A PROGRESS ON THE BINARY GOLDBACH CONJECTURE

T. AGAMA

Abstract. In this paper we develop the method of circle of partitions and associated statistics. As an application we prove conditionally the binary Goldbach conjecture. We develop series of steps to prove the binary Goldbach conjecture in full. We end the paper by proving the binary Goldbach conjecture for all even numbers exploiting the strategies outlined.

1. **Introduction**

The Goldbach conjecture dates from 1742 out of the correspondence between the Swiss mathematician Leonard Euler and the German mathematician Christian Goldbach. The problem has two folds, namely the binary case and the ternary case. The binary case ask if every even number ≥ 6 can be written as a sum of two primes, where as the ternary case ask if every odd number ≥ 7 can be written as a sum of three prime numbers. The ternary case has, however, been solved quite recently in the preprint [2] culminating several works. Though the binary problem remains unsolved as of now there has been substantive progress as well as on its variants. The first milestone in this direction can be found in (see [5]), where it is shown that every even number can be written as the sum of at most C primes, where C is an effectively computable constant. In the early twentieth century, G.H Hardy and J.E Littlewood assuming the Generalized Riemann hypothesis (see [8]), showed that the number of even numbers $\leq X$ and violating the binary Goldbach conjecture is much less than $X^{\frac{1}{2}+c}$, where c is a small positive constant. Jing-run Chen [3], using the methods of sieve theory, showed that every even number can either be written as a sum of two prime numbers or a prime number and a number which is a product of two primes. It also known that almost all even numbers can be written as the sum of two prime numbers, in the sense that the density of even numbers representable in this manner is one [7], [6]. It is also known that there exist a constant K such that every even number can be written as the sum of two prime numbers and at most K powers of two, where we can take $K = 13$ [4].
In [1] we have developed a method which we feel might be a valuable resource and a recipe for studying problems concerning partition of numbers in specified subsets of $\mathbb{N}$. The method is very elementary in nature and has parallels with configurations of points on the geometric circle.
Let us suppose that for any $n \in \mathbb{N}$ we can write $n = u + v$ where $u, v \in \mathbb{M} \subset \mathbb{N}$ then the new method associate each of this summands to points on the circle generated in a certain manner by $n > 2$ and a line joining any such associated points on the circle. This geometric correspondence turns out to useful in our development, as

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the results obtained in this setting are then transformed back to results concerning the partition of integers.

2. **The Circle of Partition**

Here we repeat the base results of the method of circles of partition developed in [1].

Definition 2.1. Let $n \in \mathbb{N}$ and $\mathbb{M} \subseteq \mathbb{N}$. We denote with

$$\mathcal{C}(n, \mathbb{M}) = \{[x] \mid x, y \in \mathbb{M}, n = x + y\}$$

the **Circle of Partition** generated by n with respect to the subset $\mathbb{M}$. We will abbreviate this in the further text as CoP. We call members of $\mathcal{C}(n, \mathbb{M})$ as points and denote them by $[x]$. For the special case $\mathbb{M} = \mathbb{N}$ we denote the CoP shortly as $\mathcal{C}(n)$. We denote with $\|[x]\| := x$ the **weight** of the point $[x]$ and correspondingly the weight set of points in the CoP $\mathcal{C}(n, \mathbb{M})$ as $\|\mathcal{C}(n, \mathbb{M})\|$. Obviously holds

$$\|\mathcal{C}(n)\| = \{1, 2, \ldots, n-1\}.$$

Definition 2.2. We denote the line $\mathbb{L}_{[x],[y]}$ joining the point $[x]$ and $[y]$ as an axis of the CoP $\mathcal{C}(n, \mathbb{M})$ if and only if $x + y = n$. We say the axis point $[y]$ is an axis partner of the axis point $[x]$ and vice versa. We do not distinguish between $\mathbb{L}_{[x],[y]}$ and $\mathbb{L}_{[y],[x]}$, since it is essentially the the same axis. The point $[x] \in \mathcal{C}(n, \mathbb{M})$ such that $2x = n$ is the **center** of the CoP. If it exists then we call it as a **degenerated axis** $\mathbb{L}_{[x]}$ in comparison to the **real axes** $\mathbb{L}_{[x],[y]}$. We denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ to a CoP $\mathcal{C}(n, \mathbb{M})$ as

$$\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{M}) \text{ which means } [x], [y] \in \mathcal{C}(n, \mathbb{M}) \text{ with } x + y = n.$$

In the following we consider only real axes. Therefore we abstain from the attribute *real* in the sequel.

Proposition 2.3. *Each axis is uniquely determined by points* $[x] \in \mathcal{C}(n, \mathbb{M})$.

Proof. Let $\mathbb{L}_{[x],[y]}$ be an axis of the CoP $\mathcal{C}(n, \mathbb{M})$. Suppose as well that $\mathbb{L}_{[x],[z]}$ is also an axis with $z \neq y$. Then it follows by Definition 2.2 that we must have $n = x + y = x + z$ and therefore $y = z$. This cannot be and the claim follows immediately. ☐

Corollary 2.4. *Each point of a CoP* $\mathcal{C}(n, \mathbb{M})$ *except its center has exactly one axis partner.*

Proof. Let $[x] \in \mathcal{C}(n, \mathbb{M})$ be a point without an axis partner being not the center of the CoP. Then holds for every point $[y] \neq [x]$ except the center

$$x + y \neq n.$$

This is a contradiction to the Definition 2.1. Due to Proposition 2.3 the case of more than one axis partners is impossible. This completes the proof. ☐

Notation. We denote by

$$\mathbb{N}_n = \{m \in \mathbb{N} \mid \ m \leq n\} \tag{2.1}$$

the **sequence** of the first n natural numbers. We denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ resp. $\mathbb{L}_{[x]}$ to a CoP $\mathcal{C}(n,\mathbb{M})$ as

$$\mathbb{L}_{[x],[y]} \ \hat{\in} \ \mathcal{C}(n,\mathbb{M}) \text{ which means } [x],[y] \in \mathcal{C}(n,\mathbb{M}) \text{ and } x+y=n \text{ resp.}$$
$$\mathbb{L}_{[x]} \ \hat{\in} \ \mathcal{C}(n,\mathbb{M}) \text{ which means } [x] \in \mathcal{C}(n,\mathbb{M}) \text{ and } 2x=n$$

and the number of real axes of a CoP as

$$\nu(n,\mathbb{M}) := \#\{\mathbb{L}_{[x],[y]} \ \hat{\in} \ \mathcal{C}(n,\mathbb{M}) \mid x<y\}.$$

Obviously holds

$$\nu(n,\mathbb{M}) = \left\lfloor \frac{k}{2} \right\rfloor, \text{ if } |\mathcal{C}(n,\mathbb{M})| = k.$$

For any $f,g:\mathbb{N}\longrightarrow\mathbb{N}$, we write $f(n)\sim g(n)$ if and only if $\lim\limits_{n\longrightarrow\infty}\frac{f(n)}{g(n)}=1$. We also write $f(n)=o(1)$ if and only if $\lim\limits_{n\longrightarrow\infty} f(n)=0$.

3. **The Density of Points on the Circle of Partition**

In this section we introduce the notion of density of points on CoP $\mathcal{C}(n,\mathbb{M})$ for $\mathbb{M}\subseteq\mathbb{N}$. We launch the following language in that regard. We consider in this section only real axes. Therefore we don't use the attribute *real* in this section.

Definition 3.1. Let be $\mathbb{H}\subset\mathbb{N}$. Then the limits

$$\mathcal{D}\left(\mathbb{H}\right) = \lim_{n\to\infty}\frac{|\mathbb{H}\cap\mathbb{N}_n|}{n}$$

denotes the density of $\mathbb{H}$ if it exists.

Definition 3.2. Let $\mathcal{C}(n,\mathbb{M})$ be CoP with $\mathbb{M}\subset\mathbb{N}$ and $n\in\mathbb{N}$. Suppose $\mathbb{H}\subset\mathbb{M}$ then by the density of points $[x]\in\mathcal{C}(n,\mathbb{M})$ such that $x\in\mathbb{H}$, denoted $\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty,\mathbb{M})})$, we mean the limit

$$\mathcal{D}\left(\mathbb{H}_{\mathcal{C}(\infty,\mathbb{M})}\right) = \lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{L}_{[x],[y]} \ \hat{\in} \ \mathcal{C}(n,\mathbb{M})| \ \{x,y\}\cap\mathbb{H}\neq\emptyset\}}{\nu(n,\mathbb{M})}$$

if it exists.

The notion of the density of points as espoused in Definition 3.2 provides a passage between the density of the corresponding weight set of points. This possibility renders this type of density as a black box in studying problems concerning partition of numbers into specialized sequences taking into consideration their density.

Proposition 3.3. *Let $\mathcal{C}(n)$ with $n\in\mathbb{N}$ be a CoP and $\mathbb{H}\subset\mathbb{N}$. Then the following inequality holds*

$$\mathcal{D}(\mathbb{H}) = \lim_{n\longrightarrow\infty}\frac{\left\lfloor\frac{|\mathbb{H}\cap\mathbb{N}_n|}{2}\right\rfloor}{\left\lfloor\frac{n-1}{2}\right\rfloor} \leq \mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)}) \leq \lim_{n\longrightarrow\infty}\frac{|\mathbb{H}\cap\mathbb{N}_n|}{\left\lfloor\frac{n-1}{2}\right\rfloor} = 2\mathcal{D}(\mathbb{H}).$$

Proof. The upper bound is obtained from a configuration where no two points $[x],[y] \in \mathcal{C}(n)$ such that $x,y \in \mathbb{H}$ lie on the same axis of the CoP. That is, by the uniqueness of the axes of CoPs with $\nu(n,\mathbb{H})=0$, we can write

$$\begin{aligned}\#\left\{\mathbb{L}_{[x],[y]} \in \mathcal{C}(n)|\ \{x,y\}\cap\mathbb{H}\neq\emptyset\right\} &= \nu(n,\mathbb{H}) + \#\left\{\mathbb{L}_{[x],[y]} \in \mathcal{C}(n)|\ x\in\mathbb{H},\ y\in\mathbb{N}\setminus\mathbb{H}\right\}\\ &= \#\left\{\mathbb{L}_{[x],[y]} \in \mathcal{C}(n)|\ x\in\mathbb{H},\ y\in\mathbb{N}\setminus\mathbb{H}\right\}\\ &= |\mathbb{H}\cap\mathbb{N}_n|.\end{aligned}$$

The lower bound however follows from a configuration where any two points $[x],[y] \in \mathcal{C}(n)$ with $x,y\in\mathbb{H}$ are joined by a axis of the CoP. That is, by the uniqueness of the axis of CoPs with $\#\left\{\mathbb{L}_{[x],[y]} \in \mathcal{C}(n)|\ x\in\mathbb{H},\ y\in\mathbb{N}\setminus\mathbb{H}\right\}=0$, then we can write

$$\begin{aligned}\#\left\{\mathbb{L}_{[x],[y]} \in \mathcal{C}(n)|\ \{x,y\}\cap\mathbb{H}\neq\emptyset\right\} &= \nu(n,\mathbb{H})\\ &= \left\lfloor \frac{|\mathbb{H}\cap\mathbb{N}_n|}{2}\right\rfloor.\end{aligned}$$

□

Proposition 3.4. *Let $\mathbb{H}\subset\mathbb{N}$ and $\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)})$ be the density of the corresponding points with weight set $\mathbb{H}$. Then the following properties hold:*

(i) $\mathcal{D}(\mathbb{N}_{\mathcal{C}(\infty)})=1$ *and* $\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)})\leq 1$ *and additionally that* $\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)})<1$ *provided* $\mathcal{D}(\mathbb{N}\setminus\mathbb{H})>0$.

(ii) $1-\lim\limits_{n\longrightarrow\infty}\dfrac{\nu(n,\mathbb{N}\setminus\mathbb{H})}{\nu(n,\mathbb{N})}=\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)})$.

(iii) *If* $|\mathbb{H}|<\infty$ *then* $\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)})=0$.

Proof. It is easy to see that the first part of **Property** (i) and (iii) are both easy consequences of the definition of density of points on the CoP $\mathcal{C}(n)$ and Proposition 3.3. We establish the second part of property (i) and **Property** (ii), which is the less obvious case. We observe by the uniqueness of the axes of CoPs that we can write

$$\begin{aligned}1 &= \lim_{n\longrightarrow\infty}\frac{\nu(n,\mathbb{N})}{\nu(n,\mathbb{N})}\\ &= \lim_{n\longrightarrow\infty}\frac{\#\{\mathbb{L}_{[x],[y]}\ \hat{\in}\ \mathcal{C}(n)|\ x\in\mathbb{H}\ ,y\in\mathbb{N}\setminus\mathbb{H}\}}{\nu(n,\mathbb{N})}\\ &+ \lim_{n\longrightarrow\infty}\frac{\nu(n,\mathbb{H})}{\nu(n,\mathbb{N})}+\lim_{n\longrightarrow\infty}\frac{\nu(n,\mathbb{N}\setminus\mathbb{H})}{\nu(n,\mathbb{N})}\\ &= \mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)})+\lim_{n\longrightarrow\infty}\frac{\nu(n,\mathbb{N}\setminus\mathbb{H})}{\nu(n,\mathbb{N})}\end{aligned}$$

and (ii) follows immediately. The second part of (i) follows from the above expression and exploiting the inequality

$$\lim_{n\longrightarrow\infty}\frac{\nu(n,\mathbb{N}\setminus\mathbb{H})}{\nu(n,\mathbb{N})}\leq\lim_{n\longrightarrow\infty}\frac{\left\lfloor\frac{|\mathbb{N}\setminus\mathbb{H}\cap\mathbb{N}_n|}{2}\right\rfloor}{\left\lfloor\frac{n-1}{2}\right\rfloor}=\mathcal{D}(\mathbb{N}\setminus\mathbb{H}).$$

□

Next we transfer the notion of the density of a sequence to the density of corresponding points on the CoP $\mathcal{C}(n)$. This notion will play a crucial role in our latter developments.

Proposition 3.5. *Let $\epsilon \in (0,1]$ and $\mathbb{H}$ be a sequence with $\mathbb{H} \subset \mathbb{N}$ and $\mathcal{C}(n)$ be a CoP. If $\mathcal{D}(\mathbb{H}) \geq \epsilon$ then $\mathcal{D}\left(\mathbb{H}_{\mathcal{C}(\infty)}\right) \geq \epsilon$.*

Proof. The result follows by exploiting the inequality in Proposition 3.3. □

3.1. **Application of Density of Points to Partitions.** In this subsection we explore the connection between the notion of density of points in a typical CoP to the possibility of partitioning number into certain sequences. This method tends to work very efficiently for sets of integers having a positive density.

Theorem 3.6. *Let $\mathbb{H} \subset \mathbb{N}$ such that $\mathcal{D}(\mathbb{H}) > \frac{1}{2}$. Then every sufficiently large $n \in \mathbb{N}$ has representation of the form*

$$n = z_1 + z_2$$

where $z_1, z_2 \in \mathbb{H}$.

Proof. Appealing to Proposition 3.3 we can write

$$\lim_{n \longrightarrow \infty} \frac{\left\lfloor \frac{|\mathbb{H} \cap \mathbb{N}_n|}{2} \right\rfloor}{\left\lfloor \frac{n-1}{2} \right\rfloor} \leq \mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)}) \leq \lim_{n \longrightarrow \infty} \frac{|\mathbb{H} \cap \mathbb{N}_n|}{\left\lfloor \frac{n-1}{2} \right\rfloor}.$$

By the uniqueness of the axes of CoPs we can write

$$\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n) |\ \{x,y\} \cap \mathbb{H} \neq \emptyset\right\} = \nu(n,\mathbb{H}) + \#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n) |\ x \in \mathbb{H},\ y \in \mathbb{N} \setminus \mathbb{H}\right\}.$$

Let us assume $\nu(n,\mathbb{H}) = 0$ then it follows by appealing to Definition 3.2 and according to the proof of Proposition 3.3

$$\begin{aligned}\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)}) &= 2\mathcal{D}(\mathbb{H}) \\ &> 2 \times \frac{1}{2} = 1.\end{aligned}$$

This contradicts the inequality $\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)}) \leq 1$ in Proposition 3.4. This proves that $\nu(n,\mathbb{H}) > 0$ for all sufficiently large values of $n \in \mathbb{N}$. □

Corollary 3.7. *Let $\mathbb{R} := \{m \in \mathbb{N} |\ \mu(m) \neq 0\}$. Then every sufficiently large $n \in \mathbb{N}$ can be written in the form*

$$n = z_1 + z_2$$

where $\mu(z_1) = \mu(z_2) \neq 0$.

Proof. By the uniqueness of the axes of CoPs we can write

$$\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n) |\ \{x,y\} \cap \mathbb{R} \neq \emptyset\right\} = \nu(n,\mathbb{R}) + \#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n) |\ x \in \mathbb{R},\ y \in \mathbb{N} \setminus \mathbb{R}\right\}.$$

Let us assume $\nu(n,\mathbb{R}) = 0$ then it follows by appealing to Definition 3.2 and Theorem 3.6

$$\begin{aligned}\mathcal{D}(\mathbb{R}_{\mathcal{C}(\infty)}) &= 2\mathcal{D}(\mathbb{R}) \\ &= \frac{12}{\pi^2} > 1\end{aligned}$$

since $\mathcal{D}(\mathbb{R}) = \frac{6}{\pi^2}$. This contradicts the inequality $\mathcal{D}(\mathbb{R}_{\mathcal{C}(\infty)}) \leq 1$ in Proposition 3.4. This proves that $\nu(n,\mathbb{R}) > 0$ for all sufficiently large values of $n \in \mathbb{N}$. □

One could ever hope and dream of this strategy to work when we replace the set $\mathbb{R}$ of square-free integers with the set of prime numbers. There we would certainly ran into complete deadlock, since the prime in accordance with the prime number theorem have density zero. Any success in this regard could conceivably work by introducing some exotic forms of the notion of density of points and carefully choosing a subset of the integers that is somewhat dense among the set of integers and covers that primes. We propose a strategy somewhat akin to the above method for possibly getting a handle on the binary Goldbach conjecture and it's variants. Before that we state and prove a conditional theorem concerning the binary Goldbach conjecture.

Theorem 3.8. *Let $\mathbb{B} \subset \mathbb{N}$ such that $\mathbb{P} \subset \mathbb{B}$ with*

$$\lim_{n\longrightarrow\infty} \frac{|\mathbb{P}\cap\mathbb{N}_n|}{\eta(n)} > \frac{1}{2}$$

where $\eta(n) = |\mathcal{C}(n,\mathbb{B})|$. Then $\nu(n,\mathbb{P}) > 0$ for all sufficiently large values of $n \in 2\mathbb{N}$.

Proof. First let us upper and lower bound the density of points in the CoP $\mathcal{C}(n,\mathbb{B})$ with weight belonging to the set of the primes $\mathbb{P}$ so that we obtain the inequality

$$\lim_{n\longrightarrow\infty} \frac{\left\lfloor \frac{|\mathbb{P}\cap\mathbb{N}_n|}{2} \right\rfloor}{\left\lfloor \frac{\eta(n)}{2} \right\rfloor} \leq \mathcal{D}(\mathbb{P}_{\mathcal{C}(\infty,\mathbb{B})}) \leq \lim_{n\longrightarrow\infty} \frac{|\mathbb{P}\cap\mathbb{N}_n|}{\left\lfloor \frac{\eta(n)}{2} \right\rfloor} = 2\lim_{n\longrightarrow\infty} \frac{|\mathbb{P}\cap\mathbb{N}_n|}{\eta(n)}$$

for all sufficiently large values of n. Appealing to the uniqueness of the axes of CoPs, we can write

$$\#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{B}) \mid \{x,y\}\cap\mathbb{P} \neq \emptyset\right\} = \nu(n,\mathbb{P}) + \#\left\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{B}) \mid x\in\mathbb{P},\ y\in\mathbb{B}\setminus\mathbb{P}\right\}.$$

Let us assume to the contrary $\nu(n,\mathbb{P}) = 0$, then it follows that no two points in the CoP $\mathcal{C}(n,\mathbb{B})$ with weight in the set $\mathbb{P}$ are axes partners, so that under the requirement

$$\lim_{n\longrightarrow\infty} \frac{|\mathbb{P}\cap\mathbb{N}_n|}{\eta(n)} > \frac{1}{2}$$

where $\eta(n)$ is the counting function of all integers belonging to the set $\mathbb{B}\cap\mathbb{N}_n$, we obtain the inequality

$$\begin{aligned}\mathcal{D}(\mathbb{P}_{\mathcal{C}(\infty,\mathbb{B})}) &= 2\lim_{n\longrightarrow\infty} \frac{|\mathbb{P}\cap\mathbb{N}_n|}{\eta(n)}\\ &> 2\times\frac{1}{2} = 1.\end{aligned}$$

This contradicts the inequality $\mathcal{D}(\mathbb{P}_{\mathcal{C}(\infty,\mathbb{B})}) \leq 1$ in Proposition 3.4. This proves that $\nu(n,\mathbb{P}) > 0$ for all sufficiently large values of $n \in 2\mathbb{N}$. ☐

3.2. **A Strategy to Prove the Binary Goldbach Conjecture by Circles of Partition.** In this subsection we propose series of steps that could be taken to confirm the truth of the binary Goldbach conjecture. We enumerate the strategies chronologically as follows:

- First construct a CoP $\mathcal{C}(n,\mathbb{B})$ such that $\mathbb{P}_n \subset ||\mathcal{C}(n,\mathbb{B})||$ and that

$$\lim_{n\longrightarrow\infty} \frac{|\mathbb{P}\cap\mathbb{N}_n|}{\eta(n)} > \frac{1}{2}$$

  where $\eta(n) = |\mathcal{C}(n,\mathbb{B})|$.

- Next we remark that the following inequality also holds and this can be obtain by replacing the weight set $||\mathcal{C}(n)||$ with the set $||\mathcal{C}(n,\mathbb{B})||$.

$$\lim_{n\longrightarrow\infty}\frac{\left\lfloor\frac{|\mathbb{P}\cap\mathbb{N}_n|}{2}\right\rfloor}{\left\lfloor\frac{\eta(n)}{2}\right\rfloor}\leq\mathcal{D}(\mathbb{P}_{\mathcal{C}(\infty,\mathbb{B})})\leq\lim_{n\longrightarrow\infty}\frac{|\mathbb{P}\cap\mathbb{N}_n|}{\left\lfloor\frac{\eta(n)}{2}\right\rfloor}=2\lim_{n\longrightarrow\infty}\frac{|\mathbb{P}\cap\mathbb{N}_n|}{\eta(n)}.$$

- Appealing to the uniqueness of the axes of CoPs, we can write

$$\begin{aligned}&\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n,\mathbb{B})|\ \{x,y\}\cap\mathbb{P}\neq\emptyset\right\}\\&=\nu(n,\mathbb{P})+\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n,\mathbb{B})|\ x\in\mathbb{P},\ y\in\mathbb{B}\setminus\mathbb{P}\right\}.\end{aligned}$$

- Let us assume $\nu(n,\mathbb{P})=0$ then it follows by appealing to Definition 3.2 and in the sense of the proof of Proposition 3.3

$$\begin{aligned}\mathcal{D}(\mathbb{P}_{\mathcal{C}(\infty,\mathbb{B})})&=2\lim_{n\longrightarrow\infty}\frac{|\mathbb{P}\cap\mathbb{N}_n|}{\eta(n)}\\&>2\times\frac{1}{2}=1.\end{aligned}$$

  This contradicts the inequality $\mathcal{D}(\mathbb{P}_{\mathcal{C}(\infty,\mathbb{B})})\leq 1$ in Proposition 3.4. This proves that $\nu(n,\mathbb{P})>0$ for all sufficiently large values of $n\in 2\mathbb{N}$.

Next we show that the binary Goldbach conjecture is true for all even numbers.

Theorem 3.9. *Let $\mathbb{P}^*$ denotes the set of all odd prime numbers. If $n\geq 6$ is an even number then the representation holds*

$$n=p+q$$

where $p,q\in\mathbb{P}^*$.

Proof. Let n be even and consider all the odd prime numbers $<n$. Let us enumerate them as

$$3<5<\cdots<p_{\pi(n)}$$

where $\pi(n)$ denotes the number of primes no more than n. We define the function

$$\kappa_n:\mathbb{N}\longrightarrow\mathbb{Z}$$

as a *mollifier* of the CoP $\mathcal{C}(n,\mathbb{B})$ with the property that $[\kappa_n(p)+p]\in\mathcal{C}(n,\mathbb{B})$ for each $[p]\in\mathcal{C}(n,\mathbb{B})$ such that

$$\kappa_n(p_i):=n-2p_i.$$

We remark that the value of the **mollifier** depends on the choice of the generator n and the weight of points with at least a prime number weight on the CoP $\mathcal{C}(n,\mathbb{B})$. Let us now construct and choose the infinite set $\mathbb{B}:=\bigcup\limits_{\substack{n\in\mathbb{N}\\n\geq 6}}\mathbb{B}_n$ with

$$\mathbb{B}_n=\cup_{i=2}^{\pi(n)}\{p_i\}\bigcup\cup_{i=2}^{\pi(n)}\{\kappa_n(p_i)+p_i\}$$

such that $\mathbb{B}_n\setminus\{p_j,n-p_j\}=||\mathcal{C}(n,\mathbb{B})||$ for any fixed $2\leq j\leq\pi(n)$ so that $|\mathcal{C}(n,\mathbb{B})|=|\mathbb{B}_n|-2\Leftrightarrow|\mathcal{C}(n,\mathbb{B})|=(1-o(1))|\mathbb{B}_n|$ and $\mathbb{P}^*\subset\mathbb{B}$. It is important to notice that our choice of CoP $\mathcal{C}(n,\mathbb{B})$ is the one where each axis has at least a prime number weight. Let $\eta(n)=|\mathcal{C}(n,\mathbb{B})|$ then it follows that

$$\eta(n)=|\mathcal{C}(n,\mathbb{B})|=|\mathbb{B}_n|-2\leq(\pi(n)-1)+(\pi(n)-1)-2\leq 2\pi(n)-4$$

and

$$\lim_{n\longrightarrow\infty}\frac{|\mathbb{P}^*\cap\mathbb{N}_n|}{\eta(n)}=\lim_{n\longrightarrow\infty}\frac{|\mathbb{P}^*\cap\mathbb{N}_n|}{(1-o(1))|\mathbb{B}_n|}>\lim_{n\longrightarrow\infty}\frac{\pi(n)-1}{2\pi(n)-2}=\frac{1}{2}$$

By leveraging the uniqueness of the axes of CoPs we can write the decomposition

$$\begin{aligned}&\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n,\mathbb{B})|\ \{x,y\}\cap\mathbb{P}^*\neq\emptyset\right\}\\&=\nu(n,\mathbb{P}^*)+\#\left\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n,\mathbb{B})|\ x\in\mathbb{P}^*,\ y\in\mathbb{B}\setminus\mathbb{P}^*\right\}.\end{aligned}$$

Let us assume to the contrary $\nu(n,\mathbb{P}^*)=0$, then it follows that no two points in the CoP $\mathcal{C}(n,\mathbb{B})$ with weight in the set $\mathbb{P}^*$ are axes partners, so that we obtain the inequality

$$\begin{aligned}\mathcal{D}(\mathbb{P}^*_{\mathcal{C}(\infty,\mathbb{B})})&=2\lim_{n\longrightarrow\infty}\frac{|\mathbb{P}^*\cap\mathbb{N}_n|}{\eta(n)}\\&>2\times\frac{1}{2}=1.\end{aligned}$$

This contradicts the inequality $\mathcal{D}(\mathbb{P}^*_{\mathcal{C}(\infty,\mathbb{B})})\leq 1$ in Proposition 3.4. This proves that $\nu(n,\mathbb{P}^*)>0$ for all values of $n\in 2\mathbb{N}$ with $n\geq 6$. $\square$

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Department of Mathematics, African Institute for mathematical sciences, Ghana.
E-mail address: `Theophilus@aims.edu.gh/emperordagama@yahoo.com`