Proof of Legendre's conjecture

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Abstract

By a consideration of this research, since we found that at least one prime number exists between n^2 and $n(n+1)$ when $n \geqq 3$, we have obtained a conclusion that Legendre's conjecture is true.

Contents

1. Introduction

This is the conjecture that there is a prime number always between n^2 and $(n+1)^2$ for arbitrary positive integer n. It was set up by the French mathematician Adrien-Marie Legendre. (Quoted from Wikipedia)

2. Proof

Let n and p be positive integers. If Legendre's conjecture is true, there is at least one prime number p satisfying the following inequalities.

$$n^2 < p < (n+1)^2 \ (n \geqq 1) \ \ldots (1)$$

Ⅰ. When $n < 3$

The inequalities (1) hold when $n = 1$ since there are prime numbers 2 and 3 between 1 and 4. In the same way, the inequalities hold when $n = 2$ since there are prime numbers 5 and 7.

Ⅱ. When $n \geqq 3$

Suppose that p satisfies the following inequalities,

$$n^2 < p < n(n+1) \ \ldots (2)$$

Let q be a positive integer. If any p is a composite number and has prime factors, the largest possible factor of p in the range of the inequalities (2) is $n(n+1)/2 - 1$ and p must be divided by q from $n+2$ to $(n^2+n-2)/2$ since n and $n+1$ cannot divide p and the product of two factors between 2 and $n-1$ cannot be p.

$$n+1 < q < n(n+1)/2 \ \dots (3)$$

We will consider the case where p is divisible by q satisfying the inequalities above. Let r be a positive integer and the quotient when p is divided. r must be satisfied the following inequalities.

$$1 < r < n \ \dots (4)$$

If it is assumed that p are all composite numbers in the range of the inequalities (2), p must be divided by q in the inequalities (3). When p is a composite number, one p corresponds to some combinations of q and r. q has a one-to-one correspondence with p since the minimum number of q, $n+2$ is greater than the number of p satisfying the inequalities (2), $n-1$ and r has a one-to-many correspondence with p.

We need to apply a rule to select the relations from p to r and consider the case when $n = 9$.

When $p = 82$, $(q, r) = (41,2)$

When $p = 84$, $(q, r) = (42,2), (28, 3), (21,4), (14, 6), (12, 7)$

When $p = 85$, $(q, r) = (17,5)$

When $p = 86$, $(q, r) = (43,2)$

When $p = 87$, $(q, r) = (29,3)$

When $p = 88$, $(q, r) = (44,2), (22, 4), (11,8)$

Define $[p, r]$ as a relation from p to r. We will select the relations between p and r so that there are all one-to-one correspondences. At first the relations are selected by r which are multiples of 2 for each p. $[82,2]$ is sorted out when $p = 82$. Then $[84,4]$ is sorted out since $r = 2$ has been selected. When $p = 86$, there is one combination $(q, r) = (43,2)$ and $r = 2$ has been taken from. In this case, we consider to use the factor 2 of 6 and think that there is a relation $[86,6]$. Then $[88,8]$ is sorted out. Next, we select the relation by 3 multiples r and $[87,3]$ is sorted out.

When r is a composite number we skip the number since we have already taken from the relations by a multiple of the prime factor of r. Next, we select the relations by multiples of prime numbers greater than or equal to 5.

Let $a(n, r)$ and $b(n, r)$ be integers and $a(n, r)$ be the number of r multiples in the range of the inequalities (2) and $b(n, r)$ be that in the range of the inequalities (4). The following inequalities hold.

$$a(n, r) \leqq b(n, r) + 1$$

When $n = 8$, $a(8,5) = 2$, $b(8,5) = 1$ and $a(8,5) > b(8,5)$ hold.
When $p = 65$, $(q, r) = (13,5)$
When $p = 66$, $(q, r) = (33,2), (22,3), (11,6)$
When $p = 68$, $(q, r) = (34,2), (17,4)$
When $p = 69$, $(q, r) = (23,3)$
When $p = 70$, $(q, r) = (35,2), (14,5), (10,7)$
In this case, $[65,5]$ can be selected since $[70,6]$ has already sorted out when $r = 2$. Let s be a prime number less than r. In the case of $a(n, r) > b(n, r)$, the actual increase in the number of relations between p and r is less than or equal to $b(n, r)$ at the time of making the selection because one of the s adjacent multiples of r is a multiple of s and the relations have already been selected by s multiples. The value of s can be considered 2 or 3 since $a(n, s) = b(n, s)$ holds as follows.

Let m be an integer.

・When $n = 2m$ and $m > 1$
$a(2m, 2) = \text{floor}(((2m)^2 + 2m - 1)/2) - \text{floor}((2m)^2/2) = m - 1$
$b(2m, 2) = \text{floor}((2m - 1)/2) = m - 1$
・When $n = 2m + 1$ and $m > 0$
$a(2m + 1,2) = \text{floor}(((2m + 1)^2 + 2m + 1 - 1)/2) - \text{floor}((2m + 1)^2/2) = m$
$b(2m + 1,2) = \text{floor}((2m + 1 - 1)/2) = m$
Therefore, $a(n, 2) = b(n, 2)$ holds when $n \geqq 3$.

・When $n = 3m$ and $m > 0$
$a(3m, 3) = \text{floor}(((3m)^2 + 3m - 1)/3) - \text{floor}((3m)^2/3) = m - 1$
$b(3m, 3) = \text{floor}((3m - 1)/3) = m - 1$
・When $n = 3m + 1$ and $m > 0$
$a(3m + 1,3) = \text{floor}(((3m + 1)^2 + 3m + 1 - 1)/3) - \text{floor}((3m + 1)^2/3) = m$
$b(3m + 1,3) = \text{floor}((3m + 1 - 1)/3) = m$
・When $n = 3m + 2$ and $m > 0$
$a(3m + 2,3) = \text{floor}(((3m + 2)^2 + 3m + 2 - 1)/3) - \text{floor}((3m + 2)^2/3) = m$
$b(3m + 2,3) = \text{floor}((3m + 2 - 1)/3) = m$
Therefore, $a(n, 3) = b(n, 3)$ holds when $n \geqq 3$.
From the above, the prime number r with $a(n, r) > b(n, r)$ satisfies $r \geqq 5$.

We will consider the case when $n = 17$.
When $n = 17$, $a(17,5) = 4$, $b(17,5) = 3$ and $a(17,5) > b(17,5)$ hold.

When $p = 290$, $(q, r) = (145,2), (58,5), (29,10)$

When $p = 291$, $(q, r) = (97,3)$

When $p = 292$, $(q, r) = (146,2), (73,4)$

When $p = 294$, $(q, r) = (147,2), (98,3), (49,6), (42,7), (21,14)$

When $p = 295$, $(q, r) = (59,5)$

When $p = 296$, $(q, r) = (148,2), (74,4), (37,8)$

When $p = 297$, $(q, r) = (99,3), (33,9), (27,11)$

When $p = 298$, $(q, r) = (149,2)$

When $p = 299$, $(q, r) = (23,13)$

When $p = 300$, $(q, r) = (150,2), (100,3), (75,4), (60,5), (50,6), (30,10), (25,12), (20,15)$

When $p = 301$, $(q, r) = (43,7)$

When $p = 302$, $(q, r) = (151,2)$

When $p = 303$, $(q, r) = (101,3)$

When $p = 304$, $(q, r) = (152,2), (76,4), (38,8), (19,16)$

When $p = 305$, $(q, r) = (61,5)$

In the beginning, we select the relations $[290,2]$, $[292,4]$, $[294,6]$, $[296,8]$, $[298,10]$, $[300,12]$, $[302,14]$ and $[304,16]$ when $r = 2$. Then we select $[291,3]$, $[297,9]$ and $[303,15]$ when $r = 3$. When $r = 5$, we should select the relations in the case of $p = 295$ and $p = 305$. However, there is only 5 for r which corresponds to p since 10 and 15 have already been taken from. With this method, we cannot select the one-to-one relations between p and r.

And so we will change the rules as follows. We select relations by multiples of the prime numbers in descending order. When $a(n, r) > b(n, r)$ holds, a composite number p can be skipped since one of the relations can later be selected by multiples of a prime number less than r. When $n = 17$, the relations are selected as follows.

When $r = 13$, $[299,13]$

When $r = 11$, $[297,11]$

When $r = 7$, $[294,7],[301,14]$

When $r = 5$, $[290,5],[295,10],[305,15]$

When $r = 3$, $[291,3],[300,6],[303,9]$

When $r = 2$, $[292,2],[296,4],[298,8],[302,12],[304,16]$

The minimum n when there exists r with $a(n, r) > b(n, r)$ is 7 and the minimum r where $a(n, r) > b(n, r)$ holds is 5. Therefore, if we select the relations this way, one-to-one correspondence with all composite numbers p can be set for r, for all n in the range of $n \geqq 3$.

However, it becomes a contradiction since the number of p in the inequalities (2), $n-1$ is greater than the number of r in the inequalities (4), $n-2$ and it does not become a one-to-one correspondence between p and r. Therefore, the assumption that p are all composite numbers in the range is false and there is at least one prime number in the range of the inequalities (2) when $n \geqq 3$. From the above Ⅰ and Ⅱ, it is proved that Legendre's conjecture is true. (Q.E.D.)

3. Complement

Oppermann's conjecture states that, for every integer $x > 1$, there is at least one prime number between $x(x-1)$ and x^2, and at least another prime between x^2 and $x(x+1)$. It is named after Danish mathematician Ludvig Oppermann, who announced it in an unpublished lecture in March 1877. (Quoted from Wikipedia)

$$x(x+1) < p < x(x+2)$$

Considering an integer p satisfying these inequalities, because $a(x,r) \leqq b(x,r)+1$ holds and the minimum x where there exists r with $a(x,r) > b(x,r)$ is 9 and the minimum r where $a(x,r) > b(x,r)$ holds is 7, we found that at least one prime number between $x(x+1)$ and $(x+1)^2$ when $x \geqq 3$ in the same way as this proof. Therefore, we conclude that Oppermann's conjecture is true.

4. Acknowledgement

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5. References

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