ON ODD PERFECT NUMBERS

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Abstract. In this note, we introduce the notion of the disc induced by an arithmetic function and apply this notion to the odd perfect number problem. We show that no odd perfect numbers exist by exploiting this concept.

1. Introduction

Let $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ denotes the sum-of-divisor function, defined as

$$\sigma(N) := \sum_{n|N} 1$$

for a fixed $N \in \mathbb{N}$. We say N is a perfect number if and only if $\sigma(N) = 2N$. If N is perfect and is odd then we say it is an odd perfect number. It is still unknown if there exist any odd perfect numbers and the problem for asserting their existence or non-existence still remains an active area of research. Much work has already been done in this area and most subtle and basic properties about odd perfect - if they exist - are now known. The eighteenth century mathematician Leonard Euler was the first to show that if any odd perfect number N exists then it must be of the form

$$N := q^{\beta} \prod_{i=1}^{n} p_i^{\alpha_i}$$

where $q, \beta \equiv 1 \pmod 4$ and $\alpha_i \equiv 0 \pmod 2$ for each $1 \leq i \leq n$. It is also know that, if an odd perfect number N exists then it must satisfy the inequality $N > 10^{1500}$ [1]. It is also known that (see [2]) an odd perfect number must not be divisible by 105 and must satisfy the congruence conditions (see [3])

$$N \equiv 1 \pmod{12} \quad \textbf{and} \quad N \equiv 117 \pmod{468} \quad N \equiv 81 \pmod{324}.$$

If there are k of the exponents α_i in the prime factorization of N with $\alpha_i \equiv 0 \pmod 2$, then it is known that the smallest prime factor of N is at most $\frac{k-1}{2}$ [4]. In this case, it has been shown that (see [5])

$$N < 2^{4^{k+1} - 2^{k+1}}$$

and with $q \prod_{i=1}^{k} p_i < 2N^{\frac{17}{26}}$ [6]. The scale of the largest and the second largest prime factor of an odd perfect number - if they exist - has also been studied quite extensively in a series of papers by several authors. It is now known that the largest prime factor of N is greater than 10^{18} (see [7]) and less than $(3N)^{\frac{1}{3}}$ [8]. It has also

Date: October 21, 2021.

2000 *Mathematics Subject Classification.* Primary 54C40, 14E20; Secondary 46E25, 20C20.

Key words and phrases. canonical; odd perfect; canonical product; disc; degenerative; non-degenerative.

been shown that the second largest prime factor of an odd perfect number N must be greater than 10^4 and less than $(2N)^{\frac{1}{5}}$ [9]. The third largest prime factor is now known to be greater than 100. All of these result could conceivably be synthesized in a nice way to study the main question of the existence or non-existence of an odd perfect number.
In this paper, by using the notion of the disc induced by arithmetic functions, we show that there exist no odd perfect numbers.

2. **Preliminary results**

Lemma 2.1 (Euler). *Let N be an odd perfect number, then N has the unique representation*

$$N = q^{\beta}\prod_{i=1}^{n} p_i^{\alpha_i}$$

where $q, \beta \equiv 1 \pmod 4$ and $\alpha_i \equiv 0 \pmod 2$ for each $1 \leq i \leq n$.

Theorem 2.2. *If N an odd perfect number then*

$$\varphi(N) \leq \lfloor \frac{N}{2} \rfloor$$

where φ denotes the Euler totient function.

Proof. Let us assume there exists an odd perfect number N. It is clear that N must be composite so that by the fundamental theorem of arithmetic and Lemma 3.4 the representation holds for N

$$N := q^{\beta}\prod_{i=1}^{n} p_i^{\alpha_i}$$

where $q, \beta \equiv 1 \pmod 4$ and $\alpha_i \equiv 0 \pmod 2$ for each $1 \leq i \leq n$. Next let us apply the sum-of-divisor function σ on N and study their internal structure

$$\sigma(N) := \sigma\bigg(q^{\beta} \times \prod_{i=1}^{n} p_i^{\alpha_i}\bigg).$$

Since the sum-of-divisor function σ is multiplicative, we obtain further the decomposition

$$\sigma(N) = \bigg(\sum_{j=0}^{\beta} q^{\beta-j}\bigg) \times \bigg(\prod_{i=1}^{n}\sum_{j=0}^{\alpha_i} p_i^{\alpha_i-j}\bigg)$$

where we have used the elementary identity $x^n - 1 = (x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)$. Since we have assumed that N is an odd perfect number, it follows that $\sigma(N) = 2N$ so that

$$2q^{\beta}\prod_{i=1}^{n} p_i^{\alpha_i} = \bigg(\sum_{j=0}^{\beta} q^{\beta-j}\bigg) \times \bigg(\prod_{i=1}^{n}\sum_{j=0}^{\alpha_i} p_i^{\alpha_i-j}\bigg).$$

By rearranging terms the following representation holds

$$\bigg(\sum_{j=0}^{\beta}\frac{1}{q^j}\bigg) \times \bigg(\prod_{\substack{i=1\\ p_i^{\alpha_i}||N}}^{n}\sum_{j=0}^{\alpha_i}\frac{1}{p_i^j}\bigg) = 2.$$

Then we have the inequality

$$2 = \left(\sum_{j=0}^{\beta}\frac{1}{q^j}\right) \times \left(\prod_{\substack{i=1 \\ p_i^{\alpha_i}||N}}^{n}\sum_{j=0}^{\alpha_i}\frac{1}{p_i^j}\right)$$
$$\leq \prod_{p|N}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots+\right)$$
$$= \frac{N}{\varphi(N)}$$

and the claimed inequality follows immediately. □

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3. The notion of the disc induced by arithmetic functions and application to the odd perfect number problem

In this section we introduce and study the notion of the disc induced by arithmetic functions. We find this notion suitable for verifying the non-existence of odd perfect numbers. We launch the following language.

Definition 3.1. Let $f:\mathbb{N}\longrightarrow\mathbb{N}$ and let $a,r\in\mathbb{N}$ be fixed. Then by the disc induced by f with center a and radius r, denoted $\mathcal{D}_f(a,r)$, we mean

$$\mathcal{D}_f(a,r) := |f(m)-a|\leq r$$

for $m\in\mathbb{N}$. We say $s\in\mathcal{D}_f(a,r)$ if and only if $|f(s)-a|\leq r$. We say the disc induced is **degenerative** if there exists some $t\in\mathcal{D}_f(a,0)$ and we call $\mathcal{D}_f(a,0)$ the degenerated disc. Otherwise we say the disc induced is non-degenerative. We say the disc induced is **uniformly** degenerative if it is degenerative for all $a\in\mathbb{N}$.

Proposition 3.2. The following properties hold

(i) Let $g:\mathbb{N}\longrightarrow\mathbb{N}$ be multiplicative and $s=uv$ with $(u,v)=1$ with $u,v>1$. If $s\in\mathcal{D}_g(a,r)$ for a fixed $r,a\in\mathbb{N}$, then $u\in\mathcal{D}_g(a,r-\epsilon)$ and $v\in\mathcal{D}_g(a,r-\delta)$ for some $\epsilon,\delta>0$.

(ii) Let $g:\mathbb{N}\longrightarrow\mathbb{N}$ be additive and $s=uv$ with $(u,v)=1$ with $u,v>1$. If $s\in\mathcal{D}_g(a,r)$ for a fixed $r,a\in\mathbb{N}$, then $u\in\mathcal{D}_g(a,r-\epsilon)$ and $v\in\mathcal{D}_g(a,r-\delta)$ for some $\epsilon,\delta>0$.

Proof. We only prove property (i) since the same approach could be adapted for property (ii). Let $s\in\mathcal{D}_g(a,r)$ and write $s=uv$ such that $(u,v)=1$ with $u,v>1$. Then since g is multiplicative we can write

$$|g(u)-a|<|g(s)-a|=|g(u)g(v)-a|\leq r$$

so that there exists some $\epsilon>0$ such that $|g(u)-a|+\epsilon\leq r$ and it follows that $u\in\mathcal{D}_g(a,r-\epsilon)$. It follows similarly that there exists some $\delta>0$ such that $v\in\mathcal{D}_g(a,r-\delta)$. □

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Remark 3.3. Now we verify an important but yet trivial preparatory observation for asserting the truth of our main result. It conveys the principal notion that no degenerated disc induced by an arithmetic function will ever contain a composite.

Lemma 3.4. *Let $g : \mathbb{N} \longrightarrow \mathbb{N}$ be multiplicative (resp. additive). If s is an odd composite, then $s \notin \mathcal{D}_g(a, 0)$.*

Proof. Since s is an odd composite we can write $s = uv$ such that $(u, v) = 1$ with $u, v \geq 3$ by the fundamental theorem of arithmetic. Let us assume to the contrary that $s \in \mathcal{D}_g(a, 0)$ then since g is multiplicative, it follows from Proposition 3.2 that

$$u \in \mathcal{D}_g(a, -\epsilon) \quad v \in \mathcal{D}_g(a, -\delta)$$

for some $\epsilon, \delta > 0$. This is impossible since the radius of each the degenerated disc is negative, thereby proving the lemma. □

Theorem 3.5 (Main theorem)**.** *There exists no odd perfect numbers.*

Proof. Suppose there exist an odd perfect number N. Since no prime number can be perfect, it follows that N must be composite and by the fundamental theorem of arithmetic, we can write $N = uv$ with $(u, v) = 1$ with $u, v > 1$. We note that the assumption that N is an odd perfect number implies $N \in \mathcal{D}_\sigma(2N, 0)$ and it follows that

$$u \in \mathcal{D}_\sigma(2N, -\epsilon) \quad v \in \mathcal{D}_\sigma(2N, -\delta)$$

for some $\epsilon, \delta > 0$. This is impossible since the radius of each the degenerated disc is negative, thereby proving the theorem. □

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