]], " ."]
Print["Check that the largest PPA ψ uncertainty σψ is less than 7°. Largest: ",
 Sort[σψn][[-1]] (360./2. π), "° ."]
```

```
There are 99 stars in the sample.

Check that the Sample obeys the data cuts:

Check that the smallest % polarization p in the sample is 0.1% or more. Smallest: 0.15% .

Check that the largest fractional uncertainty
   in % polarization, σp/p, is less than 0.25. Largest: 0.233333 .

Check that the largest PPA ψ uncertainty σψ is less than 7°. Largest: 6.7° .
```

In[53]:=
```
rSrc = Table[er[ gLONSrc[[i]], gLATSrc[[i]] ],  {i, nSrc}];(*calculated from Input.*)
eNSrc = Table[eN[ gLONSrc[[i]], gLATSrc[[i]] ], {i, nSrc}];(*calculated from Input.*)
eESrc = Table[eE[  gLONSrc[[i]], gLATSrc[[i]]  ], {i, nSrc}];(*calculated from Input.*)
```

In[56]:=
```
sourceCenter0 = 1/nSrc Sum[rSrc[[i]], {i, nSrc}];
sourceCenter = sourceCenter0/(sourceCenter0.sourceCenter0)^(1/2);
(*unit radial vector to the arithmetic center of the sources.*)
angleSourceToCenter = Table[ArcCos[rSrc[[i]].sourceCenter], {i, nSrc}];
```

The Selection Process:

The stars in the combined Heiles 2000 and Berdyugin 2014 catalog are filtered for % polarization and experimental uncertainty $\sigma\psi$. Then 5° radius regions are constructed on the 10518 grid points. There were 3632 populated regions with N stars, $314 \geq N \geq 7$, seven being the minimum required for the statistics formulas in Sec. 4 to be valid. Of these, 2983 had very significant alignment, sig $\leq$ 1% $= 1 \times 10^{-2}$, meaning at most one in a hundred samples with randomly directed polarization directions would be equally well aligned. See Fig. 2 for a plot of the significance of these very significantly aligned 5° radius regions. At each region's center point, the negative log of the significance is plotted for convenience, so the minimum value is $-\log_{10}(1 \times 10^{-2}) = +2$, which corresponds to a significance of 1%.

The stars selected for the sample studied are all the stars in all the 5° radius regions that (*i*) have 7 or more stars, (*ii*) have longitude $17° \leq$ gLON $\leq 34°$, (*iii*) have latitude $23° \leq$ gLAT $\leq 35°$, (*iv*) gLAT $<$ 23° - 1.5(gLON - 36°), and (*v*) whose stars have polarization directions aligned with a significance less than a billionth, sig $\leq 10^{-9}$. Requirement (*iv*) separated the region of interest from an adjacent peak. Look closely in Fig. 2. There are 35 regions satisfying (*i*) - (*v*) containing a total of 99 stars. The sample, shaded green in Fig. 2, is among the lower hills in Fig. 2.

Out[ ]=

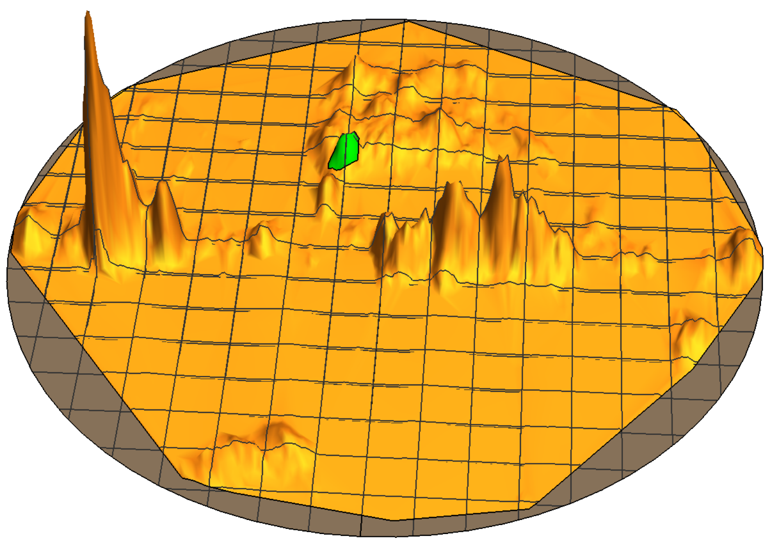


Figure 2. *A whole-sky plot of the significance of 5° radius regions.* Only very significant regions are shown. For convenience, the negative logarithm of the significance is plotted. The largest peaks occur along the Galactic Disk. The top-most peak on the left rises to a value of about 250, meaning that fewer than one in 10^{250} randomly directed regions would have better aligned polarization directions. The hill shaded Green is composed of 35 of the 5° radius regions, located off-Disk and Northeast of the Galactic Center. These 35 regions combine to make the 99-star sample considered here. The most significant of the 35 regions is better aligned than all but one in 10^{30} randomly directed regions.

In[59]:=

```
(*Get the plot*)
(*
SetDirectory[homeDirectory];
fiveDegRegionsWithClump6=Get["20210409showClump6.dat"]
*)
```

In[60]:=
```
ListPlot[Table[{-gLONSrc[[j]], gLATSrc[[j]]} (360./(2. π)), {j, nSrc}],
  PlotRange → {{-180, 180}, {-90, 90}},
  Ticks → {Table[{i, -i}, {i, -180, 180, 60}], Table[{j, j}, {j, -90, 90, 30}]},
  PlotLabel → "Sources", AxesLabel → {"°gLON", "°gLAT"}, PlotStyle → Green]
Print["Figure 3. The locations of the ", nSrc, " stars in the sample. "]
Print[
  "Sample Size: The angular separation of the furthest star from the sample center is ",
  Sort[angleSourceToCenter][[-1]] (360./(2. π)), "°."]
```

Out[60]=

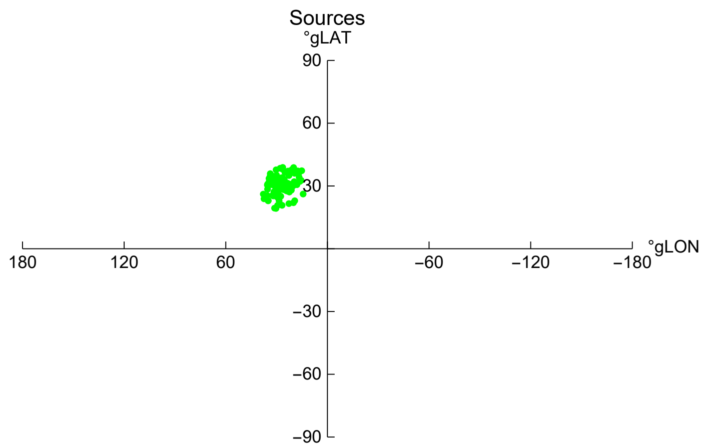


Figure 3. The locations of the 99 stars in the sample.

Sample Size: The angular separation of the furthest star from the sample center is 11.84°.

4. Probability Distributions and Significance Formulas

The problem of “significance” is to determine the likelihood that random polarizations directions would have better alignment or avoidance than the observed polarization directions. To determine the probability distributions and related formulas, in a previous notebook, we made many runs with random data and fit the results.

For samples with randomly directed polarization vectors, the basic formula, Eq. 1, looks like the sum of random numbers each restricted to the range 0 to π. Such random sums can be related to well-known Random Walk scenarios. That connection helps explain the dependence on $\sqrt{N}$ in the formulas below.

Definitions:

norm	a constant used to normalize the distribution so the integral of probability is 1.
probMIN0, probMAX0	probability distributions for alignment (MIN) and avoidance (MAX), functions of η, η_0, σ
ρciaiMIN,MAX	constants used in the formulas to mean η_0 and uncertainty σ
$\sigma\rho$ciaiMIN,MAX	uncertainty σ in the constants used in the formulas to mean η_0 and uncertainty σ
regionRadiusChoices	radii used in random runs performed elsewhere, not in this notebook
regionChoice	determines the best choice for the current sample
rgnRadius	assumed radius of the region for the purpose of selecting the statistics constants c_i and a_i
iρ	dummy variable used to select region radius

ciMIN,MAX and aiMIN,MAX parameters for statistics formulas for η_0 and σ
η0MIN, MAX function to estimate mean η_0
σMIN, MAX function to estimate uncertainty σ
probMIN, probMAX probability distributions using estimated values of η_0, σ
signiMIN0, signiMAX0significance as a function of (η, η_0, σ)
signiMIN, signiMAX significance of η using estimated values of η_0, σ

In[63]:=
```
(* y = ((η - η0)/σ); dy = dη/σ *)
(* The normalization factor "norm" is needed for the probability density *)
```
$$\text{norm} = \left(\frac{1}{(2\pi)^{1/2}} \text{NIntegrate}\left[\left(1 + e^{4\,(y-1)}\right)^{-1} e^{-\frac{y^2}{2}}, \{y, -\infty, \infty\}\right]\right)^{-1};$$
```
norm;(*Constant needed to make the integral
 of the probability distribution equal to unity.*)
```

In[65]:= $\text{probMIN0}[\eta\_, \eta0\_, \sigma\_] := \left(\frac{\text{norm}}{\sigma\,(2\pi)^{1/2}}\right)\left(1 + e^{4\,\frac{(\eta-\eta0-\sigma)}{\sigma}}\right)^{-1} e^{-\frac{1}{2}\left(\frac{\eta-\eta0}{\sigma}\right)^2}$

signiMIN0[η_, η0_, σ_] := NIntegrate[probMIN0[η1, η0, σ], {η1, -∞, η}]

In[67]:= $\text{probMAX0}[\eta\_, \eta0\_, \sigma\_] := \left(\frac{\text{norm}}{\sigma\,(2\pi)^{1/2}}\right)\left(1 + e^{-4\,\frac{(\eta-\eta0+\sigma)}{\sigma}}\right)^{-1} e^{-\frac{1}{2}\left(\frac{\eta-\eta0}{\sigma}\right)^2}$

signiMAX0[η_, η0_, σ_] := NIntegrate[probMAX0[η1, η0, σ], {η1, η, ∞}]

The significance signiMIN0[η, η0, σ] is the Integral of probMIN0, i.e. signiMIN0 = $\int_{-\infty}^{\eta} P_{MIN}(\eta)\, d\eta$.

The significance signiMAX0[η, η0, σ] is the Integral of probMAX0, i.e. signiMAX0 = $\int_{\eta}^{\infty} P_{MAX}(\eta)\, d\eta$.

The formulas for mean $\eta_0 = \frac{\pi}{4} \pm \frac{c1}{N^{a1}}$ and half-width $\sigma = \frac{c2}{4\,N^{a2}}$ estimate η_0 and σ by functions of the number N of sources. These formulas depend on the size of the region (radius ρ) by the choice of parameters c_i and a_i, i = 1,2. The following values for the parameters c_i and a_i are based on random runs. For each combination of N = {8,16,32,64,128,181,256,512} and ρ = {0°,5°,12°,24°,48°,90°}, there were 2000 random runs completed.

A notation conflict between this notebook and the article, Ref. 5, should be noted. We doubled the exponent “a” so $N^{a/2}$ appears in the article, whereas in the formulas here we see N^a. Thus $a \approx 1/2$ here, but the paper has $a_{\text{Article}} \approx 1$. That explains the “/2” in the following arrays.

In[69]:= ρciaiMIN = ;

"ρ"	"c1"	"a1"	"c2"	"a2"
90	0.9423	1.0046/2	1.061	0.954/2
48	0.9505	1.0156/2	1.166	0.9956/2
24	0.9235	1.0069/2	1.127	0.964/2
12	0.8912	1.0054/2	1.238	1.021/2
5	0.8363	1.0088/2	1.076	0.940/2
0	0.5031	1.0153/2	1.522	1.053/2

In[70]:=
```
             "ρ"   "c1"     "a1"     "c2"    "a2"
             90  0.9441  1.0055/2  1.000  0.931/2
             48  0.9572  1.0165/2  1.090  0.958/2
ρciaiMAX =   24  0.927   1.0068/2  1.101  0.964/2 ;
             12  0.9049  1.0090/2  1.228  1.018/2
             5   0.8424  1.0062/2  1.168  0.992/2
             0   0.4982  1.0093/2  1.543  1.060/2
```

In[71]:=
```
              "ρ"   "c1"     "a1"     "c2"    "a2"
              90  0.0050  0.0036/2  0.026  0.016/2
              48  0.0079  0.0057/2  0.016  0.0095/2
ρΔciaiMIN =   24  0.0024  0.0018/2  0.022  0.013/2 ;
              12  0.0034  0.0026/2  0.039  0.021/2
              5   0.0035  0.0028/2  0.030  0.019/2
              0   0.0059  0.0080/2  0.052  0.024/2
```

In[72]:=
```
              "ρ"   "c1"     "a1"     "c2"    "a2"
              90  0.0061  0.0044/2  0.038  0.025/2
              48  0.0063  0.0045/2  0.026  0.016/2
ρΔciaiMAX =   24  0.011   0.0079/2  0.019  0.011/2 ;
              12  0.0069  0.0052/2  0.039  0.022/2
              5   0.0038  0.0031/2  0.022  0.013/2
              0   0.0058  0.0080/2  0.057  0.025/2
```

In[73]:=
```
(*The region radius controls the constants ci and ai for statistics in Sec. 4.*)
regionRadiusChoices = {90, 48, 24, 12, 5, 0};(*Do not change this statement*)
regionChoice = 4; (*This is a setting. The choice 24° is 3rd in the list. *)
rgnRadius = regionRadiusChoices[[regionChoice]];
Print["The region radius ρ is set at ", rgnRadius, "°."]
```

The region radius ρ is set at 12°.

In[77]:=
```
iρ = regionChoice + 1; (* Parameters ci, ai, i = 1,2. *)
Print["These constants are for sources confined to regions with radii ρ = ",
 ρciaiMIN[[iρ, 1]], "°."]
{c1MIN, a1MIN, c2MIN, a2MIN} = Table[ρciaiMIN[[iρ, j]], {j, 2, 5}]
{c1MAX, a1MAX, c2MAX, a2MAX} = Table[ρciaiMAX[[iρ, j]], {j, 2, 5}]
```

These constants are for sources confined to regions with radii ρ = 12°.

Out[79]= {0.8912, 0.5027, 1.238, 0.5105}

Out[80]= {0.9049, 0.5045, 1.228, 0.509}

In[81]:=
```
iρ = regionChoice + 1; (* ± uncertainty for the Parameters ci and ai, i = 1,2. *)
Print["These uncertainties are for sources confined to regions with radii ρ = ",
 ρciaiMAX[[iρ, 1]], "°."]
{c1MINplusMinus, a1MINplusMinus, c2MINplusMinus, a2MINplusMinus} =
 Table[ρΔciaiMIN[[iρ, j]], {j, 2, 5}]
{c1MAXplusMinus, a1MAXplusMinus, c2MAXplusMinus, a2MAXplusMinus} =
 Table[ρΔciaiMAX[[iρ, j]], {j, 2, 5}]
```

```
These uncertainties are for sources confined to regions with radii ρ = 12°.
```

Out[83]= {0.0034, 0.0013, 0.039, 0.0105}

Out[84]= {0.0069, 0.0026, 0.039, 0.011}

In[85]:= $\eta 0MIN[nSrc\_, c1\_, a1\_] := \frac{\pi}{4} - \frac{c1}{nSrc^{a1}}$

$\sigma MIN[nSrc\_, c2\_, a2\_] := \frac{c2}{4\, nSrc^{a2}}$

In[87]:= $\eta 0MAX[nSrc\_, c1\_, a1\_] := \frac{\pi}{4} + \frac{c1}{nSrc^{a1}}$

$\sigma MAX[nSrc\_, c2\_, a2\_] := \frac{c2}{4\, nSrc^{a2}}$

The following probability distributions and significances make use of the above formulas for mean η_0 and half-width σ. They are functions of the alignment angle η and the number of sources N.

In[89]:=
```
probMIN[η_, nSrc_] := probMIN0[ η, η0MIN[nSrc, c1MIN, a1MIN], σMIN[nSrc, c2MIN, a2MIN] ]
```

In[90]:=
```
signiMIN[η_, nSrc_] := signiMIN0[η, η0MIN[nSrc, c1MIN, a1MIN], σMIN[nSrc, c2MIN, a2MIN]]
```

In[91]:=
```
probMAX[η_, nSrc_] := probMAX0[ η, η0MAX[nSrc, c1MAX, a1MAX], σMAX[nSrc, c2MAX, a2MAX] ]
signiMAX[η_, nSrc_] := signiMAX0[η, η0MAX[nSrc, c1MAX, a1MAX], σMAX[nSrc, c2MAX, a2MAX]]
```

Section Summary

In[93]:=
```
Print["The angular separation of the furthest star from the region center is ",
 Sort[angleSourceToCenter][[-1]] (360./(2. π)), "°.",
 " We choose the statistics constants ai and ci, i = 1,2, for
   sources confined to regions with radii ρ = ", ρciaiMIN[[iρ, 1]], "°."]
Print["The formulas also depend on the number of sources, nSrc = ", nSrc, "."]
Print["For this sample, but with observed replaced by random polarization
   directions, the expected smallest alignment angle η̄min  is  η̄min^Random ψ = ",
 η0MIN[nSrc, c1MIN, a1MIN] (360./(2. π)), "° ± ", σMIN[nSrc, c2MIN, a2MIN] (360./(2. π)),
 "°. (Random ψ)"]
Print["For this sample, but with observed replaced by random polarization
   directions, the expected largest avoidance angle η̄max is  η̄max^Random ψ = ",
 η0MAX[nSrc, c1MAX, a1MAX] (360./(2. π)), "° ± ", σMAX[nSrc, c2MAX, a2MAX] (360./(2. π)),
 "°. (Random ψ)"]
```

The angular separation of the furthest star from the region center is 11.84°. We choose the statistics constants a_i and c_i, $i = 1,2$, for sources confined to regions with radii $\rho = 12°$.

The formulas also depend on the number of sources, nSrc = 99.

For this sample, but with observed replaced by random polarization directions, the expected smallest alignment angle $\overline{\eta}_{min}$ is $\overline{\eta}_{min}{}^{Random\,\psi} = 39.9314° \pm 1.69829°$. (Random ψ)

For this sample, but with observed replaced by random polarization directions, the expected largest avoidance angle $\overline{\eta}_{max}$ is $\overline{\eta}_{max}{}^{Random\,\psi} = 50.1042° \pm 1.69622°$. (Random ψ)

5. Results using the Best Values ψn of the Polarization Directions

"Best" means we use the ψn that were listed in the catalog. We calculate the alignment function $\overline{\eta}$(H) at the grid points H. Given the alignment function $\overline{\eta}$(H) , one can find the smallest alignment angle $\overline{\eta}_{min}$ and the largest avoidance angle $\overline{\eta}_{max}$ and determine the significances for the alignment and avoidance of the polarization directions.

Note that, in Sec. 6 below, we consider other values of the polarization directions that are not the best values, but that are consistent with uncertainty $\sigma\psi$ in the measured values.

5a. The alignment function $\overline{\eta}$(H) .

Definitions:

vψSrc unit vectors along the polarization directions in the tangent planes of the sources
eN local unit vectors along local North
eE local unit vectors along local East
jηBarHj {j,$\overline{\eta}$(H)}, where j is the index for grid point H_j and $\overline{\eta}$(H) is the average alignment angle at H_j. See Eq. (1) in the Introduction.
sortjηBarHj {j,$\overline{\eta}$(H)}, sorted, with smallest angles $\overline{\eta}$(H) first.
jηBarMin {j,$\overline{\eta}$(H)}, the j and $\overline{\eta}$ for the smallest value of $\overline{\eta}$(H) , best alignment
ηBarMin the smallest value of $\overline{\eta}$(H) , measures alignment of the polarization directions
jηBarMax {j,$\overline{\eta}$(H)}, the j and $\overline{\eta}$ for the largest value of $\overline{\eta}$(H) , most avoided
ηBarMax the largest value of $\overline{\eta}$(H) , measures avoidance

nSxψn unit vector, $S_i \times \psi_i$, cross product of the radial vector to the source with the vector in the direction of the polarization
nSxHnj unit vector, $S_i \times H_j$, cross product of the radial vector to the source with the radial vector to the grid point H_j
ηnHj alignment angle between source and grid point H_j, see Fig. 1
ηBarHj alignment angle $\overline{\eta}(H_j)$ between source and grid point H_j, avegLONged over all sources
jηBarHj {j, $\overline{\eta}(H_j)$ }, the j and $\overline{\eta}$ for grid point H_j

sigηBarMin significance of the smallest alignment angle
sigrangeηBarMin get the range of sigs using the plus/minus values on the parameters c_i, a_i
sigSmallηBarMin the smallest of the values in sigrangeηBarMin
sigBigηBarMin the largest of the values in sigrangeηBarMin
sigηBarMax significance of the largest alignment angle (i.e. avoidance)
sigrangeηBarMax get the range if sigs using the plus/minus values on the parameters c_i, a_i

sigSmallηBarMax the smallest of the values in sigrangeηBarMax
sigBigηBarMax the largest of the values in sigrangeηBarMax
gLONHminDegrees gLON of the point $H_{\min}$where $\overline{\eta}$(H) is the smallest
gLATHminDegrees gLAT of the point $H_{\min}$where $\overline{\eta}$(H) is the smallest
gLONHmaxDegrees gLON of the point $H_{\max}$where $\overline{\eta}$(H) is the largest
gLATHmaxDegrees gLAT of the point $H_{\max}$where $\overline{\eta}$(H) is the largest

In[97]:=
```
(* vψ, eN, eE unit vectors in the tangent plane of each source Si,
pointing along the polarization direction, local North,
and local East, respectively. See Fig. 1.*)
vψSrc = Table[Cos[ ψn[[i]] ] eN[ gLONSrc[[i]], gLATSrc[[i]] ] +
    Sin[ ψn[[i]] ] eE[ gLONSrc[[i]], gLATSrc[[i]] ], {i, nSrc}];
```

In[98]:=
```
(* Analysis using Eq (5) in Ref. 5 to get η̄(Hj). First ηiH,
cos(ηiH) = |v̂H.v̂ψi|, and then η̄(Hj), by Eq. (1). *)
jηBarHj =
  Table[{j, (1/nSrc) Sum[ArcCos[ Abs[ rGrid[[j]].vψSrc[[i]] / ((rGrid[[j]] - (rGrid[[j]].
            rSrc[[i]]) rSrc[[i]]).(rGrid[[j]] - (rGrid[[j]].rSrc[[i]])
          rSrc[[i]]))^1/2 ] - 0.000001 ] , {i, nSrc}]}, {j, nGrid}];
 sortjηBarHj = Sort[jηBarHj, #1[[2]] < #2[[2]] &];
jηBarMin = sortjηBarHj[[1]]; (*  {j,η̄(Hj)} for smallest η̄(Hj) *)
ηBarMin = jηBarMin[[2]];
jηBarMax = sortjηBarHj[[-1]]; (*  {j,η̄(Hj)} for largest η̄(Hj) *)
ηBarMax = jηBarMax[[2]] ;
```

In[104]:=
```
(*Significance of the smallest alignment angle η̄min .*)
sigηBarMin = signiMIN[ηBarMin, nSrc];
sigrangeηBarMin = Sort[Partition[Flatten[Table[
      {signiMIN0[ηBarMin, η0MIN[nSrc, c1MIN + γ1 c1MINplusMinus, a1MIN + α1 a1MINplusMinus],
        σMIN[nSrc, c2MIN + γ2 c2MINplusMinus, a2MIN + α2 a2MINplusMinus]], γ1, α1, γ2, α2},
      {γ1, -1, 1}, {α1, -1, 1}, {γ2, -1, 1}, {α2, -1, 1}] ], 5 ]   ];
{sigrangeηBarMin[[1]], sigrangeηBarMin[[-1]]};
sigSmallηBarMin = sigrangeηBarMin[[1, 1]];
sigBigηBarMin = sigrangeηBarMin[[-1, 1]];
```

In[109]:=
```
(*Significance of the largest avoidance angle η̄max .*)
sigηBarMax = signiMAX[ηBarMax, nSrc];
sigrangeηBarMax = Sort[Partition[Flatten[Table[
      {signiMAX0[ηBarMax, η0MAX[nSrc, c1MAX + γ1 c1MAXplusMinus, a1MAX + α1 a1MAXplusMinus],
        σMAX[nSrc, c2MAX + γ2 c2MAXplusMinus, a2MAX + α2 a2MAXplusMinus]], γ1, α1, γ2, α2},
      {γ1, -1, 1}, {α1, -1, 1}, {γ2, -1, 1}, {α2, -1, 1}] ], 5 ]   ];
{sigrangeηBarMax[[1]], sigrangeηBarMax[[-1]]};
sigSmallηBarMax = sigrangeηBarMax[[1, 1]];
sigBigηBarMax = sigrangeηBarMax[[-1, 1]];
```

In[114]:=
```
(* Galactic coordinates (gLON,gLAT) for the hubs Hmin and Hmax .*)
gLONHminDegrees = gLONGrid[[ jηBarMin[[1]] ]] (360 / (2 π)); (*Hmin*)
gLATHminDegrees = gLATGrid[[ jηBarMin[[1]] ]] (360 / (2 π));

gLONHmaxDegrees = gLONGrid[[ jηBarMax[[1]] ]] (360 / (2 π)); (*Hmax*)
gLATHmaxDegrees = gLATGrid[[ jηBarMax[[1]] ]] (360 / (2 π));
```

In[118]:=
```
(*The names "jηBarMin", "jηBarMax" are similar to quantities below,
so save the current values labeled by "Best".*)
(* jηBar entries: 1. grid point # , 2. alignment angle .*)
{jηBarMinBest, jηBarMaxBest} = {jηBarMin, jηBarMax} ;
```

In[119]:=
```
Print["The min alignment angle is ηmin = ", jηBarMinBest[[2]] * (360. / (2. π)),
 "° , which has a significance of sig. = ", sigηBarMin, ",  plus/minus = + ",
 sigBigηBarMin - sigηBarMin, " and - ", sigηBarMin - sigSmallηBarMin,
 " , giving a range from sig. = ", sigSmallηBarMin, " to ", sigBigηBarMin, " ."]
Print["The max avoidance angle is ηmax = ", jηBarMaxBest[[2]] * (360. / (2. π)),
 "° , which has a significance of sig. = ", sigηBarMax, ",  plus/minus = + ",
 sigBigηBarMax - sigηBarMax, " and - ", sigηBarMax - sigSmallηBarMax,
 " , giving a range from sig. = ", sigSmallηBarMax, " to ", sigBigηBarMax, " ."]
Print["These uncertainties are due to the uncertainties in the constants ci, ai."]
```

The min alignment angle is ηmin = 7.00977° , which has a significance of sig. = 6.2843×10^{-84}, plus/minus = + 1.01537×10^{-71} and − 6.2843×10^{-84} , giving a range from sig. = 1.84974×10^{-98} to 1.01537×10^{-71} .

The max avoidance angle is ηmax = 83.1133° , which has a significance of sig. = 1.45299×10^{-84}, plus/minus = + 1.08832×10^{-71} and − 1.45299×10^{-84} , giving a range from sig. = 5.33528×10^{-100} to 1.08832×10^{-71} .

These uncertainties are due to the uncertainties in the constants c_i, a_i.

5b. Plot of the Alignment Angle Function $\overline{\eta}$(H)

Definitions

gLONjgLATjηBarHjTable	{gLON$_j$, gLAT$_j$, $\overline{\eta}$(H)} at each grid point $H = H_j$, in degrees
ηBarHjSmooth	interpolation of gLONjgLATjηBarHjTable yields $\overline{\eta}$(H) as a smooth function of the (gLON,gLAT) of H
xyηBarAitoffTable	{x, y, $\overline{\eta}$(x,y)} , where x,y are Aitoff coordinates and $\overline{\eta}$(x,y) is the alignment angle
xyAitoffSources	{x,y} Aitoff coordinates for the sources' locations on the sphere
dηContourPlot	separation of successive contour lines, in degrees
listCP	list contour plot of $\overline{\eta}$(H) from xyηBarAitoffTable
mapOfηBar	contour plot of the alignment angle $\overline{\eta}$(H) , adorned with source locations and labels
rCenterSrc	arithmetic average of the radial unit vectors to the sources, previously called sourceCenter
rHmin, rHmax	radial unit vectors to the alignment and avoidance hubs $H_{\min}$ and $H_{\max}$
rPerpHmin (max)	a unit vector in the plane of the great circle combining rCenterSrc and rHmin (max)
rGreatMinCircle(θ) (Max)	radial unit vector to a point on the great circle
gLONGreatMin (Max)	galactic longitude at the point for θ

gLATGreatMin (Max)	galactic latitude at the point for θ
xyAitoffGreatMin (Max)	Aitoff plot coordinates for the great circles
crossMin (Max)	unit vector perpendicular, normal to the plane of the great circle
θminMAXgreatcircles	angle between the vectors normal to the planes of the two great circles

This sample is an extreme case, both alignment and avoidance are highly significant. This not unusual with stellar polarization in the Galaxy, there are many samples that qualify as extreme in this way. By drawing the great circles connecting the sources to the hubs, we see that the two great circles are perpendicular.

We include the Great Circle from the center of the sources to the alignment hub $H_{\min}$ on the map. We also draw the Great Circle from source center to the avoidance hub $H_{\max}$ because the two Great Circles divide the sphere quite evenly, an extreme case of collective behavior of polarization directions. The two Great Circles are perpendicular at the two points where they cross, within experimental error.

Also supporting celebrity status for this sample is the near perfect coincidence of one of the Great Circles, the $H_{\min}$one, with the Celestial Equator.

In[122]:=
```
rCenterSrc0 = 1/nSrc Sum[rSrc[[i]], {i, Length[rSrc]}];
rCenterSrc = rCenterSrc0/(rCenterSrc0.rCenterSrc0)^(1/2.);
```

In[124]:=
```
rHmin = er[ gLONHminDegrees (2. π/360.) + π, -gLATHminDegrees (2. π/360.) ];
rPerpHmin0 = rHmin - (rHmin.rCenterSrc) rCenterSrc;
rPerpHmin = rPerpHmin0/(rPerpHmin0.rPerpHmin0)^(1/2.);
rGreatMinCircle[θ_] := Cos[θ] rCenterSrc + Sin[θ] rPerpHmin
gLONGreatMin[θ_] := gLONFROMr[rGreatMinCircle[θ]]
gLATGreatMin[θ_] := gLATFROMr[rGreatMinCircle[θ]]
xyAitoffGreatMin =
  Table[{xHGal[ gLONGreatMin[θ] (360 / (2 π)), gLATGreatMin[θ] (360 / (2 π)) ],
    yHGal[ gLONGreatMin[θ] (360 / (2 π)), gLATGreatMin[θ] (360 / (2 π)) ]}, {θ, 1, 360}];
```

In[131]:=
```
rHmax = er[ gLONHmaxDegrees (2. π/360.) + π, -gLATHmaxDegrees (2. π/360.) ];
rPerpHmax0 = rHmax - (rHmax.rCenterSrc) rCenterSrc;
rPerpHmax = rPerpHmax0/(rPerpHmax0.rPerpHmax0)^(1/2.);
rGreatMaxCircle[θ_] := Cos[θ] rCenterSrc + Sin[θ] rPerpHmax
gLONGreatMax[θ_] := gLONFROMr[rGreatMaxCircle[θ]]
gLATGreatMax[θ_] := gLATFROMr[rGreatMaxCircle[θ]]
xyAitoffGreatMax =
  Table[{xHGal[ gLONGreatMax[θ] (360 / (2 π)), gLATGreatMax[θ] (360 / (2 π)) ],
    yHGal[ gLONGreatMax[θ] (360 / (2 π)), gLATGreatMax[θ] (360 / (2 π)) ]}, {θ, 1, 360}];
```

In[138]:=
```
crossMin0 = Cross[rHmin, rCenterSrc];
crossMin = crossMin0 / (crossMin0.crossMin0)^1/2.;
crossMax0 = Cross[rHmax, rCenterSrc];
crossMax = crossMax0 / (crossMax0.crossMax0)^1/2.;
ΘminMAXgreatcircles = ArcCos[crossMax.crossMin] (360. / (2. π));
```

In[143]:=
```
(*The following table gLONjgLATjηBarHjTable is created to be interpolated below,
yielding a smooth function ηBarHjSmooth of the alignment angle η̅(H) over the sphere.*)
(* Table gLONjgLATjηBarHjTable
 entries: 1.  gLON  2. gLAT  3. alignment angle ηBarRgnkj at grid point (all in degrees)*)
gLONjgLATjηBarHjTable = (gLONjgLATjηBarHjTable0 = {};
   For[j = 1, j ≤ Length[jηBarHj], j++, AppendTo[ gLONjgLATjηBarHjTable0,
     {gLONGrid[[j]] * (360. / (2. π)), gLATGrid[[j]] * (360. / (2. π)),
      jηBarHj[[j, 2]] * (360. / (2. π))}]   ; If[ 180 ≥ gLONGrid[[j]] * (360. / (2. π)) > 174.,
     AppendTo[ gLONjgLATjηBarHjTable0,  {gLONGrid[[j]] * (360. / (2. π)) - 360.,
       gLATGrid[[j]] * (360. / (2. π)), jηBarHj[[j, 2]] * (360. / (2. π))}]   ] ;
    If[ -174. > gLONGrid[[j]] * (360. / (2. π)) ≥ -180., AppendTo[ gLONjgLATjηBarHjTable0,
      {gLONGrid[[j]] * (360. / (2. π)) + 360, gLATGrid[[j]] * (360. / (2. π)),
       jηBarHj[[j, 2]] * (360. / (2. π))}]   ]        ];
   gLONjgLATjηBarHjTable0);
```

In[144]:=
```
ηBarHjSmooth = Interpolation[gLONjgLATjηBarHjTable] (*The smooth alignment angle function η̅(H).*)
```

Interpolation: Interpolation on unstructured grids is currently only supported for InterpolationOrder->1 or InterpolationOrder->All. Order will be reduced to 1.

Out[144]=
```
InterpolatingFunction[ Domain: {{-186., 186.}, {-88., 88.}} Output: scalar ]
```

In[145]:=
```
(*Transcribe the alignment function  η̅(H), the location of the sources,
and the Celestial Equator onto an Aitoff plot.*)
xyηBarAitoffTable =
  Partition[Flatten[Table[{xHGal[gLON, gLAT], yHGal[gLON, gLAT], ηBarHjSmooth[gLON, gLAT]},
     {gLON, -178., 178., 2.}, {gLAT, -88., 88., 2.}]], 3];
(* The smooth alignment angle function η̅(H)  = ηBarHjSmooth mapped
  onto a 2D Aitoff projection of the sphere. *)

xyAitoffSources = Table[{xHGal[ gLONSrc[[n]] (360 / (2 π)), gLATSrc[[n]] (360 / (2 π)) ],
    yHGal[ gLONSrc[[n]] (360 / (2 π)), gLATSrc[[n]] (360 / (2 π)) ]}, {n, nSrc}];
 (*The Aitoff coordinates for the sources' locations.*)
```

In[147]:=
```
(* Contour plot of the alignment function ηBarHjSmooth. *)
dηContourPlot = 10 ;
(*, in degrees. *)listCP = ListContourPlot[Union[xyηBarAitoffTable(*,{{xHGal[gLONHminDegrees,
      gLATHminDegrees],yHGal[gLONHminDegrees,gLATHminDegrees],ηBarMin*(360./(2.π))-1.0}},
   {{xHGal[gLONHmaxDegrees,gLATHmaxDegrees],yHGal[gLONHmaxDegrees,gLATHmaxDegrees],
    ηBarMax*(360./(2.π))+1.0}}*)], AspectRatio → 1/2, Contours → Table[η,
   {η, Floor[jηBarMin[[2]] * (360. / (2. π))] + 1, Ceiling[jηBarMax[[2]] * (360. / (2. π))] - 1,
    dηContourPlot}], ColorFunction → "TemperatureMap", PlotRange → {{-7, 7}, {-3, 3}},
  Axes -> False, Frame → False, PlotLabel → "The alignment function η̅(H)", PlotLegends → Automatic] ;
```

In[148]:=
```
(*Construct the map of η̅(H).*)
mapOfηBar =
  Show[{listCP, Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]},
      {gLAT, -90, 90}, PlotStyle → {Black, Thickness[0.002]}, (*Mesh→{11,5,0}
      (*{23,11,0}*),MeshStyle→Thick,*)PlotPoints → 60], {gLON, -180, 180, 30}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLON, -180, 180},
      PlotStyle → {Black, Thickness[0.002]}, (*Mesh→{11,5,0}(*{23,11,0}*),
      MeshStyle→Thick,*)PlotPoints → 60], {gLAT, -60, 60, 30}],
    Graphics[{PointSize[0.004], Text[StyleForm["N", FontSize -> 10, FontWeight -> "Plain"],
       {0, 1.85}], (*Sources S:*)Green, Point[ xyAitoffSources ], Gray,
      PointSize[0.002], Point[ xyAitoffGreatMin ], Point[ xyAitoffGreatMax ], Black,
      Text[StyleForm["Hmin", FontSize → 8, FontWeight -> "Bold"], {-3.3, -1.0}],
      {Arrow[BezierCurve[{{-3.3, -1.2}, {-2.3, -2.0}, {xHGal[gLONHminDegrees + 180,
            -gLATHminDegrees], yHGal[gLONHminDegrees + 180, -gLATHminDegrees]}}]]},
      Text[StyleForm["Hmax", FontSize → 8, FontWeight -> "Bold"], {3.3, -1.0}],
      {Arrow[BezierCurve[{{3.3, -1.2}, {2.3, -2.0}, {xHGal[gLONHmaxDegrees, gLATHmaxDegrees],
           yHGal[gLONHmaxDegrees, gLATHmaxDegrees]}}]]},
      Text[StyleForm["Hmax", FontSize → 8, FontWeight -> "Bold"], {-3.3, 1.0}],
      {Arrow[BezierCurve[{{-3.3, 1.2}, {-2.3, 2.0}, {xHGal[gLONHmaxDegrees + 180,
            -gLATHmaxDegrees], yHGal[gLONHmaxDegrees + 180, -gLATHmaxDegrees]}}]]},
      Text[StyleForm["Hmin", FontSize → 8, FontWeight -> "Bold"], {3.3, 1.0}] ,
      {Arrow[BezierCurve[{{3.3, 1.2}, {2.3, 2.0}, {xHGal[gLONHminDegrees, gLATHminDegrees],
           yHGal[gLONHminDegrees, gLATHminDegrees]}}]]}
     }]}, ImageSize → 2 × 432];
```

Section Summary

In[149]:=
```
mapOfηBar
Print[
 "Figure 4: The alignment function η̄(H), Eq. (1). The map is centered on (gLON,gLAT)=(0°,0°),"]
Print["with gLON = +180° on the left and gLON = -180° on the right."]
Print["The sources are located at the dots, shaded ", Green, " ."]
Print["The smallest alignment angle is η̄min = ",
 Round[jηBarMinBest[[2]] (360./(2.π))], "°, located at the"]
Print["alignment hubs Hmin and -Hmin in the areas shaded ", Blue, " . "]
Print["The hubs are located at (gLON,gLAT) = ", Round[{gLONHminDegrees, gLATHminDegrees }],
 " and ", Round[{gLONHminDegrees + 180, -gLATHminDegrees }], " , in degrees."]
Print["The largest avoidance angle is η̄max = ",
 Round[jηBarMaxBest[[2]] (360./(2.π))], "°, located at the"]
Print["avoidance hubs Hmax and -Hmax in the areas shaded ", Red, " . "]
Print["The hubs are located at (gLON,gLAT) = ",
 Round[{gLONHmaxDegrees + 180, -gLATHmaxDegrees }], " and at ",
 Round[{gLONHmaxDegrees, gLATHmaxDegrees }], " , in degrees."]
Print["To guide the eye, two Great Circles are plotted, one through the sources' center and the
   avoidance hubs Hmax and -Hmax. The other connects the center of the sources' locations
   with the alignment hubs Hmin and -Hmin. The Great Circles are shaded Gray, ", Gray, " ."]
Print["Notes: Although somewhat obscured by the distortion needed to plot a
   sphere on a flat surface, the function η̄(H) is symmetric across diameters.
   Diametrically opposite points -H and H have the same alignment angle η̄(H)."]
```

Out[149]=

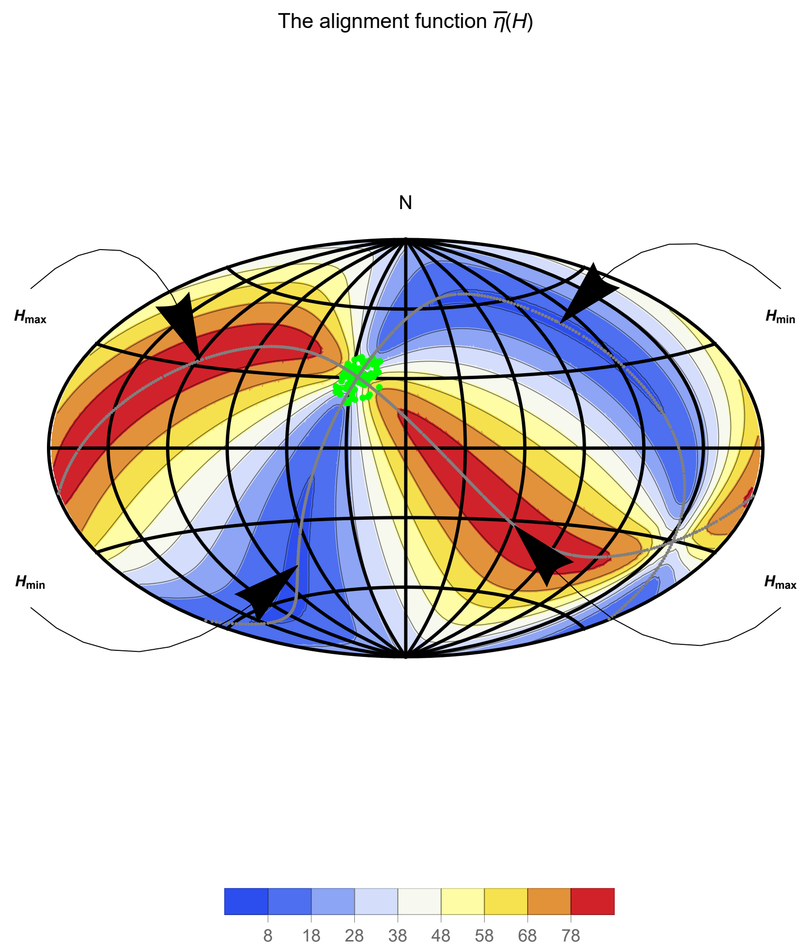


Figure 4: The alignment function $\overline{\eta}(H)$, Eq. (1). The map is centered on (gLON,gLAT) = (0°,0°), with gLON = +180° on the left and gLON = -180° on the right.

The sources are located at the dots, shaded ■ .

The smallest alignment angle is $\overline{\eta}_{min}$ = 7°, located at the alignment hubs H_{min} and $-H_{min}$ in the areas shaded ■ .

The hubs are located at (gLON,gLAT) = {-107, 48} and {73, -48} , in degrees.

The largest avoidance angle is $\overline{\eta}_{max}$ = 83°, located at the avoidance hubs H_{max} and $-H_{max}$ in the areas shaded ■ .

The hubs are located at (gLON,gLAT) = {118, 32} and at {-62, -32} , in degrees.

To guide the eye, two Great Circles are plotted, one through the sources' center and the avoidance hubs H_{max} and $-H_{max}$. The other connects the center of the sources' locations with the alignment hubs H_{min} and $-H_{min}$. The Great Circles are shaded Gray, ■ .

Notes: Although somewhat obscured by the distortion needed to plot a sphere on a flat surface, the function $\overline{\eta}(H)$ is symmetric across diameters. Diametrically opposite points -H and H have the same alignment angle $\overline{\eta}(H)$.

In[161]:=
```
(*Statistics*)
Print["Statistics of the Alignment Function η̅(H) :"]
Print[" "]
Print["The number of sources: N = ", nSrc]
Print["The min alignment angle, ηmin = ", jηBarMinBest[[2]] * (360. / (2. π)),
 "°, is ", (η0MIN[nSrc, c1MIN, a1MIN] - jηBarMinBest[[2]]) * (360. / (2. π)),
 "° below the most likely value, ",
 η0MIN[nSrc, c1MIN, a1MIN] * (360. / (2. π)), "°, for random runs."]
Print["Since the uncertainty σ is ", σMIN[nSrc, c2MIN, a2MIN] * (360. / (2. π)),
 "°, the difference ", (η0MIN[nSrc, c1MIN, a1MIN] - jηBarMinBest[[2]]) * (360. / (2. π)),
 "° is ", (η0MIN[nSrc, c1MIN, a1MIN] - jηBarMinBest[[2]]) / σMIN[nSrc, c2MIN, a2MIN],
 "σs from the most likely random run value."]
Print["Thus, the smallest alignment angle η̅min is ",
 (η0MIN[nSrc, c1MIN, a1MIN] - jηBarMinBest[[2]]) / σMIN[nSrc, c2MIN, a2MIN],
 "σs below the most likely random run value."]
Print[""]
Print["The largest avoidance angle, ηmax = ", jηBarMaxBest[[2]] * (360. / (2. π)),
 "°, is ", -(η0MAX[nSrc, c1MAX, a1MAX] - jηBarMaxBest[[2]]) * (360. / (2. π)),
 "° above the most likely value, ",
 η0MAX[nSrc, c1MAX, a1MAX] * (360. / (2. π)), "°, for random runs."]
Print["Since the uncertainty σ is ", σMAX[nSrc, c2MAX, a2MAX] * (360. / (2. π)),
 "°, the difference ", -(η0MAX[nSrc, c1MAX, a1MAX] - jηBarMaxBest[[2]]) * (360. / (2. π)),
 "° is ", -((η0MAX[nSrc, c1MAX, a1MAX] - jηBarMaxBest[[2]]) / σMAX[nSrc, c2MAX, a2MAX]),
 "σs from the most likely random run value." ]
Print["Thus, the largest avoidance angle η̅max is ",
 (jηBarMaxBest[[2]] - η0MAX[nSrc, c1MAX, a1MAX]) / σMAX[nSrc, c2MAX, a2MAX],
 "σs above the most likely random run value."]
```

Statistics of the Alignment Function $\overline{\eta}(H)$:

The number of sources: N = 99

The min alignment angle, ηmin = 7.00977°, is
32.9216° below the most likely value, 39.9314°, for random runs.

Since the uncertainty σ is 1.69829°, the difference
32.9216° is 19.3852σs from the most likely random run value.

Thus, the smallest alignment angle $\overline{\eta}_{min}$ is 19.3852σs below the most likely random run value.

The largest avoidance angle, ηmax = 83.1133°, is
33.0091° above the most likely value, 50.1042°, for random runs.

Since the uncertainty σ is 1.69622°, the difference
33.0091° is 19.4604σs from the most likely random run value.

Thus, the largest avoidance angle $\overline{\eta}_{max}$ is 19.4604σs above the most likely random run value.

In[171]:=
```
Print["The center of the sources is a point that makes a great circle, shaded ",
  Gray, " in Fig. 4, with the alignment hub Hmin."]
Print["The center of the sources makes a second great circle, shaded ",
  Gray, " in Fig. 4, with the avoidance hub Hmax."]
Print["The angle between the planes of the two great circles is ",
  θminMAXgreatcircles, "°."]
```

```
The center of the sources is a point that makes a great circle, shaded
 ■ in Fig. 4, with the alignment hub Hmin.
The center of the sources makes a second great circle, shaded
 ■ in Fig. 4, with the avoidance hub Hmax.
The angle between the planes of the two great circles is 89.915°.
```

6. Uncertainty Runs

6a. Creating and Storing Uncertainty Runs

For each "uncertainty run", the polarization direction ψ for each source is allowed to differ from the best value ψn by an amount $\delta\psi$ chosen according to a Gaussian distribution with mean (best) value ψn and half-width $\sigma\psi$, $\psi = \psi n + \delta\psi$. Both values ψn and $\sigma\psi$ are taken from the catalogs.

Definitions:

rSrcxrGrid unit vector $S_i \times H_j$ in the direction of the cross product of the radial vector S_i to a source with the radial vector H_j to a grid point

μ the mean value μ of the measurement Gaussian for ψ

σ the uncertainty of the measured polarization position angle ψ

ψData polarization directions $\psi = \psi n + \delta\psi$

runData collection of data to save from the uncertainty runs, see below for content list

nRunPrint dummy index controlling when current TimeUsed and MemoryInUse are printed

ψSrc the polarization direction ψ for the run.

rSrcxψSrc unit vector, $S_i \times \psi_i$, cross product of the radial vector S_i to the source with the vector $\hat{v}_\psi$ in the direction of the polarization

jηBarToGrid {j, $\overline{\eta}(H_j)$}, where j is the index for the grid point H_j and $\overline{\eta}(H_j)$ is the alignment angle function, (1), at H_j

sortjηBarToGrid sort {j, $\overline{\eta}(H_j)$}, with the smaller angle $\overline{\eta}$(H) first.

jηBarMin1 {j,$\overline{\eta}$(H)} for the smallest value of $\overline{\eta}$(H) , best alignment

jηBarMax1 {j,$\overline{\eta}$(H)}, for the largest value of $\overline{\eta}$(H) , most avoided

ηBarMinData values of $\overline{\eta}_{\min}$ from uncertainty runs, alignment

ηBarMaxData values of $\overline{\eta}_{\max}$ from uncertainty runs, avoidance

HmingLONData values of gLON = gLON for hub $H_{\min}$ from uncertainty runs, alignment

HmingLATData values of gLAT = gLAT for hub $H_{\min}$ from uncertainty runs, alignment

HmaxgLONData values of gLON = gLON for hub $H_{\max}$ from uncertainty runs, avoidance

HmaxgLATData values of gLAT = gLAT for hub $H_{\max}$ from uncertainty runs, avoidance

Tables:

ψData	entries: 1. Run #	2. ψSrc, list of polarization position angles ψ	
runData	entries: 1. Run #	2. {$\overline{\eta}_{\min}$, {gLON,gLAT} at $H_{\min}$}	3. {$\overline{\eta}_{\max}$, {gLON,gLAT} at $H_{\max}$}

To create Uncertainty Runs, first calculate “rSrcxrGrid” and then evaluate the “For” statement in the following two cells. One can save the results with the “Put[]” statements.

Once saved, there is no need to repeat the runs. Comment out the “rSrcxrGrid” and “For” statements by enclosing each in (*comment brackets*). The data can be retrieved with the “Get” statements.

In[174]:=

```
(*
rSrcxrGrid1=Table[ Cross[  rSrc[[i]],rGrid[[j]]  ] ,    {i,nSrc},{j,nGrid}];
(*first step: gLONw cross product, not unit vectors*)
rSrcxrGrid=Table[ rSrcxrGrid1[[i,j]]/
   (rSrcxrGrid1[[i,j]].rSrcxrGrid1[[i,j]]+ 0.000001)^(1/2.) ,    {i,nSrc},{j,nGrid}];
Clear[rSrcxrGrid1];
*)

(*rSrcxrGrid: table of the unit vectors perpendicular to the plane
  of the great circle containing the source Si and the grid point Hj*)
```

In[175]:=

```
(*
nR=2000;
(*number of runs with the PPA ψ allowed by measurement uncertainty. *)
μ=ψn;σ=σψn;runData={};ψData={};nRunPrint=0;
For[nRun=1,nRun≤nR,nRun++,
 If[nRun>nRunPrint,Print["At the start of run ",nRun,", the time is ",
   TimeUsed[]," seconds and the memory in use is ",MemoryInUse[]," bytes."];
  nRunPrint=nRunPrint+200];
    ψSrc=Table[RandomVariate[NormalDistribution[μ[[i]],σ[[i]]]],{i,nSrc}];
 (*table of PPA angles ψ for the sources in region j0, in radians*)
 rSrcxψSrc = Table[ Sin[ψSrc[[i]]]eNSrc[[i]]-Cos[ψSrc[[i]]] eESrc[[i]],    {i,nSrc}];
 (*table of the cross product of rSrc and vector in direction of ψSrc,
 a unit vector*)jηBarToGrid = Table[{j,(1/nSrc)Sum[ArcCos[
       Abs[  rSrcxψSrc[[i]].rSrcxrGrid[[i,j]]  ] - 0.000001  ],{i,nSrc}]},{j,nGrid}];
 (*
 {grid point #, value of the alignment angle ηnHj[j] averaged over all sources,
  in radians}*) sortjηBarToGrid=Sort[jηBarToGrid,#1[[2]]<#2[[2]]&];
 (*jηBarToGrid, {j,ηj}, but sorted with the smallest alignment angles first
 *)
 jηBarMin1=sortjηBarToGrid[[1]]; (*  {j,ηj}, at the grid point Hj with minimum η̅*)
 jηBarMax1=sortjηBarToGrid[[-1]]; (*  {j,ηj},
 at the grid point Hj with maximum η̅*)AppendTo[ψData,{nRun,ψSrc}];
 AppendTo[runData,{nRun,{ jηBarMin1[[2]],{gLONGrid [ [ jηBarMin1[[1]]  ]],
     gLATGrid [[  jηBarMin1[[1]]  ]]}},{ jηBarMax1[[2]],{gLONGrid [[
      jηBarMax1[[1]]  ]],gLATGrid [[  jηBarMax1[[1]]  ]]}}} ](*collect data*)     ]
*)
```

Hint: You can save memory if you do not get the "ψData". The table ψData is needed to reconstruct the exact values of the runData table, but it is not needed in any following calculation.

In[176]:=

```
SetDirectory[homeDirectory];(*Save memory space; ψData is not used below.*)
(*Put[ψData,"20210407PsiDataLon30Lat30offDiskHB.dat"  ] *) (*Save a new "ψData"*)
(*ψData=Get["20210407PsiDataLon30Lat30offDiskHB.dat"]; *) (*Get an old "ψData"*)
```

Hint: Saving "runData" to a file avoids the time it takes to complete the "For" statement. Make the above "For" statement into a remark so that it doesn't evaluate.

In[177]:=

```
 SetDirectory[homeDirectory];
(*Put[runData,"20210407runDataLon30Lat30offDiskHB.dat"  ] *)
(*Save a new "runData".*)
runData = Get["20210407runDataLon30Lat30offDiskHB.dat"]; (*Get an old "runData".*)
```

In[179]:=

```
Print["The number of uncertainty runs is ", Length[runData], "."]
```

```
The number of uncertainty runs is 2000.
```

In[180]:=
```
ηBarMinData = Table[runData[[i1, 2, 1]] , {i1, Length[runData]}];
ηBarMaxData = Table[runData[[i1, 3, 1]] , {i1, Length[runData]}];
HmingLONData = Table[ runData[[i1, 2, 2, 1]] , {i1, Length[runData]}];
HmingLATData = Table[runData[[i1, 2, 2, 2]], {i1, Length[runData]}];
HmaxgLONData = Table[ runData[[i1, 3, 2, 1]] , {i1, Length[runData]}];
HmaxgLATData = Table[runData[[i1, 3, 2, 2]], {i1, Length[runData]}];
```

6b. The Effects of Uncertainty on the Smallest Alignment Angle $\overline{\eta}_{\min}$

This section fits a Gaussian distribution to the $\overline{\eta}_{\min}$ from the uncertainty runs.

Definitions

sortηBarMin	sort the list of $\overline{\eta}_{\min}$ from the uncertainty runs
η0B	estimated mean of the Gaussian fit
σB	estimated half-width of the Gaussian fit
histogramrange	{min η, max η, $\Delta\eta$} for the histogram
hl0, hl	histogram {η, bin height} tables needed to set up the NonlinearModelFit
nlmB	non-linear model fit of a Gaussian to the $\overline{\eta}_{\min}$ histogram
showNLMB	plot of Gaussian and histogram
ParametersNLMB	amplitude, half-width, and mean of the Gaussian fit
pTableNLMB	table of parameter attributes, including standard error

In[186]:=
```
sortηBarMin = Sort[ηBarMinData];
η0B = mean[ηBarMinData ]; (*Guess the mean for the Gaussian. *)
σB = stanDev[ηBarMinData ];(*Guess the half-width.*)
histogramrange = {η0B - 5 σB, η0B + 5 σB, 0.4 σB};
hl0 = HistogramList[sortηBarMin, histogramrange];
hl =
 Table[{(1/2) (hl0[[1, i1]] + hl0[[1, i1 + 1]]), hl0[[2, i1]]}, {i1, Length[ hl0[[2]] ]}];
nlmB = NonlinearModelFit[hl, a Exp[- (1/2.) ((x - x0) / b)^2],
  {{a, Length[sortηBarMin / 6]}, {b, σB}, {x0, η0B}}, x];(*x is ηBarMin*)
```

In[192]:=
```
showNLMB = Show[{Histogram[sortηBarMin, histogramrange,
    PlotLabel → "η̄min ", AxesLabel → {"η̄min, radians", "ΔR"}],
   Plot[Normal[nlmB], {x, η0B - 5 σB, η0B + 5 σB}, PlotLabel → "η̄min"],
   ListPlot[hl, PlotLabel → "η̄min"] }]
Print["Figure 5: The Gaussian fit to the alignment angle
   η̄min histogram, where the height is the number "]
Print["of runs ΔR in each bin of width Δη̄min  =  ", 0.4 σB, " radians. "]
Print["The total number of runs is R = Σ(ΔR) = ", Length[runData], "."]
```

Out[192]=

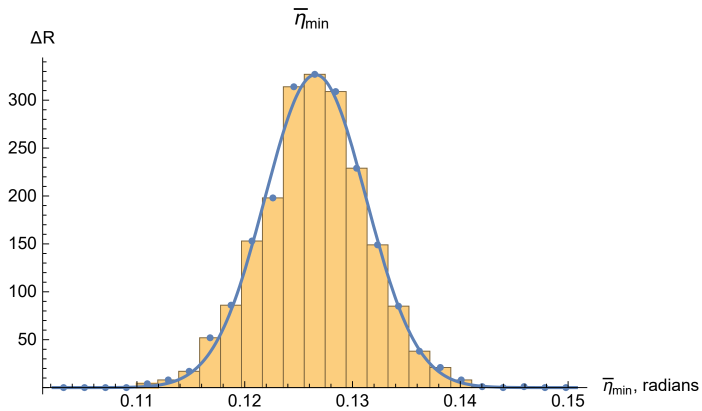


Figure 5: The Gaussian fit to the alignment angle $\bar{\eta}_{min}$ histogram, where the height is the number

of runs ΔR in each bin of width $\Delta\bar{\eta}_{min}$ = 0.00194171 radians.

The total number of runs is R = Σ(ΔR) = 2000.

In[196]:=
```
ParametersNLMB = {a, b, x0} /. nlmB["BestFitParameters"];
pTableNLMB = nlmB["ParameterTable"]
{σηBarMinFit, ηBarMinFit} = {ParametersNLMB[[2]] , ParametersNLMB[[3]] };(*radians*)
```

Out[197]=

	Estimate	Standard Error	t-Statistic	P-Value
a	326.618	5.14978	63.4237	2.07943×10^{-26}
b	0.00468516	0.0000852987	54.9265	4.83034×10^{-25}
x0	0.126568	0.0000852987	1483.83	1.66713×10^{-56}

6c. The Effects of Uncertainty on the Largest Avoidance Angle $\bar{\eta}_{max}$

This section fits a Gaussian distribution to the $\bar{\eta}_{max}$ returned by the uncertainty runs.

Definitions: Check the list of Definitions in Sec. 6b. Trade avoidance (Max) here for alignment (Min) there.

In[199]:=
```
sortηBarMax = Sort[ηBarMaxData];
η0MaxB = mean[ηBarMaxData ]; (*Guess the mean for the Gaussian. *)
σMaxB  = stanDev[ηBarMaxData ];(*Guess the half-width.*)
histogramrangeMAX = {η0MaxB - 5 σMaxB, η0MaxB + 5 σMaxB, 0.4 σMaxB};
hl0Max = HistogramList[sortηBarMax, histogramrangeMAX];
hlMax = Table[{ (1/2) (hl0Max[[1, i1]] + hl0Max[[1, i1 + 1]]), hl0Max[[2, i1]]},
  {i1, Length[ hl0Max[[2]] ]}];
nlmMaxB = NonlinearModelFit[hlMax, a Exp[- (1/2.) ((x - x0) / b)^2],
  {{a, 300.}, {b, σMaxB}, {x0, η0MaxB}}, x];(*x is ηBarMax *)
```

In[205]:=
```
showNLMMaxB = Show[{Histogram[sortηBarMax,
    histogramrangeMAX, PlotLabel → "η̄max", AxesLabel → {"η̄max, radians", "ΔR"}],
   Plot[Normal[nlmMaxB], {x, η0MaxB - 5 σMaxB, η0MaxB + 5 σMaxB}, PlotLabel → "η̄max"] ,
   ListPlot[hlMax, PlotLabel → "η̄max"]}]
Print["Figure 6: The Gaussian fit to the avoidance angle η̄max
   histogram. The bins have a width Δη̄max  =  ", 0.4 σMaxB,
 " radians and have a height equal to the number of runs ΔR in the bin."]
Print["The total number of runs is R = Σ(ΔR) = ", Length[runData], "."]
```

Out[205]=

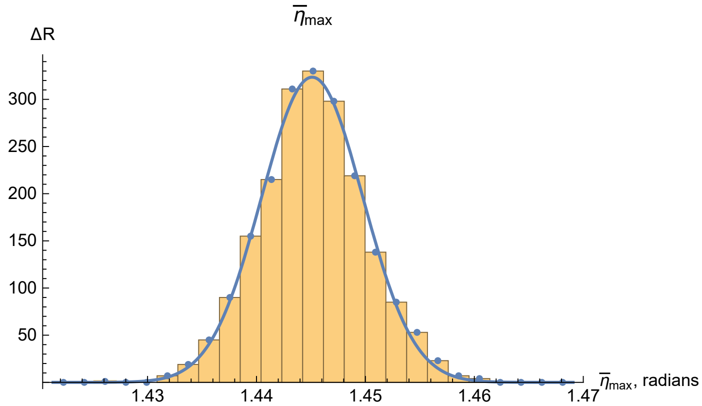


Figure 6: The Gaussian fit to the avoidance angle $\bar{\eta}_{max}$ histogram. The bins have a width $\Delta\bar{\eta}_{max}$ = 0.00190934 radians and have a height equal to the number of runs ΔR in the bin.

The total number of runs is R = Σ(ΔR) = 2000.

In[208]:=
```
ParametersNLMMaxB = {a, b, x0} /. nlmMaxB["BestFitParameters"];
pTableNLMMaxB = nlmMaxB["ParameterTable"]
{σηBarMaxFit, ηBarMaxFit} = {ParametersNLMMaxB[[2]] , ParametersNLMMaxB[[3]] };
(*radians*)
```

Out[209]=

	Estimate	Standard Error	t-Statistic	P-Value
a	323.477	4.15262	77.8972	2.30356×10^{-28}
b	0.00465072	0.0000689394	67.4609	5.38538×10^{-27}
x0	1.44511	0.0000689394	20962.1	8.33478×10^{-82}

6d. The Effects of Uncertainty on the Locations (gLON,gLAT) of the Alignment Hubs H_{min}

Each uncertainty run returns an alignment hub H_{min}. In this section, we calculate the mean and standard deviation to approximate the distribution of the locations the Alignment Hubs H_{min}.

An Issue: In any one run, the analysis produces an alignment angle $\bar{\eta}$ at each grid point. There can be just one minimum alignment angle $\bar{\eta}_{min}$, but there are two hubs, H_{min} and $-H_{min}$, by the symmetry across a diameter. So we collect all the hubs together by moving the $-H_{min}$ hubs across a diameter to join the H_{min} hubs.

Definitions

HmingLON	gLON in radians for H_{min}
HmingLAT	gLAT in radians for H_{min}
σgLONMinFit1	half-width for gLON uncertainty runs
gLONMinFit1	mean for gLON uncertainty runs

σgLATMinFit1	half-width for gLAT uncertainty runs
gLATMinFit1	mean for gLAT uncertainty runs
HmingLONAVE	average over all uncertainty runs of gLON for H_{min}
HmingLONgLAT	(gLON,gLAT) table for ListPlot
lpHmin	plot Hmin hubs from uncertainty runs
gLON1,2Min1	values needed for framing the most likely hubs
gLAT1,2Min1	ditto for latitude

In[211]:=
```
(* Gather the hubs. Move the hubs across diameters,
ΔgLON = π, or around a complete circle, ΔgLON = 360°,
if necessary, so that all hubs satisfy 0° ≤  gLON  <  180° .*)
HmingLON0 = HmingLONData;
HmingLAT0 = HmingLATData;
HmingLONBy180n = Round[HmingLON0 / π];
HmingLON1 = Table[HmingLON0[[i1]] - HmingLONBy180n[[i1]] π , {i1, Length[HmingLON0]}];
HmingLAT1 = Table[(-1)^HmingLONBy180n[[i1]] HmingLAT0[[i1]] , {i1, Length[HmingLAT0]}];
HmingLON  = Table[If[HmingLON1[[i1]] < 0, HmingLON1[[i1]] + π, HmingLON1[[i1]], "huh?"] ,
   {i1, Length[HmingLON1]}];
HmingLAT  = Table[If[HmingLON1[[i1]] < 0, -HmingLAT1[[i1]], HmingLAT1[[i1]], "huh?"] ,
   {i1, Length[HmingLAT1]}];
```

In[217]:=
```
(*Check that 0° ≤  gLON  <  180°  and -90° ≤  gLAT  <  90° *)
(*ListPlot[{Sort[HmingLON],Sort[HmingLAT]},
 PlotLabel→"gLON and gLAT for Hmin, radians",AxesLabel→{"Run #","gLON,gLAT"}]*)
```

In[218]:=
```
{σgLONMinFit1, gLONMinFit1} = {stanDev[HmingLON], mean[HmingLON]};(*radians*)
{σgLATMinFit1, gLATMinFit1} = {stanDev[HmingLAT], mean[HmingLAT]};(*radians*)
```

In[220]:=
```
(*Define quantities for the plot of the Hmin from the uncertainty runs. *)
HmingLONgLAT = Sort[Table[{-HmingLON[[i5]], HmingLAT[[i5]]}, {i5, Length[HmingLON]}]];
{HmingLONgLAT[[1]], HmingLONgLAT[[-1]]} ;(*radians*)
{HmingLONgLAT[[1]], HmingLONgLAT[[-1]]} (360. / (2. π)) ;(*degrees*)
lpHmin = ListPlot[HmingLONgLAT (360. / (2. π)), PlotRange → {{-180, 180}, {-90, 90}},
   PlotMarkers → Automatic, AxesLabel → {"-gLON, degrees", "gLAT, degrees"},
   PlotLabel → "(-gLON,gLAT) for the Hmin hubs",
   Ticks → {Table[{t, -t}, {t, -180, 180, 45}], Automatic}];
gLON1Min1 = (gLONMinFit1 - σgLONMinFit1) (360. / (2. π));
gLON2Min1 = (gLONMinFit1 + σgLONMinFit1) (360. / (2. π));
gLAT1Min1 = (gLATMinFit1 - σgLATMinFit1) (360. / (2. π));
gLAT2Min1 = (gLATMinFit1 + σgLATMinFit1) (360. / (2. π));
```

6e. The Effects of Uncertainty on the Locations (gLON,gLAT) of the Avoidance Hubs H_{max}.

Each uncertainty run returns an alignment hub H_{max}. In this section, we calculate the mean and standard deviation all such hubs to approximate the distribution of the locations of the Avoidance Hubs H_{max}.

Definitions: Explore the definitions for H_{min} at the start of Sec. 6d. Find the similarly named quantity by interchanging Max for Min. Adjust the definition to the present context.

```
In[228]:= (* Move hubs, if necessary, so that 0°  ≤  gLON  <  360° *)
HmaxgLON0 = HmaxgLONData;
HmaxgLAT0 = HmaxgLATData;
HmaxgLONBy180n = Round[HmaxgLON0 / π];
HmaxgLON1 = Table[HmaxgLON0[[i1]] - HmaxgLONBy180n[[i1]] π , {i1, Length[HmaxgLON0]}];
HmaxgLAT1 = Table[(-1)^HmaxgLONBy180n[[i1]] HmaxgLAT0[[i1]] , {i1, Length[HmaxgLAT0]}];
HmaxgLON  = Table[If[0 > HmaxgLON1[[i1]], HmaxgLON1[[i1]] + π, HmaxgLON1[[i1]], "huh?"] ,
   {i1, Length[HmaxgLON1]}];
HmaxgLAT  = Table[If[0 > HmaxgLON1[[i1]], -HmaxgLAT1[[i1]], HmaxgLAT1[[i1]], "ah"] ,
   {i1, Length[HmaxgLAT1]}];

In[234]:= (*Check that 0°  ≤  gLON  <  180°  and -90° ≤  gLAT  <  90° *)
(*ListPlot[{Sort[HmaxgLON],Sort[HmaxgLAT]},PlotRange→{-2π,2π},
 AxesLabel→{"Run #","gLON,gLAT radians"},PlotLabel→"gLONs, gLATs for Hmax"]*)

In[235]:= {σgLONMaxFit, gLONMaxFit} = {stanDev[HmaxgLON], mean[HmaxgLON]};(*radians*)
{σgLATMaxFit, gLATMaxFit} = {stanDev[HmaxgLAT], mean[HmaxgLAT]};(*radians*)

In[237]:= (* Define quantities for the plot of the
 locations of the Hmax from the uncertainty runs. *)
HmaxgLONgLAT = Table[{-HmaxgLON[[i8]], HmaxgLAT[[i8]]}, {i8, Length[HmaxgLAT ]}];
{HmaxgLONgLAT[[1]], HmaxgLONgLAT[[-1]]} ;(*radians*)
{HmaxgLONgLAT[[1]], HmaxgLONgLAT[[-1]]} (360. / (2. π)) ;(*degrees*)
lpHmax1 = ListPlot[HmaxgLONgLAT (360. / (2. π)), PlotRange → {{-180, +180}, {-90, 90}},
   PlotMarkers → Automatic, AxesLabel → {"-gLON, degrees", "gLAT, degrees"},
   PlotLabel → "Hmax hubs with the most likely region indicated ",
   Ticks → {Table[{t, -t}, {t, -180, 180, 45}], Automatic}];
gLON1Max = (gLONMaxFit - σgLONMaxFit) (360. / (2. π));
gLON2Max = (gLONMaxFit + σgLONMaxFit) (360. / (2. π));
gLAT1Max = (gLATMaxFit - σgLATMaxFit) (360. / (2. π));
gLAT2Max = (gLATMaxFit + σgLATMaxFit) (360. / (2. π));
```

6f. The Effects of Uncertainty on the angle θ between the planes of the Sample to $H_{\min}$ Great Circle and the Sample to $H_{\max}$ Great Circle.

These are the Gray lines in Fig. 4.

Definitions:

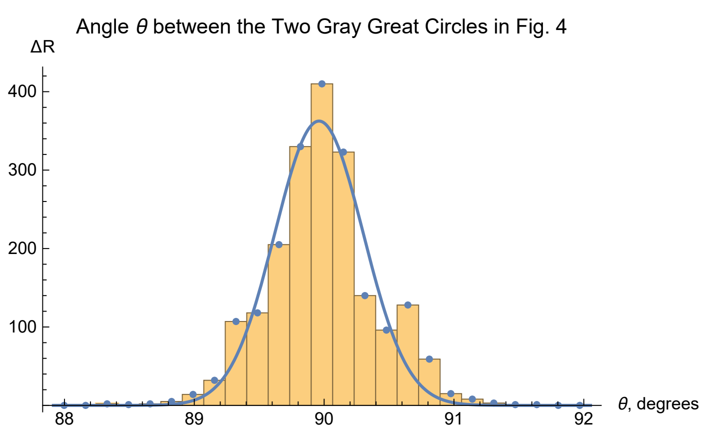
“uRuns” prefix	results from the uncertainty runs
uRunsCrossMin	unit vector normal to the Great Circle connecting the center of the source region with the alignment hub $H_{\min}$
uRunsCrossMax	unit vector normal to the Great Circle connecting the center of the source region with the alignment hub $H_{\max}$
uRunsθminMAXgreatcircles	angle between the two normals in degrees
sortθminMAX	sort “uRunsθminMAXgreatcircles”, smallest θ first
See Definitions above in Secs. 6a,6b for other quantities below. There you should find similarly named quantities.

```
In[245]:= uRunsCrossMin0 =
            Table[Cross[er[HmingLON[[i]], HmingLAT[[i]]], sourceCenter], {i, Length[HmingLON]}];
          uRunsCrossMin = Table[uRunsCrossMin0[[i]] / (uRunsCrossMin0[[i]].uRunsCrossMin0[[i]])^(1/2.),
              {i, Length[HmingLON]}];
          uRunsCrossMax0 = Table[Cross[er[HmaxgLON[[i]], HmaxgLAT[[i]]], sourceCenter],
              {i, Length[HmaxgLON]}];
          uRunsCrossMax = Table[uRunsCrossMax0[[i]] / (uRunsCrossMax0[[i]].uRunsCrossMax0[[i]])^(1/2.),
              {i, Length[HmaxgLON]}];
          uRunsθminMAXgreatcircles = Table[ArcCos[uRunsCrossMax[[i]].uRunsCrossMin[[i]]] (360./(2. π)),
              {i, Length[HmaxgLON]}];

In[250]:= sortθminMAX = Sort[uRunsθminMAXgreatcircles];
          η0θ = mean[uRunsθminMAXgreatcircles]; (*Guess the mean for the Gaussian. *)
          σθ = stanDev[uRunsθminMAXgreatcircles];(*Guess the half-width.*)
          histogramrange = {η0θ - 5 σθ, η0θ + 5 σθ, 0.4 σθ};
          hl0 = HistogramList[sortθminMAX, histogramrange];
          hl =
           Table[{(1/2) (hl0[[1, i1]] + hl0[[1, i1 + 1]]), hl0[[2, i1]]}, {i1, Length[ hl0[[2]] ]}];
          nlmθ = NonlinearModelFit[hl, a Exp[- (1/2.) ((x - x0) / b)^2],
            {{a, Length[sortθminMAX / 6]}, {b, σθ}, {x0, η0θ}}, x];(*x is θminMAX*)

In[256]:= showNLMθ = Show[{Histogram[sortθminMAX, histogramrange,
              PlotLabel → "Angle θ between the Two Gray Great Circles in Fig. 4",
              AxesLabel → {"θ, degrees", "ΔR"}],
             Plot[Normal[nlmθ], {x, η0θ - 5 σθ, η0θ + 5 σθ}], ListPlot[hl] }]
          Print["Figure 7: The Gaussian fit to the angle θ histogram,
             where the height is the number of runs ΔR in"]
          Print[" each bin of width Δθ  =  ", 0.4 σθ, " degrees."]
          Print[" The total number of runs is R = Σ(ΔR)  = ", Length[runData], "."]
```

Out[256]=



```
Figure 7: The Gaussian fit to the angle θ histogram, where the height is the number of runs ΔR in
 each bin of width Δθ  =  0.165476 degrees.
 The total number of runs is R = Σ(ΔR) = 2000.
```

In[260]:=
```
ParametersNLMθ = {a, b, x0} /. nlmθ["BestFitParameters"];
pTableNLMθ = nlmθ["ParameterTable"]
{σθminMAXFit, θminMAXFit} = {ParametersNLMθ[[2]] , ParametersNLMθ[[3]] };(*degrees*)
```

Out[261]=

	Estimate	Standard Error	t-Statistic	P-Value
a	362.52	19.4077	18.6791	5.52892×10^{-15}
b	0.34016	0.0210279	16.1766	1.06491×10^{-13}
x0	89.9605	0.0210279	4278.15	1.27292×10^{-66}

6g. Map of the Hubs for the Uncertainty Runs

In this subsection, we map the locations of the many alignment hubs H_{min} and the locations of the avoidance hubs H_{max} that are found in the uncertainty runs.

Definitions:

xyAitoffHmin	Aitoff coordinates for the alignment hubs H_{min} from the uncertainty runs
xyAitoffHmax	Aitoff coordinates for the avoidance hubs H_{max} from the uncertainty runs
xyAitoffOppositeHmin	Aitoff coordinates for the $-H_{\text{min}}$
xyAitoffOppositeHmax	Aitoff coordinates for the $-H_{\text{max}}$
mapOfσψHminHmax	plot of the alignment and avoidance hubs H_{min}, $-H_{\text{min}}$, H_{max}, and $-H_{\text{max}}$

In[263]:=
```
(*The Aitoff coordinates for the hubs Hmin locations.*)xyAitoffHmin =
  Table[{xHGal[ HmingLON [[n]] (360/(2 π)), HmingLAT[[n]] (360/(2 π)) ], yHGal[
     HmingLON [[n]] (360/(2 π)), HmingLAT[[n]] (360/(2 π)) ]}, {n, Length[HmingLAT ]}];
(*The Aitoff coordinates for the hubs Hmax locations.*)xyAitoffHmax =
  Table[{xHGal[ HmaxgLON [[n]] (360/(2 π)), HmaxgLAT[[n]] (360/(2 π)) ], yHGal[
     HmaxgLON [[n]] (360/(2 π)), HmaxgLAT[[n]] (360/(2 π)) ]}, {n, Length[HmingLAT ]}];
(*The Aitoff coordinates for the hubs -Hmin locations.*)
xyAitoffOppositeHmin = Table[{xHGal[ If[0 ≤ HmingLON [[n]] (360/(2 π)) < +180,
      HmingLON [[n]] (360/(2 π)) - 180, If[0 > HmingLON [[n]] (360/(2 π)) > -180,
       HmingLON [[n]] (360/(2 π)) + 180]], -HmingLAT[[n]] (360/(2 π)) ],
    yHGal[ If[0 ≤ HmingLON [[n]] (360/(2 π)) < +180, HmingLON [[n]] (360/(2 π)) - 180,
      If[0 > HmingLON [[n]] (360/(2 π)) > -180, HmingLON [[n]] (360/(2 π)) + 180]],
     -HmingLAT[[n]] (360/(2 π)) ]}, {n, Length[HmingLAT ]}];
(*The Aitoff coordinates for the hubs -Hmax locations.*)
xyAitoffOppositeHmax =
  Table[{xHGal[ If[0 ≤ HmaxgLON [[n]] (360/(2 π)) < +180, HmaxgLON [[n]] (360/(2 π)) - 180,
      If[0 > HmaxgLON [[n]] (360/(2 π)) > -180, HmaxgLON [[n]] (360/(2 π)) + 180]],
     -HmaxgLAT[[n]] (360/(2 π)) ],
    yHGal[ If[0 ≤ HmaxgLON [[n]] (360/(2 π)) < +180, HmaxgLON [[n]] (360/(2 π)) - 180,
      If[0 > HmaxgLON [[n]] (360/(2 π)) > -180, HmaxgLON [[n]] (360/(2 π)) + 180]],
     -HmaxgLAT[[n]] (360/(2 π)) ]}, {n, Length[HmaxgLAT ]}];
```

In[267]:=
```
(*Construct the map of uncertainty run Hmin and Hmax hubs with ± regions indicated.*)
mapOfσψHminHmax =
  Show[{Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]},
      {gLAT, -90, 90}, PlotStyle → {Black, Thickness[0.002]}, PlotPoints → 60,
      PlotRange → {{-7, 7}, {-3, 3}}, Axes → False], {gLON, -180, 180, 30}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLON, -180, 180},
      PlotStyle → {Black, Thickness[0.002]}, PlotPoints → 60], {gLAT, -60, 60, 30}],
    Graphics[{PointSize[0.007], Text[StyleForm["N", FontSize -> 10, FontWeight -> "Plain"],
       {0, 1.85}], LightBlue, (*Hmin:*)Point[ xyAitoffHmin ],
      (*-Hmin:*)Point[ xyAitoffOppositeHmin ], LightRed, (*Hmax:*)
      Point[ xyAitoffHmax ], (*-Hmax:*)Point[ xyAitoffOppositeHmax ]     }],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLAT, gLAT1Max, gLAT2Max},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {gLON, gLON1Max, gLON2Max, gLON2Max - gLON1Max}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLON, gLON1Max, gLON2Max},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {gLAT, gLAT1Max, gLAT2Max, gLAT2Max - gLAT1Max}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLAT, -gLAT2Max, -gLAT1Max},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {gLON, gLON1Max - 180, gLON2Max - 180, gLON2Max - gLON1Max}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]},
      {gLON, gLON1Max - 180, gLON2Max - 180}, PlotStyle → {Purple, Thickness[0.002]},
      PlotPoints → 60], {gLAT, -gLAT2Max, -gLAT1Max, gLAT2Max - gLAT1Max}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLAT, -gLAT2Min1, -gLAT1Min1},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {gLON, gLON1Min1 - 180, gLON2Min1 - 180, gLON2Min1 - gLON1Min1}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]},
      {gLON, gLON1Min1 - 180, gLON2Min1 - 180}, PlotStyle → {Purple, Thickness[0.002]},
      PlotPoints → 60], {gLAT, -gLAT2Min1, -gLAT1Min1, gLAT2Min1 - gLAT1Min1}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLAT, gLAT1Min1, gLAT2Min1},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {gLON, gLON1Min1, gLON2Min1, gLON2Min1 - gLON1Min1}],
    Table[ParametricPlot[{xHGal[gLON, gLAT], yHGal[gLON, gLAT]}, {gLON, gLON1Min1, gLON2Min1},
      PlotStyle → {Purple, Thickness[0.002]}, PlotPoints → 60],
     {gLAT, gLAT1Min1, gLAT2Min1, gLAT2Min1 - gLAT1Min1}]},
   ImageSize → 2 × 432, PlotLabel → "The Hubs Found from the Uncertainty Runs"];
```

Section Summary

In[268]:=
```
Print["To estimate the effects of experimental uncertainty, there were ",
  Length[runData], " uncertainty runs."]
Print["Uncertainty runs have polarization directions ψ = ψn + δψ, ",
  "where δψ is chosen with a normal
    distribution of half-width σψ about the best value ψn."]
Print["The uncertainty runs determine the smallest alignment angle to be  η̄min = ",
  ηBarMinFit (360. / (2. π)), "° ± ", σηBarMinFit (360. / (2. π)), "°."]
Print["The uncertainty runs determine the largest avoidance angle to be  η̄max = ",
  ηBarMaxFit (360. / (2. π)), "° ± ", σηBarMaxFit (360. / (2. π)), "°."]
Print["The uncertainty runs give the location for
    one of the alignment hub Hmin  as  (gLON, gLAT)  = ",
  {gLONMinFit1 (360. / (2. π)), gLATMinFit1 (360. / (2. π))}, " ± ",
  {σgLONMinFit1 (360. / (2. π)), σgLATMinFit1 (360. / (2. π))}, ", in degrees."]
Print["The other hub, -Hmin , is located diametrically opposite from Hmin ."]
Print[
  "The uncertainty runs give the location of the avoidance hub Hmax  as  (gLON, gLAT)  = ",
  {gLONMaxFit (360. / (2. π)), gLATMaxFit (360. / (2. π))}, " ± ",
  {σgLONMaxFit (360. / (2. π)), σgLATMaxFit (360. / (2. π))}, ", in degrees."]
Print["The other hub, -Hmax , is located diametrically opposite from Hmax ."]
Print["The uncertainty runs determine the angle Θ between the two
    grey Great Circles to be  Θ = ", ΘminMAXFit, "° ± ", σΘminMAXFit, "°."]
Print["For Θ, see the caption to Fig. 4."]
```

To estimate the effects of experimental uncertainty, there were 2000 uncertainty runs.

Uncertainty runs have polarization directions $\psi = \psi n + \delta\psi$,
where $\delta\psi$ is chosen with a normal distribution of half-width $\sigma\psi$ about the best value ψn.

The uncertainty runs determine the smallest alignment angle to be $\overline{\eta}_{\text{min}}$ = 7.25184° ± 0.26844°.

The uncertainty runs determine the largest avoidance angle to be $\overline{\eta}_{\text{max}}$ = 82.7989° ± 0.266467°.

The uncertainty runs give the location for one of the alignment hub H_{min} as (gLON, gLAT) = {77.8876, −50.612} ± {6.91636, 3.96099}, in degrees.

The other hub, $-H_{\text{min}}$, is located diametrically opposite from H_{min}.

The uncertainty runs give the location of the avoidance hub H_{max} as (gLON, gLAT) = {120.901, 31.} ± {4.84243, 2.31603}, in degrees.

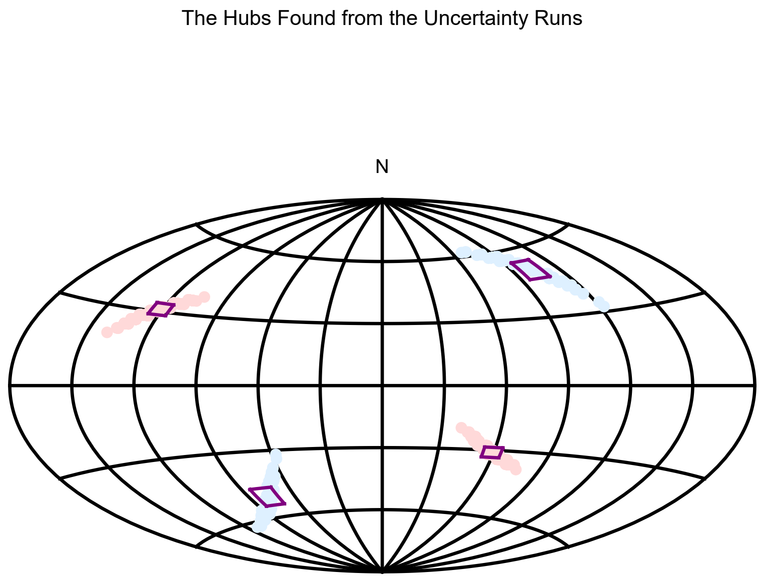
The other hub, $-H_{\text{max}}$, is located diametrically opposite from H_{max}.

The uncertainty runs determine the angle Θ between the two grey Great Circles to be Θ = 89.9605° ± 0.34016°.

For Θ, see the caption to Fig. 4.

In[278]:=
```
mapOfσψHminHmax
Print["Figure 8: The ", Length[runData], " sets of hubs found for the uncertainty runs."]
Print["The alignment hubs H_min and -H_min are plotted as light blue dots, ", LightBlue, ". "]
Print["The avoidance hubs H_max and -H_max are plotted as pink dots, ", LightRed, "."]
Print["The most likely locations of the hubs are outlined in purple, ", Purple, "."]
```

Out[278]=




```
Figure 8: The 2000 sets of hubs found for the uncertainty runs.
The alignment hubs H_min and -H_min are plotted as light blue dots, ■.
The avoidance hubs H_max and -H_max are plotted as pink dots, ■.
The most likely locations of the hubs are outlined in purple, ■.
```

As a final image, we superimpose the map of the uncertainty run hubs $H_{\min}$, $-H_{\min}$, $H_{\max}$, and $-H_{\max}$ in Fig. 8 on the graph of the alignment angle function $\overline{\eta}(H)$, Fig. 4.

In[283]:=

```
Show[{mapOfηBar, mapOfσψHminHmax}]
Print[
 "Figure 9: Overlay Fig. 8, Uncertainty Run Hubs, onto Fig. 4, Alignment Function η̄(H)
   using Best Values ψn.  Note that the light blue alignment hubs from the uncertainty
   runs closely follow the areas of convergence (blue) for the best values ψn. And
   the pink avoidance hubs follow the areas of extreme divergence (red). One sees
   that shifting the polarization directions slightly due to experimental uncertainty,
   shifts the locations of the hubs slightly. The shifted hubs favor areas, in blue
   and red, that are close to the extremes for the alignment function η̄(H) in Fig.4."]
```

Out[283]=

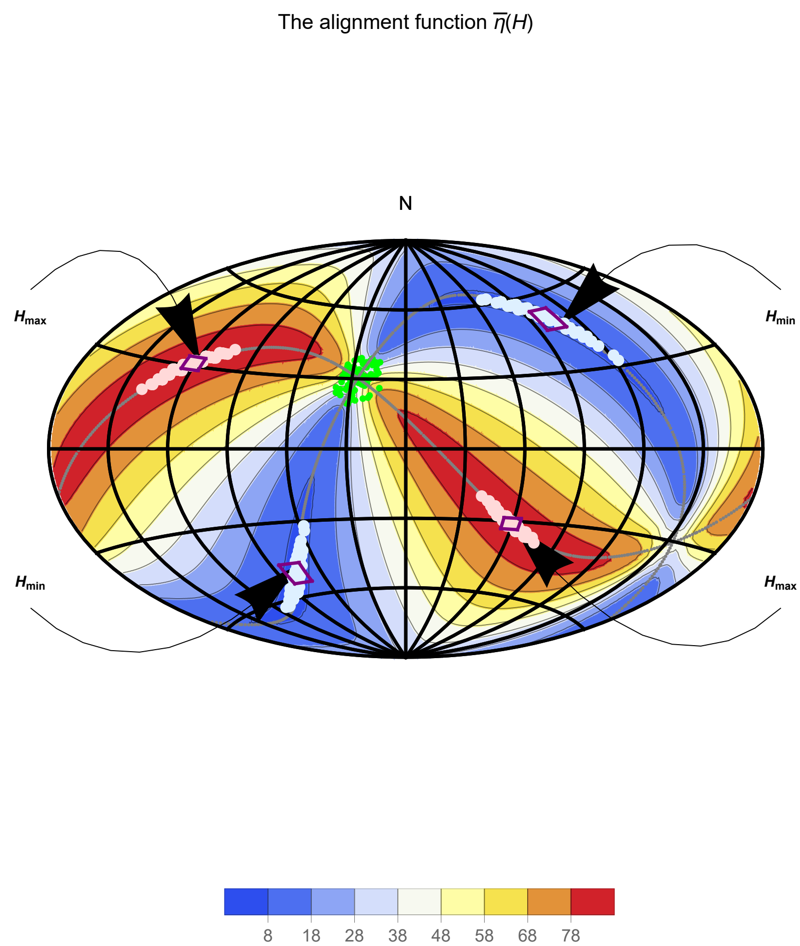


Figure 9: Overlay Fig. 8, Uncertainty Run Hubs, onto Fig. 4, Alignment Function $\overline{\eta}(H)$ using Best Values ψn. Note that the light blue alignment hubs from the uncertainty runs closely follow the areas of convergence (blue) for the best values ψn. And the pink avoidance hubs follow the areas of extreme divergence (red). One sees that shifting the polarization directions slightly due to experimental uncertainty, shifts the locations of the hubs slightly. The shifted hubs favor areas, in blue and red, that are close to the extremes for the alignment function $\overline{\eta}(H)$ in Fig.4.

7. Concluding Remarks

The polarization of starlight is a well-known phenomenon that has been important in understanding the structure of the magnetic field of the Milky Way Galaxy. The scale of the magnetic fields is large enough that starlight from regions containing large numbers of stars should confront similar environments on the way to being detected. So, it is not surprising to find that the polarization directions for stars in many good-sized regions of the Galaxy are well aligned.

The stars in the sample and the surrounding region are well-known to be polarized in the general direction from southeast to northwest, see Fig. 4 in Ref. 7 and Fig. 3 in Ref. 12, for example. If the 99 stars in the sample had randomly directed polarization directions, they would have more widely scattered polarization directions with an alignment angle near 40° and an avoidance angle near 50°. The observed polarization directions converge to a value of $\overline{\eta}_{\rm min}$ of about 7° and an avoidance angle $\overline{\eta}_{\rm max}$ of about 83°. Both for alignment and for avoidance, *i.e.* for the observed $\overline{\eta}_{\rm min}$ and $\overline{\eta}_{\rm max}$, the results occur about 20σs from the results with random polarization values. The significance is infinitesimal, about 10^{-83} or less. One concludes that the alignment is not explained by chance. Note that the Hub Test supplies numerical values to help make that determination.

The 99 stars in this region have polarization correlations that illustrate an extreme case. With samples of Galactic stellar polarization, the extreme case may not be unusual. However, when, as here, the polarization directions are all nearly equal, the smallest alignment angles $\overline{\eta}(H)$ arrange themselves along an “equator” , a Great Circle moving away from the sources in the direction of the best convergence point at the alignment hubs $H_{\rm min}$ and $-H_{\rm min}$. The sample is an extreme case because the directions perpendicular to the polarization directions are also well-correlated, making a second Great Circle through the avoidance hubs $H_{\rm max}$ and $-H_{\rm max}$. See the Grey circles in Fig. 4. The two Great Circles are perpendicular within experimental uncertainty at the points of intersection. The Hub Test treats alignment and avoidance equally, with symmetry between avoidance and alignment. In the extreme case here, both alignment and avoidance are remarkably strong.

References

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3. Wolfram Research, Inc., Mathematica, Version 12.1, Champaign, IL (2020).
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13. Berdyugin A., *et al.*, The catalog, ReadMe, and other files related to Ref. 12 are available online at VizieR; http://cdsarc.u-strasbg.fr/viz-bin/ReadMe/J/A+A/561/A24?format=html&tex=true