On completeness of one analytical solution in electrodynamics

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Abstract

An analytical solution of the equation $\mathrm{d}\,{}^*\mathrm{d}\alpha = 0$ where the 1-form α stands for "vector potential" of electromagnetic field of uniformly accelerated charge presented in the work [2], was obtained in an incomplete coordinate system. Incompleteness of the system used gives rise to doubts about correctness of the solution because of possible presence of extra sources of the filed beyond the chart of the covered by coordinates. A rigorous criterion of existence or non-existence of extra sources of this sort is proposed which was appliel to the solution. As a result, it is found that no extra sources beyond the chart exist and hence, the solution describes the field on uniformly accelerated charge properly. However, this fact discloses another discrepancy in foundations of the field theory.

1 Introduction

Uniformly accelerated motion is interesting from several points of view. First, any motion can locally be represented as a uniformly accelerated one. Second, world line of uniformly accelerated motion is an orbit of symmetry transformation of the flat spact-time and hence, this motion possesses some special properties. Third, the field of uniformly accelerated point-like charge underlies the classical theory of radiation.

There exist two distinct representations of the field of a uniformly accelerated point-like charge. One was obtained by the method of retarded potentials and presently remains the generally-accepted one [3]. Another was done as a straightforward solution of the field equation in a uniformly accelerated frame. Both are dubious and contradict each other. The earlier is nothing but integral of the Green function for the d'Alembert equation and the latter is obtained in an incomplete coordinate system. Below they will be discussed.

The method of retarded potentials is based on the equation

$$\Box A_\mu = 0, \tag{1}$$

which usually termed "d'Alembert equation for the vector potential". Below we show in what point this foundation is broken. Note that the d'Alembert operator can only be applied to scalars, otherwise it yields a non-covariant expression. As such, the equation (1) is non-covariant. Non-covariant equations are out of physical meaning, hense, this equation is physically meaningless. No other critiques of the method of retarded potentials is needed.

As for our straightforward solution of the field equation [2]

$$\mathrm{d}\,{}^*\mathrm{d}\alpha, \quad \alpha = A_\mu \mathrm{d}x^\mu, \tag{2}$$

it was obtained [2, 1] in a coordinate system which covers only a quarter of the space-time specified in standard Cartesian coordinates by the inequality

$$z \geq |t|. \tag{3}$$

This domain lies in the future from the half-hyperplane

$$z = -t \geq 0, \tag{4}$$

hence, in principle, there well may present an extra source of the field in the wedge of the past $z \leq -t \geq 0$. If there exists such an extra source, the solution obtained describes the field of two sources instead of the desired one. It will be shown below that our solution does describe the field of uniformly accelerated point-like charge.

2 Uniformly accelerated coordinate systems

When solving a problem of mathematical physics it is important to select of the most appropriate coordinate system. In case of the problem of the field of uniformly accelerated charge it is reasonable to select a system adapted to the charge world line. By coordinate system adapted to a curve we mean one which contains the curve as one of coordinate lines. In our work [2] the problem was solved in uniformly accelerated spherical coordinates. Below we outline the way it has been built.

Let $\{t, z, \rho, \varphi\}$ be the well-known round cylinder coordinate system extended with Lorenzian time t. Then the following transformation

$$\zeta = \sqrt{z^2 - t^2}, \qquad \xi = \operatorname{artanh}\frac{t}{z} \tag{5}$$

$$t = \zeta \sinh \xi, \qquad z = \zeta \cosh \xi \tag{6}$$

yields the simplest axially-symmetric uniformly accelerated coordinate system $\{\xi, \zeta, \rho, \varphi\}$ with time-like coordinate lines being world lines of uniformly accelerated motion. Each point specified by constant values of coordinates moves with acceleration equal to ζ^{-1}. Coordinate surfaces $\xi = \text{const}$ are 3-planes orthogonal to the curves. The transformation (5) itself is analogue of passage from Cartesian $\{z, \rho\}$ to polar $\{\zeta, \xi\}$ coordinates on pseudo-euclidean plane with ζ and ξ as radius and angle correspondingly. Differentiation of these equations gives the following expressions for the differentials $\mathrm{d}\zeta$ and $\mathrm{d}\xi$:

$$\mathrm{d}\zeta = \frac{z\mathrm{d}z - t\mathrm{d}t}{\sqrt{z^2 - t^2}}, \quad \mathrm{d}\xi = \frac{z\mathrm{d}t - t\mathrm{d}z}{z^2 - t^2}. \tag{7}$$

Note that this coordinate system covers only the quarters $|z| \geq |t|$ of the space-time, hence, is incomplete.

This coordinate system underlies other uniformly accelerated systems which possess rotational symmetry and hence, share the coordinates ξ and φ with it. One of them is the system of uniformly accelerated spherical coordinates $\{\xi, u, v, \varphi\}$ used in our works [2, 1]. Uniformly accelerated spherical coordinates were built by the following transformation:

$$e^{-2u} = \tfrac{(\zeta - a)^2 + \rho^2}{(\zeta + a)^2 + \rho^2}, \qquad \tan v = \frac{2a\rho}{\zeta^2 - a^2 + \rho^2} \tag{8}$$

$$\zeta = \frac{a \sinh u}{\cosh u + \cos v}, \quad \rho = \frac{a \sin v}{\cosh u + \cos v}. \tag{9}$$

This transformation is known to turn Cartesian coordinates $\{z, \rho\}$ into bipolar ones $\{u, v\}$ on plane and underlies bi-spherical and toroidal systems [4]. The value $u = \infty$ of the coordinate u corresponds to the poles $z = \pm a$, $\rho = 0$ and other values of this coordinate do to circles whose centres lie on the z-axis. In the space-time this point moves with acceleration equal to a^{-1}. Each point of a sphere $u = con$ also moves with constant acceleration equal to ζ^{-1} so that spherical shape in its frame conserves.

3 Possibility of presence of an extra source of the field

Solution of the equation (2) obtained in our work [2], which was represented to describe the field of point-like charge, or, in general, of a charged spherical shell, has the form

$$\alpha = q(\cosh u - 1). \tag{10}$$

However, since the coordinate system (8) used is incomplete, a question arises, if there is another source of the field beyond the chart, thus, under

$$0 \leq z \leq -t. \tag{11}$$

In this section, it will be shown that there are no extra sources of the field represented by our solution. In other words, we prove that the solution obtained does not provide the genuine expression of the field of uniformly accelerated point-like charge.

To do this, note that, according to causality principle, the field of the charge is non-zero only in points which are in causal connection with points of the charge world line. The world line in question can be represented in coordinates $\{t, z, \rho, \varphi\}$ by the equation

$$z^2 - t^2 - a^2.$$

So, the half-space $z + t \leq 0$ has no causal connections with point of this line. Therefore, the field of the charge is identically zero in this half-space and on its boudary. In particular, the field is zero on the half-plane

$$z + t = 0, \quad z \geq 0. \tag{12}$$

This fact allows to establish presence of and extra source of the field, which might exist in the half-space (11). Indeed, as the charge does not produce non-zero field strength on the half plane, such a strength can only be produced by a source which can only exist there. Consequently, to prove that there is no such a source, it suffices to show that the field strength $\mathrm{d}\alpha$ where the 1-form α is given by the equation (10), is zero. Differentiation of the 1-form α yields:

$$\mathrm{d}\alpha = -q \sinh u \mathrm{d}\xi \wedge \mathrm{d}u, \tag{13}$$

so, the next task is to find it on the half-plane (12).

4 Calculations and proof

Differentiation of the first equation (8) yields

$$\begin{aligned} -2e^{-2u}\mathrm{d}u = \; & 2\frac{[(\zeta + a)^2 + \rho^2][(\zeta - a)\mathrm{d}\zeta + \rho\mathrm{d}\rho] - [(\zeta - a)^2 + \rho^2][(\zeta + a)\mathrm{d}\zeta + \rho\mathrm{d}\rho]}{[(\zeta + a)^2 + \rho^2]^2} - \\ & -2\frac{[(\zeta - a)^2 + \rho^2][(\zeta + a)\mathrm{d}\zeta + \rho\mathrm{d}\rho] - [(\zeta + a)^2 + \rho^2][(\zeta - a)\mathrm{d}\zeta + \rho\mathrm{d}\rho]}{[(\zeta - a)^2 + \rho^2]^2} \end{aligned}$$

Note that the expression obtained is odd on a, therefore the factor at $\mathrm{d}\rho$ disappears and the result is

$$-2e^{-2u}\mathrm{d}u = 4\frac{(\zeta^2 - a^2 - \rho^2)\cdot 2a\mathrm{d}\zeta}{[(\zeta + a)^2 + \rho^2]^2}.$$

Now, we pass to inertial round cylinder coordinates:

$$e^{-2u}\mathrm{d}u = 2\frac{(\rho^2 + a^2 + t^2 - z^2)\cdot 2a(z\mathrm{d}z - t\mathrm{d}t)}{[(\sqrt{z^2 - t^2} + a)^2 + \rho^2]^2}.$$

At the same time,

$$\sinh u = \frac{1}{2}\left(\sqrt{\frac{(\zeta + a)^2 + \rho^2}{(\zeta - a)^2 + \rho^2}} - \sqrt{\frac{(\zeta - a)^2 + \rho^2}{(\zeta + a)^2 + \rho^2}}\right) = \tag{14}$$

$$= \frac{2a\zeta}{\sqrt{[(\zeta - a)^2 + \rho^2][(\zeta + a)^2 + \rho^2]}} =$$

$$= \frac{2a\sqrt{z^2 - t^2}}{\sqrt{[(\sqrt{z^2 - t^2} - a)^2 + \rho^2][(\sqrt{z^2 - t^2} + a)^2 + \rho^2]}}.$$

Now, we compose explicitly the 2-form (7,13) in standard round cylinder coordinates:

$$e^{-2u}\mathrm{d}\alpha = \frac{2a\sqrt{z^2 - t^2}}{\sqrt{[(\sqrt{z^2 - t^2} - a)^2 + \rho^2][(\sqrt{z^2 - t^2} + a)^2 + \rho^2]}}\times$$

$$\frac{(\rho^2 + a^2 + t^2 - z^2)\cdot 2a(z\mathrm{d}z - t\mathrm{d}t) + 2a\sqrt{z^2 - t^2}\,\rho\mathrm{d}\rho}{[(\sqrt{z^2 - t^2} + a)^2 + \rho^2]^2}\wedge\frac{z\mathrm{d}t - t\mathrm{d}z}{z^2 - t^2}.$$

This 2-form is to be taken on the half-plane (12) on which $u = 0\, z = t$. Note that $(z\mathrm{d}z - t\mathrm{d}z)\wedge(z\mathrm{d}t - t\mathrm{d}z) = (z^2 - t^2)\mathrm{d}t\wedge\mathrm{d}z$. Hence,

$$\mathrm{d}\alpha = \frac{2a\sqrt{z^2 - t^2}}{\sqrt{[(\sqrt{z^2 - t^2} - a)^2 + \rho^2][(\sqrt{z^2 - t^2} + a)^2 + \rho^2]}}\times$$

$$\times\frac{(\rho^2 + a^2 + t^2 - z^2)\cdot 2a(z^2 - t^2)\mathrm{d}t\wedge\mathrm{d}z + 2a\sqrt{z^2 - t^2}\,\rho\mathrm{d}\rho\wedge\,(z\mathrm{d}t - t\mathrm{d}z)}{[(\sqrt{z^2 - t^2} + a)^2 + \rho^2]^2(z^2 - t^2)}$$

that vanishes under $z = t$. Thus, the 2-form $\mathrm{d}\alpha$ is zero on the half-plane (12). Consequently, uniqueness of the uniformly accelerated charge as the source of the field presented by the 1-form α (10) is established. Similarly, the field strength is zero also on the second half-plane $z - t = 0, \quad z \geq 0$ hence, it is zero also in the wedge of future $t \geq |z|$. Thereby, correctness of the solution of the problem of the field of uniformly accelerated charge, given by this 2-form, is also established, but this fact gives rise to another problem. This problem is discussed in the next section.

5 Conclusion

The result obtained in this work, evidently contradicts common beliefs mentioned in the section **Introduction**. These beliefs read, in particular, that if in some space-time point strength of the field is not zero, so it is in any other point which lies inside the light cone from it. Contrary to this belief, the field of a uniformly accelerated charge obtained as an exact solution of the field equations, has zero strength in the wedge $t \geq |z|$ which overlaps the cone. Sine only field equations and their solutions can tell us, what is right and what is wrong in field theory, the result of our investigation signifies that the common belief mentioned above, is erroneous with all ensuing consequences. Since this belief is a straightforward consequence of alleged existence of Green functions for the electromagnetic field equation (2) and hence, Maxwell equations, our result actually signifies that these equations do not possess Green functions. This result shows also that if one needs an adequate representation of electromagnetic field, he has to solve the field equations as they stand, all the rest yields only erroneous expressions.

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