Another Look at “Faulhaber and Bernoulli”*

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Abstract

Let “Faulhaber’s formula” refer to an expression for the sum of powers of integers written with terms of n(n+1)/2. Initially, the author used Faulhaber’s formula to explain why odd Bernoulli numbers are equal to zero. Next, Cereceda gave alternate proofs of that result and then proved the converse, if odd Bernoulli numbers are equal to zero then we can derive Faulhaber’s formula. Here, the original author will give a new proof of the converse.

1 Motivation

If we knew nothing of the history of the problem and tried to discover for ourselves a general expression for

$$\sum_{k=1}^{n} k^m = 1^m + 2^m + \cdots + n^m,$$

where n, m are positive integers, we might notice there appear to be two ways to write such sums. For example,

$$\begin{aligned}\sum k &= \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n,\\ \sum k^2 &= \frac{2n+1}{3}\cdot\frac{n(n+1)}{2} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n,\\ \sum k^3 &= \left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2.\end{aligned} \tag{1}$$

The next two cases are

$$\sum k^4 = \frac{1}{5}\left[6\cdot\frac{n(n+1)}{2}-1\right]\cdot\sum k^2 = \frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n,$$
$$\sum k^5 = \frac{1}{3}\left[4\cdot\frac{n(n+1)}{2}-1\right]\left(\sum k\right)^2 = \frac{1}{6}n^6+\frac{1}{2}n^5+\frac{5}{12}n^4-\frac{1}{12}n^2. \quad (2)$$

(When no confusion will arise, we will abbreviate $\sum_{k=1}^{n} k^m$ by $\sum k^m$.)

We can write each sum using terms of $\frac{n(n+1)}{2}$ or n. Of course, if we have the former then we always can expand it into the latter. Do we always have the former?

At a later point, reading up on the matter we would learn that writing an expression for $\sum k^m$ using terms in n is associated with the name of Jakob Bernoulli (1654-1705), and writing the same expression using terms in $\frac{n(n+1)}{2}$ is associated with that of Johann Faulhaber (1580-1635) (Edwards [3], Edwards [4], Knuth [6]). Bernoulli's contribution has long overshadowed Faulhaber's, but now we know the two are linked inextricably.

2 Background

In order to write an expression for $\sum k^m$ in n, we first introduce the Bernoulli numbers. Set $B_0 = 1$ and define B_n by

$$\sum_{k=1}^{n} \binom{n+1}{k} B_k = 0,$$

where $n \geq 1$. Then we can write

$$\sum k^m = \frac{1}{m+1}\sum_{j=0}^{m} (-1)^j \binom{m+1}{j} B_j n^{m+1-j}. \quad (3)$$

For a sum in $\frac{n(n+1)}{2}$, for odd powers we have

$$\sum k^{2m+1} = \left[a_1\left(\frac{n(n+1)}{2}\right)^{m-1} - a_2\left(\frac{n(n+1)}{2}\right)^{m-2} \pm\cdots\pm a_{m-1}\cdot\frac{n(n+1)}{2} \pm a_m\right]\left(\sum k\right)^2, \quad (4)$$

and for even powers we have

$$\sum k^{2m} = \left[c_1\left(\frac{n(n+1)}{2}\right)^{m-1} - c_2\left(\frac{n(n+1)}{2}\right)^{m-2} \pm\cdots\pm c_{m-1}\cdot\frac{n(n+1)}{2} \pm c_m\right]\sum k^2, \quad (5)$$

where the a_i, c_i are rational numbers and $m \geq 1$. We will refer to these two expressions as Faulhaber's formula. (We can find explicit values for the coefficients a_i, c_i (Gessel and Viennot [5, section 12]), but what concerns us here are the overall forms of the sums.)

If we look at the expressions in (1) and (2), we notice a few powers of n are missing. The reason is because odd Bernoulli numbers are equal to zero: $B_1 = -\frac{1}{2}$, but for all $m \geq 1$, $B_{2m+1} = 0$. When we write such sums using (3), the powers of n which have odd Bernoulli numbers for coefficients drop out.

There are a number of ways to prove such a result (Rademacher [7, chapters 1-2]), but none seems to provide any deeper explanation for why this is so. The key insight of Zielinski [9] was that Faulhaber's formula implies such an outcome.

If we write $\sum k^{2m+1}$ in the two forms of (3) and (4), the coefficients for the terms of n must agree. Since (4) contains a factor of $\left(\sum k\right)^2 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$, there is no term in n. That means the last term of (3), $-B_{2m+1} \cdot n$, must be equal to zero. In other words, $B_{2m+1} = 0$ for all $m \geq 1$.

Cereceda [2] chose a different line of attack and introduced Bernoulli polynomials:

$$B_m(x) = \sum_{j=0}^{m} \binom{m}{j} B_j x^{m-j},$$

where x is a real variable and $B_m(0) = B_m$. This allows an expression for the sum of powers to be written as

$$\sum k^m = \frac{1}{m+1}\left(B_{m+1}(n+1) - B_{m-1}\right). \tag{6}$$

In the new context, the property $B_{2m+1} = 0$ is related to Bernoulli polynomials being evaluated at $x = \frac{1}{2}$. The notion of symmetry then allows such polynomials to be rewritten with terms of $\frac{x(x-1)}{2}$. Once this is done, (6) leads immediately to Faulhaber's formula. (Again, alternate proofs of the main result of Zielinski [9] are contained within the paper as well.)

Taken together, the papers lead to a surprising revelation, one which has been a long time in the making. Denote $\sum_{k=1}^{n} k^m$ by S_m. Then we have

Theorem 1. *(Zielinski-Cereceda) for positive integers m,*

$$B_{2m+1} = 0 \iff \begin{cases} S_{2m} = & S_2 \cdot P_{2m}\left(S_1\right), \\ S_{2m+1} = & S_1^2 \cdot P_{2m+1}\left(S_1\right), \end{cases}$$

where $P_{2m}\left(S_1\right)$ and $P_{2m+1}\left(S_1\right)$ are polynomials in $S_1 = \frac{n(n+1)}{2}$.

In this paper we will give a different proof of the converse. Our approach will be based on what commonly is referred to as the method of partial sums:

$$\sum_{k=1}^{n} k^{m+1} = (n+1)\sum_{k=1}^{n} k^m - \sum_{k=1}^{n}\sum_{l=1}^{k} l^m. \tag{7}$$

We state it in a pointed fashion to illustrate it serves as a generator for sums of powers (Zielinski [8, section 2]). In this context, when we use (3) to write expressions for $\sum\sum l^m$, the property $B_{2m+1} = 0$ will cause the bulk of the terms to be of even or odd parity like $\sum k^{m+1}$.

3 Main Result

We start with a proof of our method of partial sums:

Proposition 1. *for positive integers n,*

$$\sum_{k=1}^{n} k^{m+1} + \sum_{k=1}^{n} \sum_{l=1}^{k} l^m = (n+1) \sum_{k=1}^{n} k^m,$$

where m is a fixed, positive integer.

Proof. we proceed by mathematical induction. For the base case of $n = 1$,

$$\sum_{k=1}^{1} k^{m+1} + \sum_{k=1}^{1} \sum_{l=1}^{k} l^m = 1^{m+1} + 1^m = 1 + 1 = 2 \cdot 1 = (1+1) \sum_{k=1}^{1} k^m.$$

Assume the result is true for $n \geq 1$. Then

$$\begin{aligned}
\sum_{k=1}^{n+1} k^{m+1} + \sum_{k=1}^{n+1} \sum_{l=1}^{k} l^m &= \sum_{k=1}^{n} k^{m+1} + (n+1)^{m+1} + \sum_{k=1}^{n} \sum_{l=1}^{k} l^m + \sum_{l=1}^{n+1} l^m \\
&= (n+1) \sum_{k=1}^{n} k^m + \left((n+1)^{m+1} + \sum_{l=1}^{n+1} l^m \right) \\
&= (n+1) \sum_{k=1}^{n} k^m + (n+1)(n+1)^m + \sum_{l=1}^{n+1} l^m \\
&= (n+1) \sum_{k=1}^{n+1} k^m + \sum_{l=1}^{n+1} l^m.
\end{aligned}$$

Notice that $\sum_{k=1}^{n+1} k^m = \sum_{l=1}^{n+1} l^m$. The same sum is expressed in two different notations. Therefore

$$\sum_{k=1}^{n+1} k^{m+1} + \sum_{k=1}^{n+1} \sum_{l=1}^{k} l^m = (n+2) \sum_{k=1}^{n+1} k^m.$$

□

Before we prove the converse, we state, without proof, a lemma which points out critical, intermediate relationships:

Lemma 1. *for positive integers n,*

$$\begin{aligned}
\left(n + \frac{1}{2}\right) \left(\sum k\right)^2 &= \frac{3}{2} \cdot \frac{n(n+1)}{2} \cdot \sum k^2, \\
\left(n + \frac{1}{2}\right) \sum k^2 &= \left(\frac{4}{3} \cdot \frac{n(n+1)}{2} + \frac{1}{6}\right) \sum k.
\end{aligned}$$

A consequence of the second relationship is that $\left(\sum k^2\right)^2$ can be rewritten using only terms of $\sum k$, which then implies, through (5), that $\left(\sum k^{2m}\right)^2$ can be rewritten in $\sum k$ as well (Beardon [1, section 3]).

Now we give a new proof of the converse:

Proposition 2. *if odd Bernoulli numbers are equal to zero, we can derive Faulhaber's formula.*

Proof. we proceed by mathematical induction. We know

$$\begin{aligned}\sum k^2 &=1 \cdot \sum k^2, \\ \sum k^3 &=1 \cdot \left(\sum k\right)^2.\end{aligned}$$

We will assume (4) and (5) are true for all $1, 2, ..., m$ and then establish the case of $m+1$.

For $m+1$, Proposition 1 tells us

$$\sum_{k=1}^{n} k^{2m+2} = (n+1)\sum_{k=1}^{n} k^{2m+1} - \sum_{k=1}^{n}\sum_{l=1}^{k} l^{2m+1}. \tag{8}$$

We want to write $\sum l^{2m+1}$ in k. (3) tells us

$$\sum_{l=1}^{k} l^{2m+1} = \frac{1}{2m+2}\sum_{j=0}^{2m+1} (-1)^j \binom{2m+2}{j} B_j k^{2m+2-j}.$$

If we assume $B_{2m+1} = 0$ for $m \geq 1$, we can rewrite the double sum as

$$\begin{aligned}\sum_{k=1}^{n}\sum_{l=1}^{k} l^{2m+1} &= \frac{1}{2m+2}\sum_{k=1}^{n}\left\{1 \cdot B_0 k^{2m+2} - \binom{2m+2}{1} B_1 k^{2m+1}\right. \\ &\quad \left. + \binom{2m+2}{2} B_2 k^{2m} + \cdots + \binom{2m+2}{2m} B_{2m} k^2 \right\} \\ &= \frac{1}{2m+2}\sum_{k=1}^{n} k^{2m+2} + \frac{1}{2}\sum_{k=1}^{n} k^{2m+1} + b_{2m}\sum_{k=1}^{n} k^{2m} + \cdots + b_2 \sum_{k=1}^{n} k^2,\end{aligned}$$

where $b_{2m}, \ldots, b_2$ are rational numbers which don't interest us. Now we can rewrite (8) as

$$\begin{aligned}\sum k^{2m+2} &= (n+1)\sum k^{2m+1} - \frac{1}{2m+2}\sum k^{2m+2} - \frac{1}{2}\sum k^{2m+1} \\ &\quad - \left(b_{2m}\sum k^{2m} + \cdots + b_2 \sum k^2\right).\end{aligned} \tag{9}$$

By the inductive hypothesis for $\sum k^{2m}$, we can rewrite the sum in the parentheses as

$$b_{2m}\sum k^2 \cdot P_{2m} + \cdots + b_2 \sum k^2 \cdot P_2,$$

where $P_{2m}, \ldots, P_2$ are polynomials in $\frac{n(n+1)}{2}$. Together, this is just $P \cdot \sum k^2$ for another such polynomial P. (9) now becomes

$$\frac{2m+3}{2m+2} \sum k^{2m+2} = \left(n + \frac{1}{2}\right) \sum k^{2m+1} - P \cdot \sum k^2. \tag{10}$$

For the next step of the proof, first we invoke the inductive hypothesis for $\sum k^{2m+1}$. This allows us to rewrite the right side of (10) as

$$\left(n + \frac{1}{2}\right) \left(\sum k\right)^2 Q_{2m+1} - P \cdot \sum k^2,$$

where Q_{2m+1} is a polynomial in $\frac{n(n+1)}{2}$. Then we use the lemma to rewrite the expression as

$$\frac{3}{2} \cdot \frac{n(n+1)}{2} \cdot \sum k^2 \cdot Q_{2m+1} - P \cdot \sum k^2 \quad = \quad \left[\frac{3}{2} \cdot \frac{n(n+1)}{2} \cdot Q_{2m+1} - P\right] \sum k^2,$$

which is $Q \cdot \sum k^2$ for another such polynomial Q. The final form of (10) is

$$\sum k^{2m+2} \quad = \quad \frac{2m+2}{2m+3} \left(Q \cdot \sum k^2 - P \cdot \sum k^2\right) = R \cdot \sum k^2,$$

where R is a polynomial in $\frac{n(n+1)}{2}$.

The proof for $\sum k^{2m+3}$ proceeds along the same lines. We only wish to point out an important difference when rewriting the double sum using Bernoulli numbers. Starting with

$$\sum_{k=1}^{n} k^{2m+3} = (n+1) \sum_{k=1}^{n} k^{2m+2} - \sum_{k=1}^{n} \sum_{l=1}^{k} l^{2m+2},$$

the expression analogous to (10) would be

$$\frac{2m+4}{2m+3} \sum k^{2m+3} = \left(n + \frac{1}{2}\right) \sum k^{2m+2} - B_{2m+2} \sum k - P_{2m+1} \cdot \left(\sum k\right)^2. \tag{11}$$

We need to eliminate the term of $-B_{2m+2} \sum k$, which we do as follows.

The coefficient of B_{2m+2} comes out of writing $\sum l^{2m+2}$ according to (3). If we write the same expression using (5), which we just established, we get

$$\sum k^{2m+2} = [Q - c_{m+1}] \sum k^2 = [Q - c_{m+1}] \cdot \frac{2n^3 + 3n^2 + n}{6},$$

where Q is a polynomial in $\frac{n(n+1)}{2}$, of degree of at least one, and c_{m+1} is a rational number. The coefficient for the term of n is $-\frac{c_{m+1}}{6}$. Since both coefficients must agree, we have $-\frac{c_{m+1}}{6} = B_{2m+2}$.

When we invoke the lemma we get

$$\begin{aligned}
\left(n+\frac{1}{2}\right)[Q-c_{m+1}]\sum k^2 \quad &= \quad [Q-c_{m+1}]\left(\frac{4}{3}\cdot\frac{n(n+1)}{2}+\frac{1}{6}\right)\sum k \\
&= [Q-c_{m+1}]\left(\frac{4}{3}\left(\sum k\right)^2+\frac{1}{6}\sum k\right) \\
&= [Q-c_{m+1}]\cdot\frac{4}{3}\left(\sum k\right)^2+Q\cdot\frac{1}{6}\sum k-\frac{c_{m+1}}{6}\sum k \\
&= Q_{2m+1}^{'}\cdot\left(\sum k\right)^2-\frac{c_{m+1}}{6}\sum k,
\end{aligned}$$

where $Q_{2m+1}^{'}$ is a polynomial in $\frac{n(n+1)}{2}$. The term $-\frac{c_{m+1}}{6}\sum k$ cancels with that of $-B_{2m+2}\sum k$, and (11) becomes

$$\frac{2m+4}{2m+3}\sum k^{2m+3}=Q_{2m+1}^{'}\cdot\left(\sum k\right)^2-P_{2m+1}\cdot\left(\sum k\right)^2,$$

from which the desired result follows. (Note: since m can be even or odd, another case is possible: $-B_{2m+2}\sum k$ and $[Q+c_{m+1}]\sum k^2$. It is handled in the same fashion.) □

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